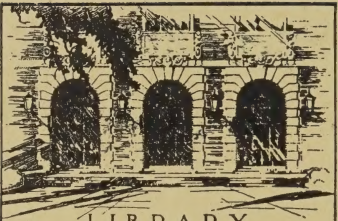

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BY

GEORGE PEACOCK, A.M. F.R.S. F.C.P.S.

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PREFACE.

The present Work was originally intended as a Supplement to a Translation of an Elementary Treatise, on the Differential and Integral Calculus, from the French of Lacroix, which was published in 1816: in the course of its progress, however, it was thought proper to enlarge the plan, and to make it a collection of Examples illustrative of all the more important analytical applications of the Differential and Integral Calculus, without any particular reference to the work in question.

The principal object of this undertaking is to remedy a defect, which exists in a greater or less degree, in nearly all the foreign works on this part of analysis, in which the theory is most frequently exhibited in its most general form, without the illustration which particular examples are calculated to afford: the student is consequently bewildered in the generality of the relations which are presented to his mind, and he either rests contented with a vague and imperfect comprehension of the subject, or abandons a study which becomes repulsive and fatiguing from its obscurity and difficulty.

The works of Euler, however, form a distinguished exception to this observation, as they every where abound with the utmost fulness of illustration; his two great works on this subject, the *Institutiones Calculi Differentialis*, and the *Institutiones Calculi Integralis*, are remarkable both for the

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number and the elegance of their Examples, which have been borrowed with little scruple or acknowledgement by succeeding authors: but rich and various as are the stores which they contain, they cannot be considered such, as to render the present Work unnecessary; for not to mention their bulk and rarity, and the numerous additions to the science made since their publication by Lagrange and other analysts, they contain much that is unnecessary, at least, to a student, and much that is capable of being exhibited in a better form.

Whatever has been borrowed from other authors, has been generally acknowledged, and a very slight inspection of this work would shew the great extent of our obligations; we say generally, for we have omitted, for the most part, any reference to examples taken from the great work of Lacroix, and in some other cases when the authors from whom they were taken, were not considered worth consulting: our object being to enable the student to refer to the original authors, whenever he may find any difficulty in the solution of a problem, or may be anxious to obtain more complete information on the subject of it.

There are several portions of this Work, particularly of the first part of it, which upon more mature consideration, we should wish to see omitted or abridged, as useless or redundant; we refer more particularly to the examples connected with Maclaurin's theorem, and the developement of functions.

The integrals have been taken almost entirely from the *Integraltefeln*, or *Tables of Integral Formulæ*, of Meyer Hirsch, of Berlin, a work remarkable for its extent, accuracy, and elegance. The utility of this portion of the Work, might have been materially increased, if some notice of the method as well as the formulæ of solution, had accompanied each class of integrals.

The same anxiety for compression, induced us to omit all notice of the important subject of Definite Integrals: if ever this Work should be honoured by a second edition, we hope to be able to supply these defects, which we cannot but consider as very considerable.

There is one subject, for the omission of which we need make no apology, as the Work has already been performed by a gentleman incomparably better calculated to do justice to it, than ourselves; we mean the Calculus of Variations: as the learned and interesting treatise of Professor Woodhouse upon this subject, which has long been before the Public, is accompanied by a collection of excellent examples, it would have been useless as well as impertinent in us, to have attempted to add to their number.

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PART I.

ON THE DIFFERENTIATION OF ALGEBRAICAL FUNCTIONS OF ONE VARIABLE.

ART. 10. (1). Let $u = a x^3 + b x^2 + c x + e$: its differential, or $d u = 3 a x^2 d x + 2 b x d x + c d x$.

Art. 11 – 17. (2). Let $u = x\ (1+x)$.

$$d u = (1 + 2 x)\ d x.$$

(3). Let $u = x^3\ (a + x)^2$.

$$d u = (3 a + 5 x)\ (a + x)\ x^2\ d x.$$

(4). Let $u = (a+x)^n\ (b+x)^m$.

$$d u = \{ n b + m a + (n+m)\ x \}\ (a+x)^{n-1}\ (b+x)^{m-1}\ d x.$$

(5). Let $u = (1+x^2)^3\ (1+x)^4$.

$$d u = \{ 4 + 6 x + 10 x^2 \}\ (1+x^2)^2\ (1+x)^3\ d x.$$

(6). Let $u = x\ (1+x)\ (1+x^2)$.

$$d u = \{ 1 + 2 x + 3 x^2 + 4 x^3 \}\ d x.$$

(7). Let $u = (a+x)\ (b+x)\ (c+x)$.

$$d u = \{ a b + a c + b c + 2 (a+b+c)\ x + 3 x^2 \}\ d x.$$

(8). Let $u=(2-x)\,(3-x)\,(4-x)$.

$$du=-\{26-18x+3x^2\}\,dx.$$

(9). Let $u=x^2\,(a+x)^3\,(b-x)^4$.

$$du=\{2ab-(6a-5b)x-9x^2\}\,x\,(a+x)^2\,(b-x)^3\,dx.$$

(10). Let $u=(a+bx^n)^m\,(a'+b'x^{n'})^{m'}$.

$$du=\{mnba'x^{n-1}+m'n'b'ax^{n'-1}+(mn+m'n')\,bb'x^{n+n'-1}\}\times(a+bx^n)^{m-1}\,(a'+b'x^{n'})^{m'-1}\,dx.$$

(11). Let $u=(a+bx^n)^m\,(a'+b'x^{n'})^{m'}\,(a''+b''x^{n''})^{m''}$.

$$du=\{mna'a''bx^{n-1}+m'n'aa''b'x^{n'-1}+m''n''aa'b''x^{n''-1}+(mn+m'n')\,a''bb'x^{n+n'-1}+(mn+m''n'')\,a'bb''x^{n+n''-1}+(m'n'+m''n'')\,ab'b''x^{n'+n''-1}+(mn+m'n'+m''n'')\,bb'b''x^{n+n'+n''-1}\}\times(a+bx^n)^{m-1}\,(a'+b'x^{n'})^{m'-1}\,(a''+b''x^{n''})^{m''-1}\,dx.^*$$

---

* The differentiation of complicated functions is sometimes rendered much easier by means of the following theorems.

(1). Let $u=v^n$, where $v=f(x)$: we readily find

$$du=\frac{u}{v}.n\,dv.$$

(2). Let $u=v_1.v_2.v_3\ldots.v_n$: in this case, we have

$$du=u\left\{\frac{dv_1}{v_1}+\frac{dv_2}{v_2}+\frac{dv_3}{v_3}+\&c.+\frac{dv_n}{v_n}\right\}.$$

(3). Let $u=\dfrac{v_1.v_2.v_3\ldots v_n}{v'_1.v'_2.v'_3\ldots v'_m}$: in this case also, we find

$$\frac{du}{u}=\left\{\frac{dv_1}{v_1}+\frac{dv_2}{v_2}+\&c.+\frac{dv_n}{v_n}-\frac{dv'_1}{v'_1}-\frac{dv'_2}{v'_2}-\&c.-\frac{dv'_m}{v'_m}\right\}.$$

These theorems were first given by Maclaurin in his Fluxions, Arts. 715. and 717.

(12). Let $u = \frac{a}{x}$.

$$d u = -\frac{a\,d x}{x^2}.$$

(13). Let $u = \frac{a}{(a-x)^3}$.

$$d u = \frac{3\,a\,d x}{(a-x)^4}.$$

(14). Let $u = \frac{x}{1+x}$.

$$d u = \frac{d x}{(1+x)^2}.$$

(15). Let $u = \frac{x}{1+x^2}$.

$$d u = \frac{(1-x^2)\,d x}{(1+x^2)^2}.$$

(16). Let $u = \frac{a+x}{b+x}$.

$$d u = \frac{(b-a)\,d x}{(b+x)^2}.$$

(17). Let $u = \frac{1+x^2}{1-x^2}$.

$$d u = \frac{4\,x\,d x}{(1-x^2)^2}.$$

(18). Let $u = \frac{x^n}{(1+x)^n}$.

$$d u = \frac{n\,x^{n-1}\,d x}{(1+x)^{n+1}}.$$

(19). Let $u = \frac{(1+x)^3}{(1-x)^4}$.

$$du = \frac{(1+x)^2}{(1-x)^5}(7+x)\,dx.$$

(20). Let $u = \frac{(a+b\,x^n)^m}{(a'+b'\,x^{n'})^{m'}}$.

$$du = \frac{\{\,mnba'x^{n-1} - m'n'ab'x^{n'-1} + (mn - m'n')\,bb'x^{n+n'-1}\,\} \times (a+bx^n)^{m-1}\,dx}{(a'+b'\,x^{n'})^{m'+1}}$$

(21). Let $u = \frac{x^3+x}{x^4-x^2+1}$.

$$du = \frac{-\{\,x^6+4\,x^4-4\,x^2-1\,\}\,dx}{(x^4-x^2+1)^2}.$$

(22). Let $u = \frac{x^2-x+1}{x^2+x+1}$.

$$du = \frac{2\,(x^2-1)\,dx}{(x^2+x+1)^2}.$$

(23). Let $u = \frac{1}{x^{\frac{1}{2}}}$.

$$du = \frac{-dx}{2\,x^{\frac{3}{2}}}.$$

(24). Let $u = (a+x)^{\frac{1}{2}}$.

$$du = \frac{dx}{2\,(a+x)^{\frac{1}{2}}}.$$

(25). Let $u = \frac{1}{(1-x)^{\frac{3}{2}}}$.

$$du = \frac{3\,dx}{2\,(1-x)^{\frac{5}{2}}}.$$

(26). Let $u = (a+bx+cx^2)^{\frac{1}{3}}$.

$$du = \frac{b\,d\,\text{[illegible]}+2cx\,dx}{3(a+bx+cx^2)^{\frac{2}{3}}}.$$

Leibnitz. *Epist. ad Oldenburg. Jun.* 21, 1677.

(27). Let $u = (a-bx^{\frac{1}{2}}+cx^{\frac{3}{2}})^{\frac{3}{4}}$.

$$du = \frac{3\{3cx-b\}dx}{8x^{\frac{1}{2}}(a-bx^{\frac{1}{2}}+cx^{\frac{3}{2}})^{\frac{1}{4}}}.$$

(28). Let $u = (1+x)\surd(1-x)$.

$$du = \frac{(1-3x)dx}{2\surd(1-x)}.$$

(29). Let $u = (1-x)\surd(1+x^2)$.

$$du = \frac{-(2x^2-x+1)dx}{(1+x^2)^{\frac{1}{2}}}.$$

(30). Let $u = \surd(1-x)\surd(1+x^2)$.

$$du = \frac{-\{3x^2-2x+1\}dx}{2\surd(1-x).\surd(1+x^2)}.$$

(31). Let $u = \dfrac{1}{\surd(1-x^2)}$.

$$du = \frac{x\,dx}{(1-x^2)^{\frac{3}{2}}}.$$

(32). Let $u = \dfrac{x}{(1-x)^{\frac{1}{2}}}$.

$$du = \frac{(2-x)dx}{2(1-x)^{\frac{3}{2}}}.$$

(33). Let $u = \dfrac{x}{\sqrt{(1-x^2)}}$.

$$d u = \frac{d x}{(1-x^2)^{\frac{3}{2}}}.$$

(34). Let $u = \dfrac{x^3}{(1-x^2)^{\frac{3}{2}}}$.

$$d u = \frac{3 x^2 d x}{(1-x^2)^{\frac{5}{2}}}.$$

(35). Let $u = \dfrac{\sqrt{(1+x)}}{\sqrt{(1-x)}}$.

$$d u = \frac{d x}{(1-x)\sqrt{(1-x^2)}}.$$

(36). Let $u = \dfrac{\sqrt{(a^2-x^2)}}{x}$.

$$d u = \frac{-a^2 d x}{x^2 \sqrt{(a^2-x^2)}}.$$

(37). Let $u = \dfrac{x^2}{\sqrt{(a^4+x^4)}}$.

$$d u = \frac{2 a^4 x d x}{(a^4+x^4)^{\frac{3}{2}}}.$$

(38). Let $u = \dfrac{1}{x+\sqrt{(a^2-x^2)}}$.

$$d u = \frac{\{x-\sqrt{(a^2-x^2)}\}^3 d x}{(2 x^2-a^2)^2 \sqrt{(a^2-x^2)}},$$

multiplying numerator and denominator by $\{x-\sqrt{(a^2-x^2)}\}^2$.

(39). Let $u = \dfrac{x}{x+\sqrt{(1+x^2)}}$.

$$du = \frac{(1+2x^2)\,dx}{\sqrt{(1+x^2)}} - 2x\,dx. \qquad \text{See Ex. 38.}$$

(40). Let $u = \dfrac{\sqrt{(1+x^2)}-x}{\sqrt{(1+x^2)}+x}$.

$$du = \frac{-2\,dx}{\sqrt{(1+x^2)}\,\{\sqrt{(1+x^2)}+x\}^2}.$$

(41). Let $u = \dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$.

$$d\,u = \frac{-\,dx}{x^2\sqrt{(1-x^2)}} - \frac{dx}{x^2}.$$

(42). Let $u = \dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$.

$$d\,u = \frac{-2\,dx}{x^3\sqrt{(1-x^4)}} - \frac{2\,dx}{x^3}.$$

(43). Let $u = \dfrac{(a+bx+cx^2)^{\frac{1}{2}}}{(a'+b'x+c'x^2)^{\frac{1}{2}}}$.

$$d\,u = \frac{\{a'b-ab'+2(a'c-ac')x+(b'c-bc')x^2\}\,dx}{2(a+bx+cx^2)^{\frac{1}{2}}(a'+b'x+c'x^2)^{\frac{3}{2}}}.$$

(44). Let $u = \sqrt{\left\{\dfrac{a+x}{a-x}\right\}\left\{\dfrac{b+x}{b-x}\right\}}$.

$$du = \frac{(a+b)(ab-x^2)\,dx}{(a-x)^{\frac{3}{2}}(b-x)^{\frac{3}{2}}\sqrt{(a+x)}\sqrt{(b+x)}}.$$

(45). Let $u = \frac{a+x}{b+x} \sqrt{\left(\frac{b^2+x^2}{a^2+x^2}\right)}$.

$$du = \frac{(a-b)\,\{(a+b)(ab+x^2)x-(ab-x^2)^2\}\,dx}{(b+x)^2(a^2+x^2)^{\frac{3}{2}}\sqrt{(b^2+x^2)}}.$$

(46). Let $u = \frac{\sqrt{\{x-\sqrt{(1-x^2)}\}}}{(x^2+x^3)^{\frac{1}{3}}}$.

$$du = \frac{-\{3x^3+x^2-6x-4+x(1+3x)\sqrt{(1-x^2)}\}\,x\,dx}{6\sqrt{(1-x^2)}\sqrt{\{x-\sqrt{(1-x^2)}\}}\,(x^2+x^3)^{\frac{4}{3}}}.$$

(47). Let $u = \sqrt{\{x-\sqrt{(x-\sqrt{(1-x^2)})}\}}$.

$$du = \frac{\{2\sqrt{(1-x^2)}\sqrt{(x-\sqrt{(1-x^2)})}-(x+\sqrt{(1-x^2)})\}\,dx}{4\sqrt{(1-x^2)}\sqrt{\{x-\sqrt{(1-x^2)}\sqrt{(x-\sqrt{(x-\sqrt{(1-x)})}\}}}}.$$

(48).* Let $u = \sqrt{\{x+\sqrt{\{x+\sqrt{\{x+\text{\&c. in infinitum}\}}}}}$.

$$du = \pm\frac{dx}{2\sqrt{\left(x+\frac{1}{4}\right)}}.$$

(49).* Let $u = \cfrac{x^2}{1+\cfrac{x^2}{1+\cfrac{x^2}{1+\text{\&c. in infinitum.}}}}$

$$du = \pm\frac{x\,dx}{\sqrt{\left(x^2+\frac{1}{4}\right)}}.$$

---

* See Wood's Algebra, Arts. 374, 375.

(50). Let $u = \left\{ 1 - \frac{x}{\sqrt{(1-x)}} + \frac{\sqrt{(1-x)}}{x} \right\}^{\frac{1}{4}}$.

$$d\,u = \frac{-(2-x)\left\{\frac{1}{(1-x)^{\frac{3}{2}}} + \frac{1}{x^2\sqrt{(1-x)}}\right\} d\,x}{8\left\{1 - \frac{x}{\sqrt{(1-x)}} + \frac{\sqrt{(1-x)}}{x}\right\}^{\frac{3}{4}}}.$$

On successive Differentiations.

ART. 18. (1). Let $u = a\,x^3$.

The differential coefficients are,

$$\frac{d\,u}{d\,x} = 3\,a\,x^2;\ \frac{d^2\,u}{d\,x^2} = 3\,.\,2\,a\,x,\ \frac{d^3\,u}{d\,x^3} = 3\,.\,2\,.\,1\,.\,a.$$

(2). Let $u = b + c\,(x-a)^n$

$$\frac{d\,u}{d\,x} = n\,c\,(x-a)^{n-1};$$

$$\frac{d^2\,u}{d\,x^2} = n\,(n-1)\,c\,(x-a)^{n-2};$$

&c. = &c.

$$\frac{d^n\,u}{d\,x^n} = n\,(n-1)\,\ldots\,3\,.\,2\,.\,1\,.\,c.$$

(3). Let $u = a\,x^4 + a_1\,x^3 + a_2\,x^2 + a_3\,x + a_4$.

$$\frac{d\,u}{d\,x} = 4\,a\,x^3 + 3\,a_1\,x^2 + 2\,a_2\,x + a_3;$$

$$\frac{d^2 u}{d x^2} = 4 \cdot 3\, a x^2 + 3 \cdot 2\, a_1 x + 2 \cdot 1 \cdot a_2 ;$$

$$\frac{d^3 u}{d x^3} = 4 \cdot 3 \cdot 2\, a x + 3 \cdot 2 \cdot 1 \cdot a_1 ;$$

$$\frac{d^4 u}{d x^4} = 4 \cdot 3 \cdot 2 \cdot 1 \cdot a.$$

(4). Let $u = \frac{1}{x}$.

$$\frac{d u}{d x} = -\frac{1}{x^2}.$$

$$\frac{d^2 u}{d x^2} = \frac{2 \cdot 1}{x^3}.$$

$$\frac{d^3 u}{d x^3} = -\frac{3 \cdot 2 \cdot 1}{x^4}.$$

&c. = &c.

$$\frac{d^n u}{d x^n} = (-1)^n \frac{n\,(n-1) \ldots . 3 \cdot 2 \cdot 1}{x^{n+1}}.$$

(5). Let $u = \frac{a^2}{a^2 + x^2}$.

$$\frac{d u}{d x} = -\frac{2 a^2 x}{(a^2 + x^2)^2}.$$

$$\frac{d^2 u}{d x^2} = \frac{-2 a^4 + 6 a^2 x^2}{(a^2 + x^2)^3}.$$

$$\frac{d^3 u}{d x^3} = \frac{24 a^4 x - 24 a^2 x^3}{(a^2 + x^2)^4}.$$

$$\frac{d^4 u}{d x^4} = \frac{24 a^6 - 240 a^4 x^2 + 120 a^2 x^4}{(a^2 + x^2)^5}.$$

$$\frac{d^5 u}{d x^5} = \frac{-720\, a^6 x + 2400\, a^4 x^3 - 720\, a^2 x^5}{(a^2+x^2)^6}.$$

$$\&c. = \&c.$$

Euler. *Institutiones Calculi Differentialis.* Pars I. Cap. V. Art. 177.

(6). Let $u=(a+b\,x+c\,x^2)^r$: to find an expression for $\frac{d^n u}{d x^n}$.

Assume

$$p = a + b\,x + c\,x^2, \text{ and } p' = b + 2\,c\,x:$$

it is obvious (Art. 21 and Note B) that $\frac{d^n u}{d x^n}$, is the coefficient of $\frac{d x^n}{1\,.\,2\,.\,3\,\ldots\ldots\,n}$, in the developement of

$$\{p + p'\,d\,x + c\,d\,x^2\}^r:$$

consequently

$$\frac{d^n u}{d x^n} = r\,(r-1)\ldots\ldots(r-n+1)\,p^{r-n}\,p'^n \Big\{ 1 + \frac{n\,(n-1)}{1\,(r-n+1)}\cdot\frac{c\,p}{p'^2} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,(r-n+1)\,(r-n+2)}\cdot\frac{c^2 p^2}{p'^4} + \frac{n\,(n-1)\ldots\ldots(n-5)}{1\,.\,2\,.\,3\,(r-n+1)\,(r-n+2)\,(r-n+3)}\cdot\frac{c^3 p^3}{p'^6} + \&c. \Big\} \ldots\ldots (a).$$

This expression may be put under a more convenient form, in which p will only appear as a factor common to all its terms.

Assume

$$e = 4\,a\,c - b^2 = 4\,p\,c - p'^2:$$

this assumption gives us

$$(p + p'\,d\,x + c\,d\,x^2)^r =$$

$$p^r\left\{1+\frac{2p'}{2p}dx+\frac{e+p'^2}{4p^2}dx^2\right\}^r=$$

$$p^r\left\{\left(1+\frac{p'}{2p}dx\right)^2+\frac{e}{4p^2}dx^2\right\}^r.$$

The developement of this expression will give us,

$$\frac{d^nu}{dx^n}=2r(2r-1)\ldots.(2r-n+1)\left(\frac{p'}{2}\right)^n p^{r-n}\left\{1+\frac{r}{1}\cdot\frac{n(n-1)}{2r(2r-1)}\frac{e}{p'^2}+\frac{r(r-1)}{1\,.\,2}\cdot\frac{n(n-1)(n-2)(n-3)}{2r(2r-1)(2r-2)(2r-3)}\cdot\frac{e^2}{p'^4}+\frac{r(r-1)(r-2)}{1\,.\,2\,.\,3}\cdot\frac{n(n-1)\ldots\ldots(n-5)}{2r(2r-1)\ldots\ldots(2r-5)}\cdot\frac{e^3}{p'^6}+\&c.\right\}\quad(b)$$

This series will terminate after $\frac{n}{2}+1$ or $\frac{n+1}{2}$ terms, according as n is an even or an odd number.

If $2r$ be a positive whole number less than n, and of the form $2m+1$, the formula just given becomes $=\frac{0}{0}$, and is therefore not applicable to the determination of the expression for $\frac{d^nu}{dx^n}$, without some previous modification (Art. 52). In this case, we shall find, when $2r=2m+1$,

$$\frac{d^nu}{dx^n}=\frac{(-1)^n.\,n(n-1)\ldots\ldots(m+2)(2m+1)(2m-1)\ldots\ldots3\,.\,1}{2^{n+m+1}}$$

$$p'^{\,n-2m-2}\,p^{r-n}\,e^{m+1}\left\{1-\frac{(n-2m-2)(n-2m-3)}{2\,.\,2\,.\,(m+2)}\cdot\frac{e}{p'^2}+\frac{(n-2m-2)(n-2m-3)\ldots\ldots(n-2m-5)}{2^2\,.\,2\,.\,3\,.\,(m+2)(m+3)}\cdot\frac{e^2}{p'^4}-\frac{(n-2m-2)(n-2m-3)\ldots\ldots(n-2m-7)}{2^3\,.\,2\,.\,3\,.\,4\,.\,(m+2)(m+3)(m+4)}\cdot\frac{e^3}{p'^6}+\&c.\right\}\quad(c)$$

This series will terminate after $\frac{n-2m}{2}$ or $\frac{n-2m-1}{2}$ terms, according as n is an even or an odd number.

The formulæ (a) and (b), were first given by Lagrange in the *Mémoires de l'Académie de Berlin* for 1772.

(7). Let $u=(a+bx)^r$: by the formula (a), we get

$$\frac{d^n u}{dx^n}=r(r-1)\ldots\ldots\ldots(r-n+1)\,b^n(a+bx)^{r-n}. \qquad (1).$$

Also, by the second formula (b), we find

$$\frac{d^n u}{dx^n}=2r(2r-1)\ldots\ldots(2r-n+1)\left(\frac{b}{2}\right)^n(a+bx)^{r-n}$$

$$\left\{1-\frac{r}{1}\cdot\frac{n(n-1)}{2r(2r-1)}+\frac{r(r-1)}{1\,.\,2}\cdot\frac{n(n-1)(n-2)(n-3)}{2r(2r-1)(2r-2)(2r-3)}\right.$$

$$\left.-\ \&c.\right\} \qquad (2).$$

From these two results, it appears that

$$r(r-1)\ldots\ldots(r-n+1)=\frac{2r(2r-1)\ldots\ldots(2r-n+1)}{2^n}$$

$$\left\{1-\frac{r}{1}\cdot\frac{n(n-1)}{2r(2r-1)}+\frac{r(r-1)}{1\,.\,2}\cdot\frac{n(n-1)(n-2)(n-3)}{2r(2r-1)(2r-2)(2r-3)}\right.$$

$$\left.-\ \&c.\right\} \qquad (3).$$

If $n=r$, we have

$$2^n=\frac{2n(2n-1)\ldots\ldots(n+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,n}\left\{1-\frac{n(n-1)}{1\,.\,2\,(2n-1)}+\right.$$

$$\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,2^2.(2n-1)(2n-3)}-\frac{n(n-1)\ldots\ldots\ldots(n-5)}{1.2.3.2^3.(2n-1)(2n-3)(2n-5)}$$

$$\left.+\ \&c.\right\} \qquad (4).$$

Since

$$2^n = 1 + n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \&c.$$

and

$$\frac{2n(2n-1)\ldots\ldots(n+1)}{1 \cdot 2 \cdot 3 \ldots\ldots n} = 1^2 + n^2 + \left(\frac{n(n-1)}{1 \cdot 2}\right)^2 +$$

$$\left(\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\right)^2 + \&c.$$

the last result may be put under the following curious form,

$$\frac{1 + n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \&c.}{1^2 + n^2 + \left(\frac{n(n-1)}{1 \cdot 2}\right)^2 + \left(\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\right)^2 + \&c.} =$$

$$1 - \frac{n(n-1)}{1 \cdot 2 \cdot (2n-1)} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 2^2 \cdot (2n-1)(2n-3)} -$$

$$\frac{n(n-1)\ldots\ldots(n-5)}{1 \cdot 2 \cdot 3 \cdot 2^3 \cdot (2n-1)(2n-3)(2n-5)} + \&c. \qquad (5).$$

(8). Let $u = (cx^2)^r$: by the formula (*a*), we find

$$\frac{d^n u}{dx^n} = r(r-1)\ldots\ldots(r-n+1)\,2^n c^r x^{2r-n} \Big\{ 1 +$$

$$\frac{n(n-1)}{1(r-n+1)} \cdot \frac{1}{2^2} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot (r-n+1)\,r-n+2)} \cdot \frac{1}{2^4} +$$

$$\frac{n(n-1)\ldots\ldots(n-5)}{1 \cdot 2 \cdot 3 \cdot (r-n+1)(r-n+2)} \cdot \frac{1}{2^6} + \&c. \Big\} \qquad (1).$$

Also, by the formula (*b*), we have

$$\frac{d^n u}{dx^n} = 2r(2r-1)\ldots\ldots(2r-n+1)\,c^r x^{2r-n}. \qquad (2).$$

Consequently, by equating these two results, we get

$$2r(2r-1)\dots\dots(2r-n+1)=r(r-1)\dots\dots(r-n+1)2^n\left\{1+\frac{n(n-1)}{1(r-n+1)}\cdot\frac{1}{2^2}+\frac{n(n-1)(n-2)(n-3)}{1.2.(r-n+1)(r-n+2)}\cdot\frac{1}{2^4}+\&c.\right\}\ (3).$$

If we make $r=n$, we get

$$\frac{2n(2n-1)\dots\dots(n+1)}{1.2\dots\dots n.2^n}=1+\frac{n(n-1)}{1^2}\cdot\frac{1}{2^2}+\frac{n.(n-1)(n-2)(n-3)}{1^2.2^2}\cdot\frac{1}{2^4}+\frac{n(n-1)\dots\dots(n-5)}{1^2.2^2.3^2}\cdot\frac{1}{2^6}+\&c.$$

$$=\frac{1^2+n^2+\left(\frac{n(n-1)}{1.2}\right)^2+\left(\frac{n(n-1)(n-2)}{1.2.3}\right)^2+\&c.}{1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\&c.}\quad(4)$$

(9). Let $u=\frac{1}{\sqrt{(1-x^2)}}$: by formula ($b$), we obtain

$$\frac{d^n u}{dx^n}=\frac{1.2\dots\dots n.x^n}{(1-x^2)^{n+\frac{1}{2}}}\left\{1+\frac{1}{2}\cdot\frac{n(n-1)}{1.2.x^2}+\frac{1.3}{2.4}\cdot\frac{n(n-1)(n-2)(n-3)}{1.2.3.4.x^4}+\frac{1.3.5}{2.4.6}\cdot\frac{n(n-1)\dots\dots(n-5)}{1.2.3.4.5.6.x^6}+\&c.\right\}=\frac{1.2\dots\dots n.x^n}{(1-x^2)^{n+\frac{1}{2}}}\left\{1+\frac{1}{2^2}\cdot\frac{n(n-1)}{x^2}+\frac{1}{2^2.4^2}\cdot\frac{n(n-1)(n-2)(n-3)}{x^4}+\&c.\right\}$$

Euler. *Inst. Calc. Diff.* Pars. I. Cap. V. Art. 177, where this expression is found by induction: Lagrange, *Mémoires de l'Acad. de Berlin.* 1772: Lacroix, *Traitè du Calcul. Diff. et Integ.* Chap. I. Art. 37.

(10). Let $u=\sqrt{(1+x^2)}$: by formula (c), we get

$$\frac{d^n u}{d x^n}=\frac{(-1)^n \cdot n(n-1) \ldots . 2 \cdot 1 \cdot x^{n-2}}{2(1+x^2)^{n-\frac{1}{2}}}\left\{1-\frac{(n-2)(n-3)}{2 \cdot 2^2 x^2}\right.$$

$$\left.+\frac{(n-2)(n-3)(n-4)(n-5)}{2^2 \cdot 2^2 \cdot 3^2 \cdot x^4}-\frac{(n-2)(n-3) \ldots .(n-7)}{2^3 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot x^6}+\&c.\right\}$$

In this case, n must not be less than 2.

(11). Let $u=(1-x^2)^{\frac{5}{2}}$; by the same formula, we find

$$\frac{d^n u}{d x^n}=\frac{(-1)^{n-1} \cdot n(n-1) \ldots . 4 \cdot 5 \cdot 3 \cdot 1 \cdot x^{n-6}}{2^3(1-x^2)^{n-\frac{5}{2}}}\left\{1+\right.$$

$$\left.\frac{(n-6)(n-7)}{2 \cdot 2 \cdot 4 \cdot x^2}+\frac{(n-6)(n-7)(n-8)(n-9)}{2^2 \cdot 2 \cdot 3 \cdot 4 \cdot 5 x^4}+\&c.\right\}$$

In this case, n must not be less than 6.

(12). Let $u=(2x-x^2)^{\frac{3}{2}}$: in a similar manner, we get

$$\frac{d^n u}{d x^n}=\frac{(-1)^n \cdot n(n-1) \ldots \ldots 3 \cdot 3 \cdot 1(1-x)^{n-4}}{2^2(1-x^2)^{n-\frac{3}{2}}}\left\{1+\right.$$

$$\left.\frac{(n-4)(n-5)}{2 \cdot 2 \cdot 3(1-x)^2}+\frac{(n-4)(n-5)(n-6)(n-7)}{2^2 \cdot 2 \cdot 3 \cdot 3 \cdot 4(1-x)^4}+\&c.\right\}$$

In this case, n must not be less than 4. The expressions for the other values of n may be found by formula (a) or (b).

(13). Let $u=\frac{1}{(1+x^2)}$: by formula (b), we obtain

$$\frac{d^n u}{d x^n}=\frac{(-1)^n \cdot 2 \cdot 3 \ldots \ldots (n+1)x^n}{(1+x^2)^{n+1}}\left\{1-\frac{n(n-1)}{2 \cdot 3 \cdot x^2}+\right.$$

$$\frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot x^4} - \frac{n(n-1) \ldots\ldots (n-5)}{2 \cdot 3 \ldots\ldots 7 \cdot x^6} + \&c. \Big\}$$

(See Example 5).

(14). Let $u = \frac{1}{(a\,x + x^2)^{\frac{3}{2}}}$: by formula ($b$), we find

$$\frac{d^n u}{d x^n} = \frac{(-1)^n \cdot 3 \cdot 4 \ldots\ldots (n+2)\,(a+2\,x)^n}{2^n\,(a\,x + x^2)^{n+\frac{3}{2}}} \Big\{ 1 +$$

$$\frac{3}{2} \cdot \frac{n(n-1)}{3 \cdot 4}\left(\frac{a}{a+2\,x}\right)^2 + \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{n(n-1)(n-2)(n-3)}{3 \cdot 4 \cdot 5 \cdot 6}\left(\frac{a}{a+2\,x}\right)^4$$

$$+ \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} \cdot \frac{n(n-1) \ldots\ldots (n-5)}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}\left(\frac{a}{a+2\,x}\right)^6 + \&c. \Big\}$$

(15). Let $u = \frac{1+x}{1-x}$: in this case

$\frac{d\,u}{d\,x} = \frac{2}{(1-x)^2}$, and consequently $\frac{d^n u}{d\,x^n} = 2 \cdot \frac{d^{n-1} \cdot \frac{1}{(1-x)^2}}{d\,x^{n-1}}$:

therefore, by formula (a),

$$\frac{d^n u}{d\,x^n} = \frac{2 \cdot 2 \cdot 3 \ldots\ldots n}{(1-x)^{n+1}}$$

(16). Let $u = \frac{x}{\sqrt{(1-x^2)}}$.

Since

$$\frac{d\,u}{d\,x} = \frac{1}{(1-x^2)^{\frac{3}{2}}},$$

we have

$$\frac{d^n x}{d\,x^n} = \frac{d^{n-1} \cdot \frac{1}{(1-x^2)^{\frac{3}{2}}}}{d\,x^{n-1}},$$

and therefore, by formula (b),

$$\frac{d^n u}{d x^n} = \frac{3 \cdot 4 \ldots\ldots (n+1)\, x^{n-1}}{(1-x^2)^{n+\frac{1}{2}}} \left\{ 1 + \frac{3}{2} \cdot \frac{(n-1)(n-2)}{3 \cdot 4 \cdot x^2} + \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{(n-1)(n-2)(n-3)(n-4)}{3 \cdot 4 \cdot 5 \cdot 6\, x^4} + \&c. \right\}$$

(17). Let $u = \frac{(a+b\,x)^r}{(a'+b'\,x)^{r'}}$.

Assume

$$p = a + b\,x,\ p' = a' + b'\,x,\ e = \frac{b'\,p}{b\,p'}:$$

by a process somewhat similar to that employed in deducing formula (a), we shall get

$$\frac{d^n u}{d x^n} = \frac{r\,(r-1) \ldots\ldots (r-n+1)\, b^n\, p^{r-n}}{p'^{\,r'}} \left\{ 1 - \frac{r'}{1} \cdot \frac{n}{(r-n+1)} \cdot e + \frac{r'\,(r'+1)}{1 \cdot 2} \cdot \frac{n\,(n-1)}{(r-n+1)\,(r-n+2)}\, e^2 - \frac{r'\,(r'+1)\,(r'+2)}{1 \cdot 2 \cdot 3} \cdot \frac{n\,(n-1)\,(n-2)}{(r-n+1)\,(r-n+2)\,(r-n+3)}\, e^3 + \&c. \right\} (d).$$

This series evidently terminates after $(n+1)$ terms.

This formula may likewise be put under the following form:

$$\frac{d^n u}{d x^n} = \frac{(-1)^{n+1}\, r'\,(r'+1) \ldots\ldots (r'+n-1)\, b'^n\, p^r}{p'^{\,r'+n}} \left\{ 1 - \frac{r}{1} \cdot \frac{n}{r'+n-1} \cdot \frac{1}{e} + \frac{r\,(r-1)}{1 \cdot 2} \cdot \frac{n\,(n-1)}{(r'+n-1)\,(r'+n-2)} \cdot \frac{1}{e^2} - \frac{r\,(r-1)\,(r-2)}{1 \cdot 2 \cdot 3} \cdot \frac{n\,(n-1)\,(n-2)}{(r'+n-1)\,(r'+n-2)\,(r'+n-3)} \cdot \frac{1}{e^3} + \&c. \right\} (e).$$

If r be a positive whole number less than n, the formula (d) will become

$$\frac{u}{x^n}=\frac{(-1)^{n-r}\,n\,(n-1)\ldots\ldots(n-r+1)\,r'\,(r'+1)\ldots\ldots(r'+n-r-1)\,b^{r}\,b'^{\,n-r}}{p^{r'+n-r}}$$

$$\Big\{1-\frac{r}{1}\cdot\frac{(r'+n-r)}{(n-r+1)}e+\frac{r\,(r-1)}{1\,.\,2}\cdot\frac{(r'+n-r)\,(r'+n-r+1)}{(n-r+1)\,(n-r+2)}\,.\,e^2$$

$$-\frac{r\,(r-1)\,(r-2)}{1\,.\,2\,.\,3}\cdot\frac{(r'+n-r)\,(r'+n-r+1)\,(r'+n-r+2)}{(n-r+1)\,(n-r+2)\,(n-r+3)}\,.\,e^3$$

$$+\,\&c.\Big\}\qquad (f).$$

This series will terminate after $(r+1)$ terms.

(18). Let $u=\sqrt{\left(\frac{1+x}{1-x}\right)}$: by formula ($d$), we get

$$\frac{d^n u}{d\,x^n}=\frac{(-1)^{n+1}.\,1\,.\,1\,.\,3\ldots.(2\,n-3)}{2^n.\,(1+x)^{n-\frac{1}{2}}\sqrt{(1-x)}}\Big\{1-\frac{n}{(2\,n-3)}\cdot\left(\frac{1+x}{1-x}\right)$$

$$+\frac{1\,.\,3}{1\,.\,2}\cdot\frac{n\,(n-1)}{(2\,n-3)\,(2\,n-5)}\cdot\left(\frac{1+x}{1-x}\right)^2-$$

$$\frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\cdot\frac{n\,(n-1)\,(n-2)}{(2\,n-3)\,(2\,n-5)\,(2\,n-7)}\cdot\left(\frac{1+x}{1-x}\right)^3+\&c.\Big\}$$

(19). Let $u=\frac{(1+x)^{\frac{3}{2}}}{(1-x)^{\frac{5}{2}}}$: by formula ($d$), we find

$$\frac{d^n u}{d\,x^n}=\frac{(-1)^n.\,1\,.\,1\,.\,3\ldots.(2\,n-5)}{2^n\,(1+x)^{n-\frac{3}{2}}\,(1-x)^{\frac{5}{2}}}\Big\{1-\frac{5}{1}\cdot\frac{n}{(2\,n-5)}\cdot\left(\frac{1+x}{1-x}\right)$$

$$+\frac{5\cdot 7}{1\,.\,2}\cdot\frac{n\,(n-1)}{(2\,n-5)\,(2\,n-7)}\cdot\left(\frac{1+x}{1-x}\right)^2-$$

$$\frac{5\,.\,7\cdot 9}{1\cdot 2\,.\,3}\cdot\frac{n\,(n-1)\,(n-2)}{(2\,n-5)\,(2\,n-7)\,(2\,n-9)}\cdot\left(\frac{1+x}{1-x}\right)^3+\&c.\Big\}$$

(20). Let $u = \frac{1}{\sqrt{(1-x)}\sqrt{(1+x)}} = \frac{1}{\sqrt{(1-x^2)}}$: the same formula will give us

$$\frac{d^n u}{d x^n} = \frac{(-1)^n . 1 . 3 \ldots . (2n-1)}{2^n (1-x)^n \sqrt{(1-x^2)}} \Big\{ 1 - \frac{n}{(2n-1)} . \left(\frac{1+x}{1-x}\right)$$

$$+ \frac{1 . 3}{1 . 2} \frac{n(n-1)}{(2n-1)(2n-3)} . \left(\frac{1+x}{1-x}\right)^2 -$$

$$\frac{1 . 3 . 5}{1 . 2 . 3} . \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} . \left(\frac{1+x}{1-x}\right)^3 + \&c. \Big\}$$

(21). Let $u = \frac{(1+x)^3}{(1-x)^4}$: by formula (*f*), we find

$$\frac{d^n u}{d x^n} = \frac{n(n-1)(n-2) . 4 . 5 \ldots . n}{(1-x)^{n+1}} \Big\{ 1 + \frac{3}{1} . \frac{n+1}{n-2} . \left(\frac{1+x}{1-x}\right)$$

$$+ \frac{3 . 2}{1 . 2} . \frac{(n+1)(n+2)}{(n-2)(n-1)} . \left(\frac{1+x}{1-x}\right)^2 +$$

$$\frac{3 . 2 . 1}{1 . 2 . 3} . \frac{(n+1)(n+2)(n+3)}{(n-2)(n-1)n} . \left(\frac{1+x}{1-x}\right)^3 \Big\}.$$

In this case n must not be less than 4.

(22). Let $u = \frac{(2+3x)^4}{(3+4x)^{10}}$: by the same formula, we get

$$\frac{d^n u}{d x^n} = \frac{(-1)^n n(n-1)(n-2)(n-3) 10 . 11 \ldots . (n+5) 3^4 4^{n-4}}{(3+4x)^{n+6}}$$

$$\Big\{ 1 - 4 . \frac{4}{3} . \frac{(n+6)}{(n-3)} . \left(\frac{2+3x}{3+4x}\right) +$$

$$6 . \left(\frac{4}{3}\right)^2 . \frac{(n+6)(n+7)}{(n-3)(n-2)} . \left(\frac{2+3x}{3+4x}\right)^2 -$$

$$4 \, . \left(\frac{4}{3}\right)^3 . \frac{(n+6)(n+7)(n+8)}{(n-3)(n-2)(n-1)} . \left(\frac{2+3x}{3+4x}\right)^3 +$$

$$\left.\left(\frac{4}{3}\right)^4 . \frac{(n+6)(n+7)(n+8)(n+9)}{(n-3)(n-2)(n-1)\,n} . \left(\frac{2+3x}{3+4x}\right)^4 \right\}.$$

It is evident that n must not be less than 5.

It is possible to construct formulæ for the n^{th} differential coefficients of functions much more complicated than any we have yet given: of this kind are the functions

$$\frac{(a+b\,x+c\,x^2)^r}{(a'+b'\,x+c'\,x^2)^{r'}} \text{ and } \frac{(a+b\,x+c\,x^2+d\,x^3)^r}{(a'+b'\,x+c'\,x^2+d'\,x^3)^{r'}}.$$

The preceding examples, however, are sufficient for the exercise of the student, and it is unnecessary to embarrass him with the investigation and application of the formulæ corresponding to these cases, which are of great extent and very little simplicity.

Art. 19. The theorem which is deduced in this article, was first given by Maclaurin in his *Fluxions*, Art. 751: it is of considerable use in the developement of functions, though it does not always enable us to discover the general term of the resulting series. The reader will find other examples under Arts. 36. and 37.

(1). Let $u = (a+x)^3$: In this case
$U_0 = a^3$, $U_1 = 3\,a^2$, $U_2 = 3 \, . \, 2\,a$, $U_3 = 3 \, . \, 2 \, . \, 1.$, $U_4 = 0$, &c.
Consequently

$$u = a^3 + 3\,a^2\frac{x}{1} + 3 \, . \, 2\,\frac{a\,x^2}{1 \, . \, 2} + 3 \, . \, 2 \, . \, 1 \, . \, \frac{x^3}{1 \, . \, 2 \, . \, 3}.$$

(2). Let $u = (1+x)^{\frac{1}{2}}$: In this case

$$U_0 = 1,\ U_1 = \frac{1}{2},\ U_2 = -\frac{1}{2^2},\ U_3 = \frac{3}{2^3},\ U_4 = -\frac{3 \, . \, 5}{2^4},\ U_5 = \frac{3 \, . \, 5 \, . \, 7}{2^5},$$

&c. therefore

$$u = 1 + \frac{1}{2} \cdot \frac{x}{1} - \frac{1}{2^2} \cdot \frac{x^2}{1 \cdot 2} + \frac{3}{2^3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} - \frac{3 \cdot 5}{2^4} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{3 \cdot 5 \cdot 7}{2^5} \cdot \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

(3). Let $u = \frac{1}{1-x}$: the same process gives us

$$u = 1 + \frac{x}{1} + 1 \cdot 2 \cdot \frac{x^2}{1 \cdot 2} + 1 \cdot 2 \cdot 3 \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

(4). Let $u = (a + b\,x + c\,x^2)^{\frac{1}{2}}$: by means of the formulæ given under the preceding articles, we get

$$u = a^{\frac{1}{2}} \left\{ 1 + \frac{b}{2\,a} \cdot \frac{x}{1} \overset{+}{+} \frac{(4\,a\,c - b^2)}{2^2 \cdot a^2} \cdot \frac{x^2}{1 \cdot 2} \overset{-}{\cancel{+}} \frac{3\,b\,(4\,a\,c - b^2)}{2^3 \cdot a^3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} \right.$$

$$+ \frac{3\,(4\,a\,c - b^2)\,(4\,a\,c - 5\,b^2)}{2^4 \cdot a^4} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} \overset{-}{\cancel{+}}$$

$$\left. \frac{5 \cdot 3 \cdot b\,(4\,a\,c - b^2)\,(12\,a\,c - 7\,b^2)}{2^5\,a^5} \cdot \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c. \right\}$$

the general term being

$$= \frac{(-1)^n \cdot n\,(n-1) \ldots\ldots 2 \cdot 1 \cdot b^{n-2}\,(4\,a\,c - b^2)}{2^{n+1}\,a^{n-\frac{1}{2}}}$$

$$\left\{ 1 - \frac{(n-2)\,(n-3)}{2 \cdot 2^2} \cdot \frac{(4\,a\,c - b^2)}{b^2} + \right.$$

$$\frac{(n-2)\,(n-3)\,(n-4)\,(n-5)}{2^2 \cdot 2^2 \cdot 3^2} \cdot \frac{(4\,a\,c - b^2)^2}{b^4} -$$

$$\left. \frac{(n-2)\,(n-3) \ldots\ldots (n-7)}{2^3 \cdot 2^2 \cdot 3^2 \cdot 4^2} \cdot \frac{(4\,a\,c - b^2)^3}{b^6} + \&c. \right\} \frac{x^n}{1 \cdot 2 \cdot 3 \ldots n}.$$

It is evident that n must not be less than 2.

(5). Let $u = (a^5 + a^4 x - x^5)^{\frac{1}{5}}$:

Newton. *Epistola ad Oldenburg.* Jan. 26, 1676.

$$u = a + \frac{1}{5} \cdot \frac{x}{1} - \frac{4}{5^2 a} \cdot \frac{x^2}{1 . 2} + \frac{4 . 9}{5^3 a^2} \cdot \frac{x^3}{1 . 2 . 3} - \frac{4 . 9 . 14}{5^4 a^3} \cdot \frac{x^4}{1 . 2 . 3 . 4}$$

$$+ \frac{4 . 9 . 14 . 19 - 1 . 2 . 3 . 4 . 5 . 5^4}{5^5 a^4} \cdot \frac{x^5}{1 . 2 . 3 . 4 . 5} - \&c.$$

the general term being

$$= (-1)^{n-1} \Big\{ 1 . 4 \ldots . (5n - 6) -$$

$$\frac{(n-4)}{1} . 1 . 4 \ldots . (5n - 26)\, n\, (n-1)(n-2)(n-3)\, 5^4 +$$

$$\frac{(n-8)(n-9)}{1 . 2} . 1 . 4 \ldots . (5n - 46)\, n\, (n-1) \ldots . (n-7)\, 5^8 -$$

$$\frac{(n-12)(n-13)(n-14)}{1 . 2 . 3} . 1 . 4 \ldots (5n - 66)\, n\, (n-1) \ldots (n-11)\, 5^{12}$$

$$+ \&c. \Big\} \times \frac{x^n}{5^n a^{n-1} . 1 . 2 \ldots . n} .$$

(6). Let $u = \sqrt{\left\{\frac{1 + a x^2}{1 - b x^2}\right\}}$.

Newton "*De Analysi per Æquationes numero terminorum infinitas.*" 1669.

$$u = 1 + (a + b) \frac{x^2}{1 . 2} + 3 . (3 b^2 + 2 a b - a^2) \frac{x^4}{1 . 2 . 3 . 4} +$$

$$5 . 3 . 3 (5 b^3 + 3 b^2 a - b a^2 + a^3) \frac{x^6}{1 . 2 . 3 . 4 . 5 . 6} +$$

$$7 . 5 . 3 . 3 (5 . 7 b^4 + 5 . 4 b^3 a - 2 . 3 . b^2 a^2 + 4 b a^3 - 5 a^4)$$

$$\frac{x^8}{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8} + \&c.$$

the general term being

$$=\Big\{1\,.\,3\,....(2\,n-1)\,b^{n}+1\,.\,3\,....(2\,n-3)\,n\,.\,\frac{1}{1}\,.\,b^{n-1}\,a-$$

$$1\,.\,3\,....\,(2\,n-5)\,n\,(n-1)\,.\,\frac{1\,.\,1}{1\,.\,2}\,.\,b^{n-2}\,a^{2}+$$

$$1\,.\,3\,....\,(2\,n-7)\,n\,(n-1)\,(n-2)\,.\,\frac{1\,.\,1\,.\,3}{1\,.\,2\,.\,3}\,b^{n-3}\,a^{3}-\&c.$$

$$...+(-1)^{n}\,1.n.1.3...(2n-5)\,b\,a^{n-1}+(-1)^{n+1}\,a^{n}\,1.3...(2\,n-3)\Big\}$$

$$\frac{x^{2\,n}}{2^{n}\,.\,1\,.\,2\,....\,n}.$$

(7). Let $u=\frac{(1+x)^{3}}{(1+x)^{4}}$: by means of Exam. 20, Art. 17, we get

$$u=1+7\,x+56\,\frac{x^{2}}{1\,.\,2}+378\,\frac{x^{3}}{1\,.\,2\,.\,3}+3096\,\frac{x^{4}}{1\,.\,2\,.\,3\,.\,4}+$$

$$27720\,\frac{x^{5}}{1\,.\,2\,.\,3\,.\,4\,.\,5}+271440\,\frac{x^{6}}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}+\&c.$$

the general term being

$$=4\,.\,5\,....\,n\,\{n\,(n-1)\,(n-2)+3\,(n+1)\,n\,(n-1)+$$
$$3\,(n+2)\,(n+1)\,n+(n+3)\,(n+2)\,(n+1)\,\}\,.$$

(8). Let $u=\frac{1}{\sqrt{(a\,x-x^{2})}}$: it is evident that the theorem we are now exemplifying, is not directly applicable to the developement of this function, since u becomes infinite, when $x=0$: a very simple artifice, however, will remove this difficulty. Assume

$$u_{1}=u\sqrt{x}=\frac{1}{\sqrt{(a-x)}}:$$

expand u_1 in the ordinary manner, and we shall get

$$u = \frac{u_1}{\sqrt{x}} = \frac{1}{\sqrt{(a\,x)}}\left\{1 + \frac{1}{2}\cdot\frac{x}{a} + \frac{1\,.\,3}{2^2}\cdot\frac{x^2}{1\,.\,2\,.\,a^2} + \frac{1\,.\,3\,.\,5}{2^3}\cdot\frac{x^3}{1\,.\,2\,.\,3\,.\,a^3} + \&c.\right\}$$

(9). Let $u = \frac{1+x}{x\,(1-x)^{10}}$: in this case, u being multiplied by x, we get

$$u_1 = u\,x = \frac{1+x}{(1-x)^{10}}:$$

if we develope u_1, as in the preceding examples, we find

$$u = \frac{u_1}{x} = \frac{1}{x} + 11 + 130\,.\,\frac{x}{1\,.\,2} + 1650\,.\,\frac{x^2}{1\,.\,2\,.\,3} + 22440\,.\,\frac{x^3}{1\,.\,2\,.\,3\,.\,4} + \&c.$$

the general term being

$$10\,.\,11\,\ldots\,(n+8)\,(2\,n+9)\,.\,\frac{x^{n-1}}{1\,.\,2\,.\,3\,\ldots\,n}.$$

(10). Let u be an implicit function of x, determined by the equation

$$u^2 - x\,u - a = 0:$$

we shall readily find

$$U_0 = \pm\sqrt{a},\ U_1 = \frac{1}{2},\ U_2 = \pm\frac{1}{4\sqrt{a}},\ U_3 = 0,\ U_4 = \mp\frac{1^2\,.\,3}{4^2\,a^{\frac{3}{2}}},$$

$$U_5 = 0,\ U_6 = \pm\frac{1^2\,.\,3^2\,.\,5}{4^3\,a^{\frac{5}{2}}},\ \&c.$$

and therefore

$$u = \pm\sqrt{a} + \frac{1}{2}\cdot\frac{x}{1} \pm \frac{1}{4\sqrt{a}}\cdot\frac{x^2}{1\,.\,2} \mp \frac{1^2\,.\,3}{4^2\,.\,a^{\frac{3}{2}}}\cdot\frac{x^4}{1\,.\,2\,.\,3\,.\,4} \pm$$

$$\frac{1^2 . 3^2 . 5}{4^3 a^{\frac{5}{2}}} \cdot \frac{a^6}{1 . 2 . 3 . 4 . 5 . 6} \mp \frac{1^2 . 3^2 . 5^2 . 7}{4^4 a^{\frac{7}{2}}} \cdot \frac{x^8}{1 . 2 . 3 \ldots . 8}$$

$\pm$ &c.

This double series expresses the two values of u, which are contained in the given quadratic equation.

This however is not the only form in which this developement may be effected: by considering u as a function of a, we shall find

$$u = x + \frac{1}{x} \cdot \frac{a}{1} - \frac{1 . 2}{x^3} \cdot \frac{a^2}{1 . 2} + \frac{1 . 3 . 2^2}{x^5} \cdot \frac{a^3}{1 . 2 . 3} - \frac{1 . 3 . 5 . 2^3}{x^7} \cdot \frac{a^4}{1 . 2 . 3 . 4} + \&c.$$

or

$$u = - \frac{1}{x} \cdot \frac{a}{1} + \frac{1 . 2}{x^3} \cdot \frac{a^2}{1 . 2} - \frac{1 . 3 . 2^2}{x^5} \cdot \frac{a^3}{1 . 2 . 3} + \&c.$$

The first series results from considering U_0 as equal to x, and the second from considering U_0 as equal to 0, it being evidently susceptible of both those values.

(11). Let u be an implicit function of x, determined by the equation

$$u^3 - x u - 1 = 0.$$

In this case, we find

$$U_0 = 1,\ U_1 = \frac{1}{3},\ U_2 = 0,\ U_3 = \frac{-2}{3^3},\ U_4 = \frac{2 . 4}{3^4},\ \&c.$$

and therefore

$$u = 1 + \frac{x}{3} - \frac{x^3}{3^4} + \frac{x^4}{3^5} - \&c. \qquad (1).$$

The other two values of u, are

$$u = \alpha + \frac{x}{3\,\alpha} - \frac{x^3}{3^4} + \frac{x^4}{3^5\,\alpha} - \&c. \qquad (2).$$

and

$$u = \alpha^2 + \frac{x}{3\,\alpha^2} - \frac{x^3}{3^4} + \frac{x^4}{3^5\,\alpha^2} - \&c. \qquad (3).$$

where $\alpha = \frac{-1+\sqrt{(-3)}}{2}$ and $\alpha^2 = \frac{-1-\sqrt{(-3)}}{2}$, which are two cube roots of unity.

If the equation which determined u had been

$$u^3 - x\,u - a = 0,$$

we should find

$$u = a^{\frac{1}{3}} + \frac{x}{3\,a^{\frac{1}{3}}} - \frac{x^3}{3^4\,a} + \frac{x^4}{3^5\,a^{\frac{4}{3}}} - \&c.$$

a series which evidently admits of three values, arising from the successive substitution of the three cube roots of a.

This theorem, however, is not very readily applicable to the developement of implicit functions, and its use for that purpose is entirely superseded by the more general theorem of Lagrange: we have given the two preceding examples merely as introductory to those which will be given in illustration of the theorem to which we have just alluded. See *Note E.* and *Examples to it.*

Art. 21, 22. The series, which is the subject of these Articles, was first given by Brook Taylor in his *Method of Increments*, published in 1715: he does not appear to have been aware of its importance, but dismisses it without remark in a Corollary to the Proposition from which it is deduced. The use of this theorem in the developement of functions is nearly superseded by the theorem we have just been considering, and by other methods of greater generality: we shall put down however a few examples of its application, in addition to the one which is given by our author, particularly as they sometimes lead to curious results: the student is referred to Art. 37. for other instances involving transcendental functions.

(1). Let $u = (a+x)^n$: in this case

$$\frac{du}{dx} = n\,(a+x)^{n-1},\ \frac{d^2 u}{dx^2} = n\,(n-1)\,(a+x)^{n-2},$$

$$\frac{d^3 u}{dx^3} = n\,(n-1)\,(n-2)\,(a+x)^{n-3},\ \&\text{c.}$$

and therefore

$$(a+x+h)^n = (a+x)^n + n\,(a+x)^{n-1}\,h + n\,(n-1)\,(a+x)^{n-2}\,\frac{h^2}{1.2}$$

$$+ n\,(n-1)\,(n-2)\,(a+x)^{n-3}\,\frac{h^3}{1\,.\,2\,.\,3} + \&\text{c.}$$

If we make $h = -x$, we shall get

$$a^n = (a+x)^n - \frac{n}{1}\,.\,(a+x)^{n-1}\,x + \frac{n\,(n-1)}{1\,.\,2}\,.\,(a+x)^{n-2}\,x^2 - \&\text{c.}$$

$$= (a+x)^n \left\{ 1 - \frac{n}{1}\,.\,\left(\frac{x}{a+x}\right) + \frac{n\,(n-1)}{1\,.\,2}\,.\,\left(\frac{x}{a+x}\right)^2 - \right.$$

$$\left. \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,.\,\left(\frac{x}{a+x}\right)^3 + \&\text{c.} \right\}$$

or

$$(a+x)^{-n} = a^{-n} \left\{ 1 - \frac{n}{1}\,.\,\left(\frac{x}{a+x}\right) + \frac{n\,(n-1)}{1\,.\,2}\,.\left(\frac{x}{a+x}\right)^2 - \right.$$

$$\left. \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,.\,\left(\frac{x}{a+x}\right)^3 + \&\text{c.} \right\} \qquad (1).$$

By a similar process, we shall find

$$(a+x)^n = a^n \left\{ 1 + n\,.\,\left(\frac{x}{a+x}\right) + \frac{n\,(n+1)}{1\,.\,2}\,.\,\left(\frac{x}{a+x}\right)^2 + \right.$$

$$\left. \frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3}\,.\,\left(\frac{x}{a+x}\right)^3 + \&\text{c.} \right\} \qquad (2).$$

If we put e^m in the place of a, and $\frac{1}{m}$ in the place of n, we shall find

$$(e^m + x)^{\frac{1}{m}} = e\left\{1 + \frac{1}{m}\cdot\left(\frac{x}{e^m + x}\right) + \frac{1\,.\,(m+1)}{1\,.\,2\,.\,m^2}\cdot\left(\frac{x}{e^m + x}\right)^2 + \frac{1\,.\,(m+1)\,(2\,m+1)}{1\,.\,2\,.\,3\,.\,m^3}\cdot\left(\frac{x}{e^m + x}\right)^3 + \&c.\right\} \qquad (3).$$

or making x negative

$$(e^m - x)^{\frac{1}{m}} = e\left\{1 - \frac{1}{m}\cdot\left(\frac{x}{e^m - x}\right) + \frac{1\,.\,(m+1)}{1\,.\,2\,.\,m^2}\cdot\left(\frac{x}{e^m - x}\right)^2 - \frac{1\,.\,(m+1)\,(2\,m+1)}{1\,.\,2\,.\,3\,.\,m^3}\cdot\left(\frac{x}{e^m - x}\right)^3 + \&c.\right\} \qquad (4).$$

These series converge with great rapidity when e^m is very large compared with x, and consequently furnish formulæ sufficiently convenient for computing the roots of numbers.

Thus, suppose it was required to extract the square root of 50: in this case $a = 7$, $x = 1$, and therefore

$$\sqrt{(50)} = 7\left\{1 + \frac{1}{2}\cdot\frac{1}{50} + \frac{3}{8}\cdot\frac{1}{50^2} + \frac{3\,.\,5}{8\,.\,6}\cdot\frac{1}{50^3} + \&c.\right\};$$

collecting together the four first terms, we find

$$\sqrt{(50)} = 7\,.\,0710675,$$

which is accurate to 6 places of decimals.

Again, suppose it was required to extract the cube root of 750: we shall find

$$e = 9, \text{ and } x = 21:$$

consequently

$$\sqrt[3]{(750)} = 9\left\{1 + \frac{1}{3}\cdot\left(\frac{21}{750}\right) + \frac{4}{3^2 . 1 . 2}\cdot\left(\frac{21}{750}\right)^2 + \right.$$

$$\left.\frac{4 . . 5}{3^3 . 1 . 2 . 3}\cdot\left(\frac{21}{750}\right)^3 + \&c.\right\}$$

the addition of the four first terms of this series gives

$$\sqrt[3]{(750)} = 9 . 0855923911, \text{ nearly,}$$

a result, which is accurate to 6 places of decimals.

This subject, however, is foreign to our present design, and we have merely mentioned the preceding method of approximating to the roots of numbers as an incidental use of the series which we had investigated. The reader will find other and more convenient formulæ for this purpose, in Euler's *Institutiones Calc. Differentialis*, Pars. II, Arts. 237, 238 and 240.

The preceding formulæ were deduced from the general series of Taylor, by making $h = -x$: as it will be frequently very convenient to make this hypothesis, it may be worth while to remark the form which that series assumes in consequence of it. Since

$$u' = f(x+h) = u + \frac{d\,u}{d\,x}\cdot\frac{h}{1} + \frac{d^2\,u}{d\,x^2}\cdot\frac{h^2}{1 . 2} + \frac{d^3\,u}{d\,x^3}\cdot\frac{h^3}{1 . 2 . 3} + \&c. \qquad (\alpha).$$

if we make $h = -x$, we shall get

$$f(x-x) = f(0) = U_0 = u - \frac{d\,u}{d\,x}\cdot\frac{x}{1} + \frac{d^2\,u}{d\,x^2}\cdot\frac{x^2}{1 . 2} - \frac{d^3\,u}{d\,x^3}\cdot\frac{x^3}{1 . 2 . 3} + \&c. \qquad (\beta).$$

(2). Let $u = \surd\{a + bx + cx^2\}$: by formula ($\beta$), and those given under Art. 18, we shall get

$$\surd a = \surd\{a+bx+cx^2\} - \frac{1}{2} \cdot \frac{(b+2cx)x}{\surd\{a+bx+cx^2\}} -$$

$$\frac{1}{1.2.2^2} \cdot \frac{(b^2-4ac)x^2}{\{a+bx+cx^2\}^{\frac{3}{2}}} - \frac{3}{1.2.3.2^3} \cdot \frac{(b^2-4ac)(b+2cx)x^3}{\{a+bx+cx^2\}^{\frac{5}{2}}}$$

$$- \frac{3(b^2-4ac)\{4ac-5b^2-16cx(b-cx)\}x^4}{1.2.3.4.2^4\{a+bx+cx^2\}^{\frac{7}{2}}} - \&c.$$

Multiplying the whole equation by

$$\frac{\surd\{a+bx+cx^2\}}{\surd a},$$

we shall get

$$\surd\{a+bx+cx^2\} = \frac{1}{\surd a}\Big\{a + \frac{bx}{2} - \frac{1}{1.2.2^2} \cdot \frac{(b^2-4ac)x^2}{\{a+bx+cx^2\}}$$

$$- \frac{3}{1.2.3.2^3} \cdot \frac{(b^2-4ac)(b+2cx)x^3}{(a+bx+cx^2)^2} - \&c.\Big\}$$

Euler *Inst. Calc. Diff.* Pars. II. Art. 77.

(3). Let $u = \frac{1}{\surd(1-x^2)}$: by means of Ex. 9, Art. 18, and formula (β), we get

$$\surd(1-x^2) = 1 - \frac{x^2}{1-x^2} + \frac{(2x^2+1)}{(1-x^2)^2} \cdot \frac{x^2}{1.2} -$$

$$\frac{3(2x^3+3x)}{(1-x^2)^3} \cdot \frac{x^3}{1.2.3} + \frac{3(8x^4+24x^2+3)}{(1-x^2)^4} \cdot \frac{x^4}{1.2.3.4} - \&c.$$

The student will find other exemplifications of this formula under Arts. 36 and 37.

On the Differentiation of Transcendental Functions.

ART. 30, 31. (1). Let $u = \log. \{x + \sqrt{(x^2 - 1)}\}$.

$$du = \frac{dx}{\sqrt{(x^2 - 1)}}.$$

(2.) Let $u = \log. \left\{\frac{1 + \sqrt{(1 - x^2)}}{x}\right\}$.

$$du = \frac{-dx}{x\sqrt{(1 - x^2)}}.$$

(3). Let $u = \log. \{x\sqrt{(-1)} + \sqrt{(1 - x^2)}\}$.

$$du = \frac{-dx}{\sqrt{(x^2 - 1)}}.$$

(4). Let $u = \log. \{x\sqrt{(-1)} - \sqrt{(1 - x^2)}\}$.

$$du = \frac{dx}{\sqrt{(x^2 - 1)}}.$$

(5). Let $u = \log. \{\sqrt{(1 + x^2)} + \sqrt{(1 - x^2)}\}$.

$$du = \frac{dx}{x} - \frac{dx}{x\sqrt{(1 - x^4)}}.$$

(6). Let $u = \log. \left\{\frac{1 + x}{1 - x}\right\}$.

$$du = \frac{2dx}{1 - x^2}.$$

(7). Let $u = \log \left\{ \frac{\sqrt{(x^2+1)}-1}{\sqrt{(x^2+1)}+1} \right\}$.

$$du = \frac{2\,dx}{x\sqrt{(x^2+1)}}.$$

(8). Let $u = \log \left\{ \frac{\sqrt{(x^2-1)}-1}{\sqrt{(x^2-1)}+1} \right\}$.

$$du = \frac{2\,x\,dx}{(x^2-2)\sqrt{(x^2-1)}}.$$

(9). Let $u = \log \left\{ \frac{x+\sqrt{(1-x^2)}}{x-\sqrt{(1-x^2)}} \right\}$.

$$du = \frac{-2\,dx}{(2\,x^2-1)\sqrt{(1-x^2)}}.$$

(10). Let $u = \frac{-1}{\sqrt{(c)}} \cdot \log \left\{ -\frac{b}{2\sqrt{(c)}} - x\sqrt{(c)} + \sqrt{(a+bx+cx^2)} \right\}$.

$$du = \frac{dx}{\sqrt{(a+bx+cx^2)}}. \quad \text{(Art. 162).}$$

(11). Let $u = \frac{1}{\sqrt{(b^2-a^2)}} \cdot \log \left\{ \frac{\sqrt{(b+a)}+\sqrt{(b-a)}\,x}{\sqrt{(b+a)}-\sqrt{(b-a)}\,x} \right\}$.

$$du = \frac{2\,dx}{a+b+(a-b)\,x^2}. \quad \text{(Note } K\text{).}$$

(12). Let $u = x^n\,a^x$.

$$du = x^{n-1}\,a^x\,dx\,\{x \log a + n\}.$$

(13). Let $u = \epsilon^x\,(x-1)$.

$$du = \epsilon^x\,x\,dx.$$

Euler. *Inst. Calc. Diff.* Pars I. Art. 190.

(14). Let $u = \epsilon^x (x^2 - 2x + 2)$.

$$du = \epsilon^x x^2 dx. \qquad \textit{Id.}$$

(15). Let $u = \epsilon^x (x^3 - 3x^2 + 6x - 6)$.

$$du = \epsilon^x x^3 dx, \qquad \textit{Id.}$$

(16). Let $u = \dfrac{\epsilon^x}{1+x}$.

$$du = \frac{\epsilon^x x\, dx}{(1+x)^2}. \qquad \text{(Art. 191).}$$

(17). Let $u = \dfrac{\epsilon^x x}{1-x}$.

$$du = \frac{\epsilon^x dx}{(1-x)^2} \cdot (1 + x - x^2).$$

(18). Let $u = x \log x$.

$$du = dx \{ \log x + 1 \}.$$

Euler. *Inst. Calc. Diff.* Pars I. Art. 185.

(19). Let $u = x^m \log x - \dfrac{x^m}{m}$.

$$du = m x^{m-1} \log x \cdot dx. \quad \textit{Id.}$$

(20). Let $u = x^m (\log x)^n$.

$$du = x^{m-1} (\log x)^{n-1} \{ m \log x + n \} dx. \quad \textit{Id.}$$

(21). Let $u = \dfrac{x^4 (\log x)^2}{4} - \dfrac{x^4 \log x}{8} + \dfrac{x^4}{32}$.

$$du = x^3 dx (\log x)^2.$$

(22). Let $u = \log \log x = \log {}^2 x$.

$$du = \frac{dx}{x \log x}.$$

(23). Let $u = \log^5 x$.

$$d\,u = \frac{d\,x}{x \log x \,.\, \log^2 x \,.\, \log^3 x \,.\, \log^4 x}.$$

(24). Let $u = x \log^2 x$.

$$d\,u = d\,x \log^2 x + \frac{d\,x}{\log x}.$$

(25). Let $u = \epsilon^x \log x$.

$$d\,u = \epsilon^x\, d\,x \left\{ \frac{1}{x} + \log x \right\}.$$

(26). Let $u = \epsilon^{\log x}$.

$$d\,u = \epsilon^{\log x} \frac{d\,x}{x}.$$

(27). Let $u = x^n \epsilon^{\log x}$.

$$d\,u = (n+1)\, x^{n-1}\, \epsilon^{\log x}\, d\,x.$$

(28). Let $u = \frac{\epsilon^x - 1}{\epsilon^x + 1}$.

$$d\,u = \frac{2\, \epsilon^x\, d\,x}{(\epsilon^x + 1)^2}.$$

(29). Let $u = \log \left\{ \frac{\epsilon^x - 1}{\epsilon^x + 1} \right\}$.

$$d\,u = \frac{2\, \epsilon^x\, d\,x}{\epsilon^{2x} - 1}.$$

(30). Let $u = \epsilon^{\epsilon^x}$.

$$d\,u = \epsilon^{\epsilon^x} \epsilon^x\, d\,x.$$

Art. 32, 33. (31). Let $u = \sin 2\,x$.

$$d\,u = 2 \cos 2\,x\, d\,x.$$

(32). Let $u = (\sin x)^n$.

$$d\,u = n\,(\sin x)^{n-1} \cos x\, d\,x.$$

(33). Let $u = \cos x \sin 2\,x$.

$$d\,u = (\cos x \cos 2\,x + \cos 3\,x)\, d\,x.$$

(34). Let $u = (\tan x)^n$.

$$d\,u = \frac{n\,(\tan x)^{n-1}\, d\,x}{(\cos x)^2}.$$

(35). Let $u = \log \sin x$.

$$d\,u = \cot x \, . \, d\,x.$$

(36). Let $u = \log \sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)}$.

$$d\,u = \frac{d\,x}{\cos x}. \quad \text{(Art. 208.)}$$

(37). Let $u = \log \sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)}$.

$$d\,u = \frac{d\,x}{\sin x}. \quad \text{(Art. 208).}$$

(38). Let $u = \frac{x}{2} - \frac{\sin x \, . \cos x}{2}$.

$$d\,u = (\sin x)^2\, d\,x.$$

(39). Let $u = \frac{\sin n\,x}{(\sin x)^n}$.

$$d\,u = \frac{-\,n \sin (n-1)\,x \, . \, d\,x}{(\sin x)^{n+1}}.$$

(40). Let $u = \frac{\sin n\,x}{(\cos x)^n}$.

$$d\,u = \frac{n \cos (n-1)\, x\, d\,x}{(\cos x)^{n+1}}.$$

(41). Let $u = \epsilon^{\sin x}$.

$$d\,u = \epsilon^{\sin x} \cos x\,d\,x.$$

(42). Let $u = \log \{ \cos x + \sqrt{(-1)} \cdot \sin x \}$.

$$d\,u = + d\,x \sqrt{(-1)}$$

(43). Let $u = \log \{ \cos x - \sqrt{(-1)} \sin x \}$.

$$d\,u = - d\,x \sqrt{(-1)}.$$

(44). Let $u = \log \left\{ \frac{1 + \sqrt{(-1)} \tan x}{1 - \sqrt{(-1)} \tan x} \right\}$.

$$d\,u = 2\,d\,x \sqrt{(-1)}.$$

(45). Let $u = \log \{ 1 - \sqrt{(1 - \epsilon^{-\frac{a}{\sin x}})} \}$.

$$d\,u = \frac{a\,\epsilon^{-\frac{a}{\sin x}} \cos x\,d\,x}{2\,(\sin x)^2 \sqrt{(1 - \epsilon^{-\frac{a}{\sin x}})} \,.\, \{ 1 - \sqrt{(1 - \epsilon^{-\frac{a}{\sin x}})} \}}.$$

Euler. *Inst. Calc. Diff.* Pars I, Art. 207.

(46). Let $u = \log \left\{ \frac{\epsilon^{x\sqrt{(-1)}} + \epsilon^{-x\sqrt{(-1)}}}{2} \right\} = \log \cos x$.

$$d\,u = \frac{(1 - \epsilon^{2x\sqrt{(-1)}})\,d\,x}{\sqrt{(-1)}\,(1 + \epsilon^{2x\sqrt{(-1)}})} = \tan x\,.\,d\,x.$$

(47). Let $u = \cos \left(\log \frac{1}{x} \right)$.

$$d\,u = \frac{d\,x}{x} \sin \left(\log \frac{1}{x} \right).$$

(48). Let $u = \cos (\sin x) = \cos \sin x$.

$$d\,u = - d\,x \cos x\,.\,\sin \sin x.$$

$$= - d\,x \cos x\,.\,\sin^2 x.$$

(49). Let $u = \sin \sin \sin x = \sin^3 x$.

$$d u = d x \cos x \cos \sin x \cos \sin^2 x.$$

(50). Let $u = \sin \tan x$.

$$d u = \frac{d x}{(\cos x)^2} \cos \tan x.$$

(51). Let $u = \cos \log \sin x$.

$$d u = - d x \cot x \sin \log \sin x.$$

(52). Let $u = \sin^{-1} \frac{1-x^2}{1+x^2}$, or in other words, let

$$u = \text{arc} \left(\text{whose sine} = \frac{1-x^2}{1+x^2}\right)$$

$$\text{Make } \frac{1-x^2}{1+x^2} = v;$$

consequently,

$$d u = \frac{d v}{\sqrt{(1-v^2)}} = -\frac{2 d x}{1+x^2}.$$

Euler. *Inst. Calc. Diff.* Pars I, Art. 207.

(53). Let $u = \sin^{-1} \sqrt{\left(\frac{1-x}{2}\right)}$.

$$d u = \frac{- d x}{2 \sqrt{(1-x^2)}}. \quad \textit{Id.}$$

(54). Let $u = \sin^{-1} \frac{x}{\sqrt{(1+x^2)}}$.

$$d u = \frac{d x}{1+x^2}.$$

(55). Let $u = \cos^{-1} (4 x^3 - 3 x)$.

$$d u = \frac{- 3 d x}{\sqrt{(1-x^2)}}.$$

(56). Let $u = \cos.^{-1} (32\,x^6 - 48\,x^4 + 18\,x^2 - 1)$.

$$d\,u = \frac{-6\,d\,x}{\surd\{1 - x^2\}}.$$

(57). Let $u = \tan^{-1} \frac{2\,x}{1 - x^2}$.

$$d\,u = \frac{2\,d\,x}{1 + x^2}.$$

(58). Let $u = \tan^{-1} \frac{5\,x - 10\,x^3 + x^5}{1 - 10\,x^2 + 5\,x^4}$.

$$d\,u = \frac{5\,d\,x}{1 + x^2}.$$

(59). Let $u = \sec^{-1} \frac{1}{2\,x^2 - 1}$.

$$d\,u = \frac{-2\,d\,x}{\surd(1 - x^2)}.$$

(60). Let $u = \tan^{-1} \left\{\frac{\surd\{1 + x^2\} - 1}{x}\right\}$.

$$d\,u = \frac{d\,x}{2\,(1 + x^2)}.$$

(61). Let $u = \tan^{-1}\{\surd(1 + x^2) - x\}$.

$$d\,u = \frac{-d\,x}{2\,(1 + x^2)}.$$

(62). Let $u = \frac{1}{\surd\{a^2 - b^2\}} \cos^{-1} \left\{\frac{b + a \cos x}{a + b \cos x}\right\}$.

$$d\,u = \frac{d\,x}{a + b \cos x}. \qquad \text{(Note } K\text{).}$$

(63). Let $u = \frac{1}{\surd(a^2 - b^2)} \tan^{-1} \left\{\frac{\sin x \surd(a^2 - b^2)}{b + a \cos x}\right\}$.

$$d\,u = \frac{d\,x}{a + b \cos x}. \qquad \text{(Note } K\text{).}$$

(64). Let $u = \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1} \left\{ \frac{\sqrt{(a-b)} \sqrt{(1-\cos x)}}{\sqrt{(a+b)} \sqrt{(1+\cos x)}} \right\}.$

$$d u = \frac{d x}{a+b \cos. x}. \qquad \text{(Note } K\text{).}$$

(65). Let $u = \epsilon^{\sin^{-1} x}$.

$$d u = \epsilon^{\sin^{-1} x} \frac{d x}{\sqrt{(1-x^2)}}.$$

(66). Let $u = \log \sin^{-1} x$.

$$d u = \frac{d x}{\sin^{-1} x . \sqrt{\{1-x^2\}}}.$$

(67). Let $u = \sqrt{(1-x^2)} \sin^{-1} x$.

$$d u = d x \left\{ 1 - \frac{x \sin.^{-1} x}{\sqrt{(1-x^2)}} \right\}.$$

(68). Let $u = \frac{1}{2\sqrt{3}} \log \left(\frac{1+x\sqrt{3+x^2}}{1-x\sqrt{3+x^2}} \right) + \frac{1}{3} \tan^{-1} \frac{3x(1-}{1-4x^2-}$

$$d u = \frac{d x}{1+x^6}.$$

(69). Let $u = \log \left(\frac{\sqrt{(1+x^4)} + x\sqrt{2}}{1-x^2} \right) + \sin^{-1} \frac{x\sqrt{2}}{1+x^2}.$

$$d u = \frac{4\sqrt{2}\, x^2\, d x}{(1-x^4)\sqrt{(1+x^4)}}.$$

(70). Let $u = \left\{ \epsilon^{\epsilon^x} x^n \sin x \cos^{-1} x \right\}^{\frac{1}{x}}$.

$d u = \{ \epsilon^{\epsilon^x} x^n \sin x \cos^{-1} x \}^{\frac{1}{x} - 1} . \epsilon^{\epsilon^x} x^{n-2} d x \Big\{ \{ \epsilon^x (x-1) +$
$n(1-\log x) - \log \sin x - \log \cos^{-1} x \} \sin x \cos^{-1} x + x$
$\{ \cos x \cos^{-1} x - \frac{\sin x}{\sqrt{(1-x^2)}} \} \Big\}.$

Arts. 36, 37. and Note D. In addition to the examples given by our author of the application of Maclaurin's theorem to the developement of transcendental functions, we subjoin the following, which may serve as good exercises for the student, in the various processes of differentiation.

(1). $u = (\cos x)^n$

$$= 1 - \frac{n x^2}{1.2} + n(3n-2)\frac{x^4}{1.2.3.4} -$$

$$n\{15(n-1)^2+1\}\frac{x^6}{1.2.3.4.5.6} +$$

$$n(105n^3 - 420n^2 + 588n - 272)\frac{x^8}{1.2\ldots.8} - \&\text{c.}$$

The general term of this series

$$= \frac{(-1)^{m+1}x^{2m}}{2^{n-1}.1.2\ldots.(2m)}\left\{n^{2m} + n(n-2)^{2m} + \right.$$

$$\left.\frac{n(n-1)}{1.2}.(n-4)^{2m} + \frac{n(n-1)(n-2)}{1.2.3}.(n-6)^{2m} + \&\text{c.}\right\}; \; (1).$$

the number of terms being limited to $\frac{n}{2}$ or $\frac{n+1}{2}$, according as n is an even or odd number.

Or under another form, it is

$$= \frac{U_{2m}x^{2m}}{1.2..(2m)} = \frac{x^{2m}}{1.2..(2m)}\left\{\frac{(m-1-n)}{m}.\frac{2m(2m-1)}{1.2}.U_{2m-2}\right.$$

$$-\frac{(m-2-2n)}{m}.\frac{2m(2m-1)(2m-2)(2m-3)}{1.2.3.4}.U_{2m-4}$$

$$+\frac{(m-3-3n)}{m}.\frac{2m(2m-1)\ldots.(2m-5)}{1.2.3.4.5.6}.U_{2m-6} - \&\text{c.}$$

$$+(-1)^m.\frac{\{1-(m-1)n\}}{m}.\frac{2m(2m-1)}{1.2}.U_2 +$$

$$\left.(-1)^m.n.U_0\right\}. \qquad (2).$$

where U_0, U_2, U_4 .... U_{2m}, are the values of

$$u, \frac{d^2 u}{d x^2}, \frac{d^4 u}{d x^4}, \dots \frac{d^{2m} u}{d x^{2m}}, \text{ when } x=0.$$

This formula may be exhibited under a form still more convenient for calculation, if we make a_0, a_2, a_4, ... a_{2m} respectively, equal to

$$U_0, \frac{U_2}{1 . 2}, \frac{U_4}{1 . 2 . 3 . 4}, \dots \frac{U_{2m}}{1 . 2 \dots (2m)},$$

or the several coefficients of x in the series: this assumption gives us the general term

$$= a_{2m} x^{2m} = \frac{x^{2m}}{m} \left\{ \frac{(m-1-n)}{1 . 2} . a_{2m-2} - \frac{(m-2-2n)}{1 . 2 . 3 . 4} . a_{2m-4} \right.$$

$$+ \frac{(m-3-3n)}{1 . 2 . 3 . 4 . 5 . 6} . a_{2m-6} - \&c. \dots\dots +$$

$$\left. \frac{(-1)^m . \{1-(m-1)n\}}{1 . 2 \dots (2m-2)} . a_2 + \frac{(-1)^m . mn}{1 . 2 \dots (2m)} . a_0 \right\}. \qquad (3).$$

(2). $u = \sec x = (\cos x)^{-1}$

$$= 1 + \frac{x^2}{1 . 2} + \frac{5 x^4}{1 . 2 . 3 . 4} + \frac{61 x^6}{1 . 2 . 3 . 4 . 5 . 6} + \frac{1385 x^8}{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8}$$

$$+ \frac{50521 x^{10}}{1 . 2 \dots 10} + \frac{2702765 x^{12}}{1 . 2 \dots 12} + \&c.$$

This series was first given by James Gregory, in a letter to Oldenburgh, in 1671: See *Commercium Epistolicum Johannis Collins et aliorum de Analysi promota*.

The general term of this series is

$$= \frac{(-1)^m 2 x^{2m}}{1 . 2 \dots (2m)} \left\{ 1^{2m} - 3^{2m} + 5^{2m} - 7^{2m} + \&c. \right\}:$$

or,

$$= \frac{2^{2m+2} x^{2m}}{\pi^{2m+1}} \left\{ 1 - \frac{1}{3^{2m+1}} + \frac{1}{5^{2m+1}} - \frac{1}{7^{2m+1}} + \&c. \right\}:$$

or, from the formula (3), in the last example, we find it

$$= a_{2m} x^{2m} = x^{2m} \left\{ \frac{a_{2m-2}}{1 \cdot 2} - \frac{a_{m-4}}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{a_{m-6}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \right.$$

$$\left. - \&c. \ldots\ldots + \frac{(-1)^m a_2}{1 \cdot 2 \ldots\ldots (2m-2)} + \frac{(-1)^{m+1} a_0}{1 \cdot 2 \ldots\ldots (2m)} \right\}.$$

(3). $u = \log \sec x = - \log \cos x$.

$$= \frac{x^2}{1 \cdot 2} + \frac{2 x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{16 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{16 \times 17 x^8}{1 \cdot 2 \ldots\ldots 8} +$$

$$\frac{128 \times 62 x^{10}}{1 \cdot 2 \ldots\ldots 10} + \&c.$$

James Gregory, Letter to Oldenburgh. 1671.

The general term of this series is

$$= \frac{(-1)^{m+1} x^{2m}}{1 \cdot 2 \ldots\ldots (2m)} \left\{ 1^{2m-1} - 2^{2m-1} + 3^{2m-1} - 4^{2m-1} + \&c. \right\}.$$

or, under another form,

$$= a_{2m} x^{2m} = \frac{x^{2m}}{m} \left\{ \frac{(m-1)}{1 \cdot 2} \cdot a_{2m-2} - \frac{(m-2)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot a_{2m-4} + \right.$$

$$\frac{(m-3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot a_{2m-6} - \&c. \ldots\ldots + \frac{(-1)^m \cdot a_2}{1 \cdot 2 \ldots\ldots (2m-2)}$$

$$\left. + \frac{(-1)^{m+1}}{2 \cdot 1 \cdot 2 \ldots\ldots (2m-1)} \right\}.$$

(4). Let $u = (\sin x)^n$: it is evident from the series for $\sin x$ (Art. 36), that the first term of this developement will involve x^n, or that the first n terms of the series of values

$U_0, U_1 \ldots . U_{n-1}$ will be severally equal to nothing: it will therefore render the operation much easier, if we assume

$$u' = \frac{u}{x^n} = \frac{(\sin x)^n}{x^n},$$

and then develope u' in the ordinary way: we shall thus get

$$u = u' \; x^n = x^n - \frac{n \, x^{n+2}}{1 . 2 . 3} + \frac{n\,(5\,n-2)}{3} . \frac{x^{n+4}}{1 . 2 . 3 . 4 . 5} -$$

$$\frac{n\,(35\,n^2 - 42\,n + 16)}{3^2} . \frac{x^{n+6}}{1 . 2 \ldots . 7} +$$

$$\frac{n\,(175\,n^3 - 420\,n^2 + 404\,n - 144)}{3 . 5} . \frac{x^{n+8}}{1 . 2 \ldots . 9} - \&c.$$

The general term of this series is

$$2^{m-1} = \frac{(-1)^m \, x^{n+2m}}{1 . 2 \ldots . (n + 2\,m)} \left\{ n^{n+2m} - n\,(n-2)^{n+2m} + \right.$$

$$\left. \frac{n\,(n-1)}{1 . 2}\,(n-4)^{n+2m} - \frac{n\,(n-1)\,(n-2)}{1 . 2 . 3}\,(n-6)^{n+2m} + \&c. \right\};$$

the series being restricted to $\frac{n}{2}$ or $\frac{n+1}{2}$ terms, according as n is an even or an odd number.

Or, under another form,

$$= a_{n+2m} \, x^{n+2m}$$

$$= \frac{x^{n+2m}}{m} \left\{ \frac{(m-1-n)}{1 . 2 . 3} . a_{n+2m-2} - \frac{(m-2-2\,n)}{1 . 2 . 3 . 4 . 5} . a_{n+2m-4} \right.$$

$$+ \frac{(m-3-3\,n)}{1 . 2 \ldots . 7} . a_{n+2m-6} - \&c. \ldots .$$

$$\left. + \frac{(-1)^m \{ 1 - (m-1)\,n \}}{1 . 2 \ldots . (2\,m-1)} . a_{n+2} + \frac{(-1)^m . m\,n}{1 . 2 \ldots . (2\,m+1)} . a_n \right\}.$$

(5). Let $u = \text{cosec } x = (\sin x)^{-1}$: consequently by means of the preceding example, we obtain

$$u = \frac{1}{x} + \frac{x}{1 \,.\, 2 \,.\, 3} + \frac{14\, x^3}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} + \frac{744\, x^5}{1 \,.\, 2 \,\ldots\, 9}$$

$$+ \frac{100584\, x^7}{1 \,.\, 2 \,\ldots\, 12} + \&c.$$

The general term of this series

$$= a_{2m-1} \,.\, x^{2m-1}.$$

$$= x^{2m-1} \left\{ \frac{a_{2m-3}}{1 \,.\, 2 \,.\, 3} - \frac{a_{2m-5}}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} + \frac{a_{2m-7}}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} \quad \&c. \right.$$

$$\left. + \frac{(-1)^m a_1}{1 \,.\, 2 \ldots (2m-1)} + \frac{(-1)^{m+1} a_{-1}}{1 \,.\, 2 \ldots (2m+1)} \right\}.$$

(6). $u = (\tan x)^n$.

$$= x^n + \frac{2n}{1 \,.\, 2 \,.\, 3} x^{n+2} + \frac{2^3 n (5n+7)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} x^{n+4}$$

$$+ \frac{2^6 n (35 n^2 + 147 n + 124)}{1 \,.\, 2 \,.\, 3 \,\ldots\, 9} x^{n+6}$$

$$+ \frac{2^7 \times 11 n \{ 175 n^3 + 1470 n^2 + 3509 n + 2286 \}}{1 \,.\, 2 \,\ldots\, 12} x^{n+8} + \&c:$$

the general term is

$$= a_{n+2m}\, x^{n+2m}$$

$$= x^{n+2m} \,.\, \frac{2}{m} \left\{ \frac{(n+2m-2)}{1 \,.\, 2 \,.\, 3} a_{n+2m-2} - \frac{2^2 (n+2m-4)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \,.\right.$$

$$a_{n+2m-4} + \frac{2^4 (n+2m-6)}{1 \,.\, 2 \,\ldots\, 7} \,.\, a_{n+2m-6} - \&c.$$

$$\left. + \frac{(-1)^m 2^{2m-4} (n+2)}{1 \,.\, 2 \,\ldots\, (2m-1)} \,.\, a_{n+2} + \frac{(-1)^{m+1} 2^{2m-2}\, n}{1 \,.\, 2 \,\ldots\, (2m+1)} \right\}.$$

(7). Let $u = \cot x = (\tan x)^{-1}$: by the preceding example, we get

$$u = \frac{1}{x} - \frac{2}{1.2.3}.x - \frac{2^4}{1.2.3.4.5.6}.x^3 - \frac{2^6.12}{1.2....9}.x^5$$

$$- \frac{2^8.3.x^7}{1.2....10} - \frac{2^{10}.10x^9}{1.2....12} - \&c.$$

See Art. 411. Appendix.

(8). $u = (\sin x)^m (\cos x)^n$.

$$= x^n - \frac{(3m+n)}{1.2.3} x^{n+2} + \frac{\{5(9m^2+n^2)+30m(n-1)-2n\}}{3.1.2.3.4.5} x^{n+4}$$

$$- \{35(n^3+27m^3) - 42(n^2+45m^2) + 16(n+63m)$$

$$+ 252(5m-3n)\} \frac{x^{n+6}}{1.2....7.3^2}$$

$$+ \{175(n^4+81m^4) - 420(n^3+135m^3)$$

$$+ 4(101n^2+1984m^2) - 144(n+255m)$$

$$+ 2100(n^2+9m^2)nm - 1260(7n+33m)mn$$

$$+ 9450m^2n^2 + 23640mn\} \frac{x^{n+8}}{1.2....9 \times 3 \times 5} - \&c.$$

The law of derivation of this series admits of no very simple expression: it may be indirectly exhibited in the following manner: let

$$(\cos x)^m = 1 + a_2 x^2 + a_4 x^4 + \ldots\ldots + a_{2p}.x^{2p} + \&c.$$

and

$$(\sin x)^n = x^n + a_2 x^{n+2} + a_4 x^{n+4} + \ldots\ldots + a_{2p}.x^{n+2p} + \&c.$$

then the general term of the series

$$= A_{n+2p} x^{n+2p}$$

$$= x^{n+2p} \{ a_{2p} + \alpha_2 a_{2p-2} + \alpha_4 a_{2p-4} + \alpha_6 a_{2p-6}$$

$$+ \&c. \ldots\ldots + a_2 \alpha_{2p-2} + \alpha_{2p} \}.$$

(9). $u = \surd \{ a^2 + 2\,a\,b \cos x + b^2 \}$

$$=(a+b)\Big\{ 1 - \frac{a\,b}{(a+b)^2} \cdot \frac{x^2}{1\,.\,2} + \frac{a\,b\,(a^2 - a\,b + b^2)}{(a+b)^4} \cdot \frac{x^4}{1\,.\,2\,.\,3\,.\,4}$$

$$- \frac{a\,b\,(a^4 - 11\,a^3\,b + 21\,a^2\,b^2 - 11\,a\,b^3 + b^4)}{(a+b)^6} \cdot \frac{x^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} +$$

$$\frac{a\,b\,(a^6 - 57\,a^5\,b + 393\,a^4\,b^2 - 673\,a^3\,b^3 + 393\,a^2\,b^4 - 57\,a\,b^5 + b^6)}{(a+b)^8}$$

$$\cdot \frac{x^8}{1\,.\,2\,\ldots\,.\,8} + \&c.$$

The general term of this series

$$= a_{2m}\,x^{2m}$$

$$= - \frac{x^{2m}}{2\,m \cdot (a+b)} \Big\{ (2\,m - 2)\,a_{2m-2}\,a_2 + (2m-4)\,a_{2m-4}\,a_4 + \&c.$$

$$+ 4\,a_4\,a_{2m-4} + 2\,a_2\,a_{2m-2} + (-1)^{m+1} \frac{a\,b}{1\,.\,2\,\ldots\,.\,(2\,m-1)} \Big\}.$$

(10). $u = \cos^{-1} x$

$$= \frac{\pi}{2} - x - \frac{1}{2} \cdot \frac{x^3}{1\,.\,2\,.\,3} - \frac{1\,.\,3}{2^2} \cdot \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} -$$

$$\frac{1\,.\,3\,.\,5}{2^3} \cdot \frac{x^7}{1\,.\,2\,\ldots\,.\,7} - \frac{1\,.\,3\,.\,5\,.\,7}{2^4} \cdot \frac{x^9}{1\,.\,2\,\ldots\,.\,9} - \&c.$$

(11). $u = \epsilon^{\cos x}$

$$= \epsilon\,\Big\{ 1 - \frac{x^2}{1\,.\,2} + \frac{4\,x^4}{1\,.\,2\,.\,3\,.\,4} - \frac{31\,x^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} + \frac{379\,x^8}{1\,.\,2\,\ldots\,.\,8}$$

$$- \frac{6556\,x^{10}}{1\,.\,2\,\ldots\,.\,10} + \frac{150349\,x^{12}}{1\,.\,2\,\ldots\,.\,12} - \&c. \Big\}.$$

The general term of this series is

$$= a_{2m} x^{2m}$$

$$= \frac{-1}{2m} \Big\{ a_{2m-2} - \frac{a_{2m-4}}{1.2.3} + \frac{a_{2m-6}}{1.2.3.4.5} - \&c.$$

$$+ \frac{(-1)^{m+1} a_0}{1.2 \dots (2m-1)} \Big\}.$$

(12). $u = \epsilon^{\sin x}$

$$= 1 + x + \frac{x^2}{1.2} - \frac{3x^4}{1.2.3.4} - \frac{8x^5}{1.2.3.4.5} - \frac{3x^6}{1.2.3.4.5.6}$$

$$+ \frac{56x^7}{1.2 \dots 7} + \frac{217x^8}{1.2 \dots 8} + \frac{36x^9}{1.2 \dots 9}$$

$$- \frac{2959x^{10}}{1.2 \dots 10} - \&c.$$

The general term of this series is

$$= a_m x^m$$

$$= \frac{x^m}{m} \Big\{ a_{m-1} - \frac{a_{m-3}}{1.2} + \frac{a_{m-5}}{1.2.3.4} - \frac{a_{m-7}}{1.2.3.4.5.6} + \&c. \Big\}$$

(13). $u = \epsilon^{\epsilon^x}$

$$= \epsilon \Big\{ 1 + x + \frac{2x^2}{1.2} + \frac{5x^3}{1.2.3} + \frac{15x^4}{1.2.3.4} + \frac{52x^5}{1.2.3.4.5} +$$

$$\frac{188x^6}{1.2 \dots 6} + \frac{847x^7}{1.2 \dots 7} + \frac{4004x^8}{1.2 \dots 8} + \&c. \Big\}.$$

The general term is

$$= a_m x^m$$

$$= \frac{x^m}{m} \Big\{ a_{m-1} + a_{m-2} + \frac{a_{m-3}}{1.2} + \frac{a_{m-4}}{1.2.3} + \&c.$$

$$\ldots\ldots + \frac{a_0}{1 \,.\, 2 \ldots\ldots (m-1)}\Big\}.$$

(14). $u = \dfrac{\epsilon^x}{\cos . x}$

$$= 1 + x + \frac{2\,x^2}{1\,.\,2} + \frac{4\,x^3}{1\,.\,2\,.\,3} + \frac{12\,x^4}{1\,.\,2\,.\,3\,.\,4} + \frac{36\,x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} +$$

$$\frac{152\,x^6}{1\,.\,2 \ldots\ldots 6} + \frac{624\,x^7}{1\,.\,2 \ldots\ldots 7} + \frac{3472\,x^8}{1\,.\,2 \ldots\ldots 8} + \&c.$$

The general term is

$$= a_m\,x^m = (\text{if } m = 2\,r),$$

$$x^m \left\{ \frac{a_{m-2}}{1\,.\,2} - \frac{a_{m-4}}{1\,.\,2\,.\,3\,.\,4} + \&c. \ldots\ldots + \frac{(-1)^{\frac{m}{2}+1} a_0}{1\,.\,2 \ldots\ldots m} + \frac{1}{1\,.\,2 \ldots m} \right\};$$

or, (if $m = 2\,r + 1$),

$$= x^m \left\{ \frac{a_{m-2}}{1\,.\,2} - \frac{a_{m-4}}{1\,.\,2\,.\,3\,.\,4} + \&c. \ldots\ldots + \frac{(-1)^{\frac{m+1}{2}} a_1}{1\,.\,2 \ldots\ldots (m-1)} + \right.$$

$$\left. \frac{1}{1\,.\,2 \ldots\ldots m} \right\}.$$

(15). $u = \sqrt{(1 + \epsilon^x)}$.

$$= \sqrt{(2)}\,.\,\left\{ 1 + \frac{x}{4} + \frac{3}{4^2} \cdot \frac{x^2}{1\,.\,2} + \frac{7}{4^3} \cdot \frac{x^3}{1\,.\,2\,.\,3} + \frac{9}{4^4} \cdot \frac{x^4}{1\,.\,2\,.\,3\,.\,4} \right.$$

$$+ \frac{1}{4^5} \cdot \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} + \frac{123}{4^6} \cdot \frac{x^6}{1\,.\,2 \ldots\ldots 6} + \frac{967}{4^7} \cdot \frac{x^7}{1\,.\,2 \ldots\ldots 7}$$

$$\left. - \frac{4911}{4^8} \cdot \frac{x^8}{1\,.\,2 \ldots\ldots 8} - \&c. \right\}.$$

The general term of this series

$$= a_m x^m.$$

$$= -\frac{x^m}{2\,m\,a_0}\Big\{(2\,m-2)\,a_1\,a_{m-1} + (2\,m-4)\,a_2\,a_{m-2} + \&c.$$

$$+ 4\,a_2\,a_{m-2} + 2\,a_1\,a_{m-1} - \frac{1}{1\,.\,2\,\ldots\,.\,(m-1)}\Big\}.$$

(16). Let $u = \frac{x}{\epsilon^x - 1}$: in this case, U_0, U_1, U_2, &c. become severally equal to $\frac{0}{0}$, and require a very tedious and embarrassing reduction, in order to determine them: the following method is much more simple. Since

$$u = \frac{x}{\epsilon^x - 1} = \frac{x}{1 + x + \frac{x^2}{1\,.\,2} + \&c. - 1} = \frac{1}{1 + \frac{x}{1\,.\,2} + \frac{x^2}{1\,.\,2\,.\,3}} + \&c.$$

by the method of indeterminate coefficients (Note *D*), we get

$$u = 1 - \frac{1}{2}\,x + \frac{1}{6}\cdot\frac{x^2}{1\,.\,2} - \frac{1}{30}\cdot\frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \frac{1}{42}\cdot\frac{x^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}$$

$$- \frac{1}{30}\cdot\frac{x^8}{1\,.\,2\,\ldots\,.\,8} + \frac{5}{66}\cdot\frac{x^{10}}{1\,.\,2\,\ldots\,.\,10} - \frac{691}{2730}\cdot\frac{x^{12}}{1\,.\,2\,\ldots\,.\,12}$$

$$+ \frac{7}{6}\cdot\frac{x^{14}}{1\,.\,2\,\ldots\,.\,14} - \frac{3617}{510}\cdot\frac{x^{16}}{1\,.\,2\,\ldots\,.\,16} + \frac{43867}{798}\cdot\frac{x^{18}}{1\,.\,2\,\ldots\,.\,18}$$

$$- \frac{174611}{330}\cdot\frac{x^{20}}{1\,.\,2\,\ldots\,.\,20} + \frac{854513}{138}\cdot\frac{x^{22}}{1\,.\,2\,\ldots\,.\,22} -$$

$$\frac{236364091}{2730}\cdot\frac{x^{24}}{1\,.\,2\,\ldots\,.\,24} + \frac{8553103}{6}\cdot\frac{x^{26}}{1\,.\,2\,\ldots\,.\,26} -$$

$$\frac{23749461029}{870} \cdot \frac{x^{28}}{1 \,.\, 2 \ldots . 28} + \frac{8615841276005}{14322} \cdot \frac{x^{30}}{1 \,.\, 2 \ldots . 30}$$

$$- \frac{8480253\ 1453387}{5440} \cdot \frac{x^{32}}{1 \,.\, 2 \,..\, 32} + \&c.$$

The general term being

$$= a_{2m} x^{2m}.$$

$$= -x^{2m} \left\{ \frac{a_{2m-2}}{1 \,.\, 2} - \frac{a_{2m-4}}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \&c. \ldots . + \frac{a_1}{1 \,.\, 2 \ldots . (2m-1)} \right.$$

$$\left. + \frac{a_0}{1 \,.\, 2 \ldots . 2m} \right\}.$$

The coefficients of

$$\frac{x^2}{1 \,.\, 2}, \quad \frac{x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4}, \quad \frac{x^6}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6}, \quad \&c.$$

are the celebrated numbers of *Bernouilli:* See Art. 408. *Appendix.*

It may be proper to observe, that in the determination of many of the preceding series, the theorem of *Maclaurin* becomes practically inapplicable, in consequence of the excessive complication of the expressions for the differential coefficients: we have therefore been sometimes obliged to have recourse to other and more expeditious methods which the Differential Calculus affords us, and which will be fully exemplified in Art. 45.

We shall put down a few examples of the use of Taylor's *Series* in the developement of transcendental functions, to which we referred in Arts. 21, 22.

(1). Let $u = \log. x$: the general series gives us,

$$\log (x + h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \&c.$$

or,

$$\log\left(\frac{x+h}{x}\right)=\frac{h}{x}-\frac{h^2}{2\,x^2}+\frac{h^3}{3\,x^3}-\text{\&c.}\qquad(1).$$

If we put $a+x$ in the place of x, and make $h=x$, this series becomes

$$\log\left(\frac{a+2\,x}{a+x}\right)=\frac{x}{a+x}-\frac{1}{2}\cdot\left(\frac{x}{a+x}\right)^2+\frac{1}{3}\left(\frac{x}{a+x}\right)^3-\text{\&c.}\ (2).$$

By making $h=-x$, as in formula (3), page 30, and reducing, we get

$$\log\left(\frac{a+x}{a}\right)=\frac{x}{a+x}+\frac{1}{2}\left(\frac{x}{a+x}\right)^2+\frac{1}{3}\cdot\left(\frac{x}{a+x}\right)^3+\text{\&c.}\qquad(3).$$

If we add these two series together, we find

$$\log.\left(\frac{a+2x}{a}\right)=2\left\{\frac{x}{a+x}+\frac{1}{3}\cdot\left(\frac{x}{a\ +\ x}\right)^3\right.$$
$$\left.+\frac{1}{5}\left(\frac{x}{a\ +\ x}\right)^5+\ \text{\&c.}\right\}.\qquad(4).$$

Or, putting $\frac{x}{2}$ in the place of x, it becomes

$$\log\left(\frac{a+x}{a}\right)=2\left\{\frac{x}{2\,a+x}+\frac{1}{3}\cdot\left(\frac{x}{2\,a+x}\right)^3\right.$$
$$\left.+\ \frac{1}{5}\left(\frac{x}{2\,a+x}\right)^5+\ \text{\&c.}\right\}\qquad(5).$$

a series which is also given in Art. 29.

(2). Let $u=\sin x$: then

$$\sin(x+h)=\sin x\left(1-\frac{h^2}{1.2}+\frac{h^4}{1.2.3.4}-\frac{h^6}{1.2.3.4.5.6}+\ \text{\&c.}\right)$$

$$+ \cos x \left(h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \&c. \right)$$

$$= \sin x \cos h + \cos x \sin h.$$

(3). Let $u = \tan x$: by making $\sin x = x$, we shall get, by means of Example 16, Art. 18,

$$\tan(x + h) = \tan x + \frac{1}{(\cos x)^2} \cdot \frac{h}{1} + \frac{2 \sin x}{(\cos. x)^3} \cdot \frac{h^2}{1.2}$$

$$+ \left(\frac{6 - 4 (\cos x)^2}{(\cos x)^4} \right) \cdot \frac{h^3}{1.2.3} + \sin x \left(\frac{24 - 8 (\cos x)^2}{(\cos x)^5} \right) \cdot \frac{h^4}{1.2.3.4}$$

$$+ \left(\frac{120 - 120 (\cos x)^2 + 16 (\cos x)^4}{(\cos x)^6} \right) \cdot \frac{h^5}{1.2.3.4.5}$$

$$+ \sin x \left(\frac{720 - 480 (\cos x)^2 + 32 (\cos. x)^4}{(\cos x)^7} \right) \cdot \frac{h^6}{1.2.3.4.5.6}$$

$$+ \left(\frac{5040 - 6720 (\cos x)^2 + 2036 (\cos x)^4 - 64 (\cos x)^6}{(\cos x)^8} \right) \frac{h^7}{1.2.3.4.5.6.7}$$

$$+ \&c.$$

(4). Let $u = \sin^{-1} x$; by means of Ex. 9, Art. 18, we find

$$\sin^{-1}(x + h) = \sin^{-1} x + \frac{1}{\sqrt{(1 - x^2)}} \cdot \frac{h}{1} + \frac{x}{(1 - x^2)^{\frac{3}{2}}} \cdot \frac{h^2}{1.2}$$

$$+ \frac{(1 + 2 x^2)}{(1 - x^2)^{\frac{5}{2}}} \cdot \frac{h^3}{1.2.3} + \frac{3 x (3 + 2 x^2)}{(1 - x^2)^{\frac{7}{2}}} \cdot \frac{h^4}{1.2.3.4}$$

$$+ \frac{3 (3 + 24 x^2 + 8 x^4)}{(1 - x^2)^{\frac{9}{2}}} \cdot \frac{h^5}{1.2.3.4.5}$$

$$+ \frac{15 x (15 + 40 x^2 + 8 x^4)}{(1 - x^2)^{\frac{11}{2}}} \cdot \frac{h^6}{1.2.3.4.5.6}$$

$$+ \frac{30(5 + 90x^2 + 120x^4 + 16x^6)}{(1 - x^2)^{\frac{13}{2}}} \cdot \frac{h^7}{1.2.3.4.5.6.7}$$

$+$ &c.

Euler. *Inst. Calc. Diff.* Pars II. Art. 83.

(5). Let $u = \tan^{-1} x$; by means of Ex. 13, Art. 18, we get

$$\tan^{-1}(x + h) = \tan^{-1} x + \frac{1}{1 + x^2} \cdot \frac{h}{1} - \frac{2x}{(1 + x^2)^2} \cdot \frac{h^2}{1.2}$$

$$+ \frac{2(3x^2 - 1)}{(1 + x^2)^3} \cdot \frac{h^3}{1.2.3}$$

$$- \frac{2.3.4(x^3 - x)}{(1 + x^2)^4} \cdot \frac{h^4}{1.2.3.4} + \frac{2.3.4(5x^4 - 10x^2 + 1)}{(1 + x^2)^5} \cdot \frac{h^5}{1.2.3.4.5}$$

$$- \frac{2.3.4.5x(6x^4 - 20x^2 + 6)}{(1 + x^2)^6} \cdot \frac{h^6}{1.2.3.4.5.6}$$

$$+ \frac{2.3.4.5.6(7x^6 - 35x^4 + 21x^2 - 1)}{(1 + x^2)^7} \cdot \frac{h^7}{1.2.3.4.5.6.7} - \&c.$$

Euler by a particular artifice, has effected the developement of $\tan^{-1}(x + h)$, in a very remarkable series, from which some very curious and beautiful results are deducible: if we assume

$$u = \tan^{-1} x = \frac{\pi}{2} - z, \text{ we have}$$

$$x = \tan u = \cot. z,$$

and

$$\frac{du}{dx} = \frac{1}{1 + x^2} = (\sin z)^2 = -\frac{dz}{dx}.$$

By differentiating, we get

$$\frac{d^2 u}{dx^2} = \frac{dz}{dx} \sin 2z = -(\sin z)^2 \sin 2z.$$

Also

$$\frac{d^3 u}{1\,.\,2\,.\,d x^3} = -\frac{d z}{d x}\left\{\sin z \cos z \sin 2 z + (\sin z)^2 \cos 2 z\right\}$$

$$= -\frac{d z}{d z} \sin z \sin 3 z = (\sin z)^3 \sin 3 z,$$

$$\frac{d^4 u}{1\,.\,2\,.\,3\,.\,d x^4} = -(\sin.\ z)^4 \sin 4 z,$$

$$\frac{d^5 u}{1.2.3.4.d z^5} = (\sin z)^5 \sin 5 z, \text{ and so on.}$$

Consequently,

$$\tan^{-1}(x+h) = \tan^{-1} x + \sin z \sin z\,.\,\frac{h}{1} - (\sin z)^2 \sin 2 z\,.\,\frac{h^2}{2}$$

$$+ (\sin z)^3 \sin 3 z\,.\,\frac{h^3}{3} - (\sin z)^4 \sin 4 z\,.\,\frac{h^4}{4} + (\sin.\ z)^5 \sin 5 z\,.\,\frac{h^5}{5}$$

$$- \&c. \qquad (1).$$

If in this series, we make $h = -x$, we find

$$\tan^{-1} x = \frac{\pi}{2} - z = \frac{x}{1} \sin z \sin z + \frac{x^2}{2} \sin 2 z (\sin z)^2$$

$$+ \frac{x^3}{3} \sin 3 z (\sin z)^3$$

$$+ \frac{x^4}{4} \sin 4 z (\sin z)^4 + \&c. \qquad (2).$$

Since $x = \cot z = \frac{\cos z}{\sin z}$, this series gives us

$$\frac{\pi}{2} = z + \sin.\ z \cos z + \frac{1}{2} \sin 2 z (\cos z)^2$$

$$+ \frac{1}{3} \sin 3 z (\cos z)^3 + \&c. \qquad (3).$$

If in series (1), we make $h = -2x$, we shall get, since $\tan^{-1}(-x) = -\tan^{-1} x$,

$$2 \tan^{-1} x = \frac{2x}{1} \sin z \sin z + \frac{2^2 x^2}{2} \sin 2z (\sin z)^2$$

$$+ \frac{2^3 x^3}{3} \sin 3z (\sin z)^3 + \&c. \qquad (4).$$

By substituting $\cot z$ for x, this series gives us

$$\frac{\pi}{2} = z + \sin z \cos z + \frac{2}{2} \sin 2z (\cos z)^2 + \frac{2^2}{3} \sin 3z (\cos z)^3$$

$$+ \frac{2^3}{4} \sin 4z (\cos z)^4 + \&c. \qquad (5).$$

By making $z = \frac{\pi}{4}$, we get

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \&c.$$

By subtracting series (3) and (5) from each other, we find

$$0 = \frac{(2^0 - 1)}{1} \sin z \cos z + \frac{(2 - 1)}{2} \sin 2z (\cos z)^2$$

$$+ \frac{(2^2 - 1)}{3} \sin 3z (\cos z)^3 + \frac{(2^3 - 1)}{4} \sin 4z (\cos z)^4 + \&c. \quad (6).$$

If in series (1), we make $h = \frac{x(1+x^2)}{1-x^2} = -\frac{\cos z}{\sin z \cos 2z}$

and therefore $\tan^{-1}(x+h) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$,

we shall get

$$\tan^{-1} x = -\left\{\frac{\sin z \cos z}{1 . \cos 2z} + \frac{\sin 2z (\cos z)^2}{2 . (\cos 2z)^2} + \frac{\sin 3z (\cos z)^3}{3 (\cos 2z)^3}\right.$$

$$\left. + \frac{\sin 4z (\cos z)^4}{4 (\cos 2z)^4} + \&c.\right\} \qquad (7).$$

In the same manner by making $h = \frac{2x(1+x^2)}{1-3x^2} = -\frac{2\cos z}{\sin z(1+2\cos 2z)}$, and $\therefore$ $\tan^{-1}(x+h) = 3\tan^{-1}x$, we shall get

$$2\tan^{-1}x = -\left\{\frac{2\sin z\cos z}{1(1+2\cos 2z)} + \frac{2^2}{2}\cdot\frac{\sin 2z(\cos z)^2}{(1+2\cos 2z)^2}\right.$$
$$\left. + \frac{2^3}{3}\cdot\frac{\sin 3z(\cos z)^3}{(1+2\cos 2z)^3} + \frac{2^4}{4}\cdot\frac{\sin 4z(\cos z)^4}{(1+2\cos 2z)^4} + \&c.\right\} \quad (8).$$

If we make $h = -\left(x+\frac{1}{x}\right) = -\frac{1}{\cos z\sin z}$, and therefore

$$\tan^{-1}(x+h) = \tan^{-1}\left(-\frac{1}{x}\right) = -\tan^{-1}\frac{1}{x} = -\frac{\pi}{2} + \tan^{-1}x,$$

the same series will give us,

$$\frac{\pi}{2} = \frac{\sin z}{1\,.\cos z} + \frac{\sin 2z}{2(\cos z)^2} + \frac{\sin 3z}{3(\cos z)^3} + \frac{\sin 4z}{4(\cos z)^4} + \&c. \; (9).$$

By differentiating this series and striking out the common factors, we shall find

$$0 = 1 + \frac{\cos z}{\cos z} + \frac{\cos 2z}{(\cos z)^2} + \frac{\cos 3z}{(\cos z)^3} + \&c. \quad (10).$$

In a similar manner, by differentiating series (3), we find

$$0 = 1 + \cos 2z + \cos z\cos 3z + (\cos z)^2\cos 4z$$
$$+ (\cos z)^3\cos 5z + \&c. \quad (11).$$

If in series (1), we make $h = -\sqrt{(1+x^2)} = -\frac{1}{\sin z}$, and therefore,

$$\tan^{-1}\{x - \surd(1+x^2)\} = -\frac{1}{2}\tan^{-1}\frac{1}{x},$$

(Ex. 61. Art. 31.)

$$= \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}x\right) = -\frac{z}{2},$$

we shall get

$$\frac{\pi}{2} = \frac{z}{2} + \frac{1}{1}\sin z + \frac{1}{2}\sin 2z + \frac{1}{3}\sin 3z + \frac{1}{4}\sin 4z + \&c. \quad (11).$$

Also, since (Note *D*. p. 630.)

$$\frac{z}{2} = \sin z - \frac{1}{2}\sin 2z + \frac{1}{3}\sin 3z - \&c.$$

by adding this series to the former, we get

$$\frac{\pi}{4} = \sin z + \frac{1}{3}\sin 3z + \frac{1}{5}\sin 5z + \&c. \quad (12).$$

Euler. *Inst. Calc. Diff.* Pars. II. Art. 57 .. 93.

(6). Let $u = (\log\tan)^{-1}x$, or let u be an arc, the logarithm of whose tangent is x: consequently, $x = \log\tan u$, and therefore by the general series

$$(\log\tan)^{-1}(x+h) = u + \frac{\sin 2u}{2}\cdot\frac{h}{1} + \frac{\sin 2u\cos 2u}{2}\cdot\frac{h^2}{1.2}$$

$$+ \frac{\sin 2u\cos 4u}{2}\cdot\frac{h^3}{1.2.3} + \frac{\sin 2u}{2}(\cos 6u - \sin 2u\sin 4u)\frac{h^4}{1.2.3.4}$$

$$+ \frac{\sin 2u}{2}(\cos 8u - 2\sin 2u\sin 6u)\frac{h^5}{1.2.3.4.5}$$

$$+ \frac{\sin 2u}{2}\{\cos 10u - 7\sin 2u\sin 8u - 2(\sin 2u)^2\cos 6u\}.$$

$$\frac{h^6}{1.2.3.4.5.6} + \frac{\sin 2u}{2}\{\cos 12u - 18\sin 2u\cos 10u$$

$$- 20(\sin 2u)^2\cos 8u\}\frac{h^7}{1.2\ldots.7} + \&c.$$

Euler. *Inst. Calc. Diff.* Pars. II. Art. 100.

(7). Let $u = \epsilon^x \sin nx$: by formula (3), p. 30, we get

$$0 = \epsilon^x \sin nx \left\{ 1 - x + (1 - n^2)\frac{x^2}{1.2} - (1 - 3n^2)\frac{x^3}{1.2.3} + (1 - 6n^2 + n^4)\frac{x^4}{1.2.3.4} - (1 - 10n^2 + 5n^4)\frac{x^5}{1.2.3.4.5} + \&c. \right\}$$

$$- \epsilon^x \cos nx \left\{ nx - 2n\frac{x^2}{1.2} + n(3 - n^2)\frac{x^3}{1.2.3} - n(4 - 4n^2)\frac{x^4}{1.2.3.4} + n(5 - 10n^2 + n^4)\frac{x^5}{1.2.3.4.5} - \&c. \right\}$$

consequently,

$$\tan nx = \frac{nx - 2n\frac{x^2}{1.2} + n(3 - n^2)\frac{x^3}{1.2.3} - \&c.}{1 - x + (1 - n^2)\frac{x^2}{1.2} - (1 - 3n^2)\frac{x^3}{1.2.3} + \&c.}$$

Euler. *Id.* Art. 101, 102.

On Series for the Calculation of Logarithms.

The series of Mercator in Art. 28, for a logarithm in terms of the number corresponding to it, or

$$(1).\ \log(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \&c.$$

is of no practical use in the calculation of logarithms, since it evidently becomes divergent, when u exceeds unity. This defect, however, is partially removed in the series given in Art. 29, or

$$(2).\ \log\left(\frac{1 + u}{1 - u}\right) = 2\left\{ u + \frac{u^3}{3} + \frac{u^5}{5} + \&c. \right\}$$

which was deduced from the former by the celebrated James Gregory*; since in this case, if we assume $\frac{1+u}{1-u}$ equal to a whole number, u is always a proper fraction, and the series is convergent, though not with sufficient rapidity for the purposes of calculation, when the number whose logarithm is required, becomes considerable. We must therefore direct our principal attention to such transformations of this series, as enable us to calculate the logarithms of numbers, by means of the logarithms of other numbers which precede them. Thus, if we assume, $\frac{1+u}{1-u} = \frac{m}{n}$, and therefore, $u = \frac{m-n}{m+n}$, we get

$$(3).\ \log\frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\}$$

or,

$$\log m = \log n + 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\}$$

If $m = n + 1$, we find,

$$(4).\ \log(n+1) = \log n + \left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \&c.\right\}$$

a series which enables us to calculate the logarithms of numbers by means of those which immediately precede them.

In investigating transformations of the series (3), it is most convenient to assume m and n such functions of any whole number (x), as are resolvable into simple factors, and whose difference $m - n$ is either unity or some constant number.

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* *Exercitationes Geometricæ.* 1670.

Thus, let $\frac{m}{n} = \frac{x^2}{x^2-1}$: an hypothesis which gives

$$(5).\quad \log.(x+1) = 2\log.x - \log.(x-1) - 2\left\{\frac{1}{2x^2-1} + \frac{1}{3}\cdot\frac{1}{(2x^2-1)^3} + \frac{1}{5}\cdot\frac{1}{(2x^2-1)^5} + \&c.\right\};$$

so great is the convergency of this series, that if we assume $x = 100$, the two first terms will give a result which is accurate to fourteen places of decimals.

Let $\frac{m}{n} = \frac{x^2+px+q}{x^2+px-q} = \frac{(x-a)(x-b)(x+a+b)}{(x+a)(x+b)(x-a-b)}$; therefore

$$(6).\ \log(x+a+b) = \{\log.(x+a)+\log(x+b)+\log(x-a-b)\} - \{\log(x-a)+\log(x-b)\} + 2\left\{\frac{q}{x^3+px} + \frac{1}{3}\left(\frac{q}{x^3+px}\right)^3 + \frac{1}{5}\left(\frac{q}{x^3+px}\right)^5 + \&c.\right\}$$

If $a = b = 1$, this formula gives us

$$(7).\ \log(x+2) = \{2\log(x+1)+\log(x-2)\} - 2\log(x-1) + 2\left\{\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \frac{1}{5}\cdot\left(\frac{2}{x^3-3x}\right)^5 + \&c.\right\}$$

The following formulæ may be deduced in a similar manner.

$$(8).\quad \log(x+5) = \{\log(x+4)+\log(x+3)+\log(x-3) + \log(x-4)\} - \{2\log x + \log(x-5)\} + 2\left\{\frac{72}{x^4-25x^2+72} + \frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3 + \&c.\right\}.$$

If $x = 100$, the first term of this series is accurate to 15 places of decimals.

$$(9).\ \text{Log}\,(x+6) = \{\,2\log(x+5) + 2\log x\,\} -$$

$$\{\,\log(x-1) + \log(x+2) + \log(x+3)\,\} -$$

$$2\,.\left\{\frac{18}{x^4+10x^3+25x^2-18} + \frac{1}{3}\left(\frac{18}{x^4+10x^3+25x^2-18}\right) + \&c.\right.$$

$$(10).\ \text{Log}\,(x+10) = \{\,\log(x+9) + \log(x+7) + \log(x-1$$

$$\log(x-4) + \log(x-2)\,\} - \{\,\log(x+4) + \log(x+2) +$$

$$\log(x-7) + \log(x-9)\,\} + 2\left\{\frac{5040}{x^5-125x^3+3004x} +\right.$$

$$\left.\frac{1}{3}\left(\frac{5040}{x^5-125x^3+3004x}\right)^3 + \&c.\right\}$$

$$(11).\ \text{Log}\,(x+8) = \{\,2\log(x+7) + 2\log x +$$

$$2\log(x-7)\,\} - \{\,\log(x-3) + \log(x+3) + \log(x-5) +$$

$$\log(x+5) + \log(x-8)\,\} - 2\left\{\frac{7200}{x^6-98x^4+2401x^2-7200} +\right.$$

$$\left.\frac{1}{3}\left(\frac{7200}{x^6-98x^4+2401x^2-7200}\right)^3 + \&c.\right\}$$

If $x = 100$, the first term of this series has no significant figure among the first 8 places of decimals.

It is unnecessary to explain the uses of many of the preceding formulæ, in abridging the laborious task of calculating tables of logarithms. This object however, is now become one of speculative rather than of practical importance, as tables were calculated of sufficient extent and accuracy for most of the purposes for which they are required, at a period when none of the expedients furnished by analysis, were known.

The series, which have been enumerated, are adapted for the calculation of Napierian logarithms only, and it is of great importance to determine the constant quantity by which they must be multiplied in order to give the logarithms in ordinary use. It appears from Art. 24, that if a be the base of the system of logarithms, this constant multiplier is $\frac{1}{k}$, where

$$k = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c.$$

$$= \log a \text{ to base } \epsilon.$$

This series, however, is not convergent, though a very simple transformation will make it so: since

$$\log \sqrt[m]{a} = \frac{1}{m} \log a, \text{ and therefore } \log a = m \log \sqrt[m]{a}.$$

we have $\log a =$

$$= m \left\{ (\sqrt[m]{a} - 1) - \frac{(\sqrt[m]{a}-1)^2}{2} + \frac{(\sqrt[m]{a}-1)^3}{3} - \frac{(\sqrt[m]{a}-1)^4}{4} + \&c. \right\}$$

$= k$: it is evident that this series, by increasing the value of m, may be made to converge with any required degree of rapidity.

Thus, if $a = 10$, and $m = 2^{50}$, we have

$$\sqrt[m]{10} = 1\,.\,00000{,}00000{,}00002{,}04510{,}63891{,}20519{,}4:$$

and since, in this case, the second term of the series for k, has no significant figure among the first 28 places of decimals, we may assume

$$\log 10 = k = \frac{1}{2^{50}} \left\{ 10^{\frac{1}{2^{50}}} - 1 \right\}.$$

But $\quad \frac{1}{2^{50}} = .00000{,}00000{,}00000{,}88817{,}84197{,}00124{,}2$

and therefore

$$\frac{1}{k} = \frac{.0,88817,84197,00124,2}{.2,04510,63891,20519,4}$$

$$= .434294481903 2518.$$

On Series for the Rectification of the Circle.

Art. 37. The series for the arc (y), in terms of the sine (x), or

$$y = x + \frac{x^3}{1.2.3} + \frac{3^2.x^5}{1.2.3.4.5} + \frac{3^2.5^2.x^7}{1.2.3.4.5.6.7}$$

$$+ \frac{3^2.5^2.7^2.x^9}{1.2 \dots 9} + \&c.,$$

as well as the series for the sine and cosine in terms of the arc itself, were first given by Newton, in a paper entitled "*Analysis per Æquationes numero terminorum infinitas.*" This paper was communicated, by Collins, the Secretary of the Royal Society, in the year 1669, to James Gregory, who in 1671 discovered the series for the arc (y) in terms of the tangent (u)*, or

$$y = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \&c.$$

The first series is convergent for all the values, which x admits of, and very rapidly so, when $x = \frac{1}{2}$, and therefore $y = 30°$: it was thus that Newton calculated the circumference of a circle, whose diameter is unity, to 16 places of decimals*: the coefficients of this series, however, are not of a form the most convenient for calculation, and other series enable us to approximate to this value with much greater rapidity.

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* *Commercium Epistolicum.*

The law of Gregory's series is remarkable for its simplicity, though amongst the values y and u, which have a known relation to each other, there is only one which makes the series sufficiently convergent to be of much practical use in the rectification of the circle. For it would require the calculation of 5000000000 terms of this series, when $u = 1$, to find the length of the arc of 45° which corresponds to it, to 20 places of decimals; an operation which could not be completed by an ordinary calculator in less than 1000 years†. By supposing, however, $y = 30°$, and therefore $u = \frac{1}{\sqrt{(3)}}$, we find

$$\frac{\pi}{2} = \sqrt{3}\left\{1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \&c.\right\};$$

a series which converges with a rapidity so considerable, that the aggregation of 18 terms will give a result which is accurate to 10 places of decimals.

It was by help of this series that the laborious Abraham Sharp calculated the value of π to 74 places of decimals, a labour of appalling extent, and requiring the aggregation of 150 terms of the series and the extraction of $\sqrt{3}$ to 76 decimal places. To extend this approximation to 100 places of decimals, would require the calculation of 210 terms of the series, a work merely within the limits of human industry. This however was effected by Machin, most probably by means of the expedient which is explained by our author. In order to effect this prodigious approximation, it would be necessary to calculate 140 terms of the first of the two series for $\frac{\pi}{2}$ and about 46 of the second: an operation by no means so difficult as that undertaken and executed by Sharp, since the powers of $\frac{1}{5}$ are finite decimals, and no square root is involved.

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* *Epistola posterior Newtoni ad Oldenburgh.* 1676.

† *Ib.*.........

The only notice of the discovery of Machin's formula is contained in Jones's *Synopsis Palmariorum Matheseos*, a book little known, which was published in 1706. It was afterwards noticed and fully explained in an appendix to a "*Dissertation on the Use of the Negative Sign in Algebra*," by Baron Maseres.

Euler, who left no part of Analysis unexplored, in a Paper in the Petersburgh Acts for 1744, on the different modes of approximating to the length of the circumference of a circle, not only gave Machin's formula, but also explained a mode of finding an infinite number of similar formulæ. Thus assuming

$$\tan(a+b) = \frac{1}{e}, \ \tan a = \frac{x}{y}, \ \tan b = \frac{x'}{y'},$$

we have

$$\tan^{-1}\frac{1}{e} = \tan^{-1}\frac{x}{y} + \tan^{-1}\frac{x'}{y'};$$

and since $\tan a = \tan(a+b-b)$, we get,

$$\frac{x}{y} = \frac{\frac{1}{e} - \frac{x'}{y'}}{1 + \frac{x'}{e\,y'}}$$

an indeterminate equation, admitting of an infinite number of solutions.

Thus, if $e = 1$, $\frac{x}{y} = \frac{1}{2}$, and therefore $\frac{x'}{y'} = \frac{1}{3}$, we find

$$\tan^{-1} 1 = \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}.$$

$$= \begin{cases} \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \&c. \\ \frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \&c. \end{cases}$$

By the aggregation of 318 terms of the first series and 200 of the second, the value of $\frac{\pi}{4}$ might be determined to 200 places of decimals.

By a continuation of this process, we shall be enabled to deduce the following general formula,

$$\tan^{-1}\frac{x}{y} = \tan^{-1}\frac{ex-y}{ey+x} + \tan^{-1}\frac{e_1-e}{ee_1+1} + \tan^{-1}\frac{e_2-e_1}{e_1e_2+1}$$

$$+ \tan^{-1}\frac{e_3-e_2}{e_2e_3+1} + \ldots\ldots + \tan^{-1}\frac{e_n-e_{n-1}}{e_{n-1}e_n+1} + \tan^{-1}\cdot\frac{1}{e_n}.$$

By making $\frac{x}{y} = 1$, we may readily verify the following equations:

(1.) $\frac{\pi}{4} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}.$

(2). $\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}.$

(3). $\frac{\pi}{4} = 3\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{2}{11}.$

(4). $\frac{\pi}{4} = 2\tan^{-1}\frac{1}{5} + 2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{7}.$

(5). $\frac{\pi}{4} = 5\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}.$

(6). $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \tan^{-1}\frac{1}{32}$

$+ \tan^{-1}\frac{1}{50} + \tan^{-1}\frac{1}{72} +$ &c. *in infinitum,* where the denominators of the tangents are the doubles of the squares of the natural numbers.

$$(7).\quad \frac{\pi}{4} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{15} + \tan^{-1}\frac{1}{31}$$

$$+ \tan^{-1}\frac{1}{63} + \tan^{-1}\frac{1}{127} + \&\text{c } \textit{in infinitum};$$

the general form of the denominators of the tangents being $2^n - 1$.

$$(8).\quad \frac{\pi}{4} = \tan^{-1}\frac{1}{2} - 2\tan^{-1}\frac{1}{14} + 2\tan^{-1}\frac{1}{2786}$$

$$- \tan^{-1}\frac{1}{10812186007}.$$

$$(9).\quad \frac{\pi}{4} = 8\tan^{-1}\frac{1}{10} - \tan^{-1}\frac{1}{239} - 8\tan^{-1}\frac{1}{1030}$$

$$+ 8\tan^{-1}\frac{1}{533530} - 4\tan^{-1}\frac{1}{7589311509788795}.$$

$$(10).\quad \mathrm{Tan}^{-1}\frac{1}{a} = \tan^{-1}.\frac{b}{a(a+b)+1} +$$

$$\tan^{-1}.\frac{b}{(a+b)(a+2b)+1} + \tan^{-1}.\frac{b}{(a+2b)(a+3b)+1}$$

$$+ \tan^{-1}.\frac{b}{(a+3b)(a+4b)+1} + \&\text{c. } \textit{in infinitum}^{*}.$$

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* Euler, "*De progressionibus arcuum circularium quorum tangentes secundum certam legem procedunt.*" Novi Commentarii Petropol, Tom. IX. 1764.

If we assume a and b in such a manner, that b may be a divisor of a^2+1, this formula will furnish the sums of series of inverse tangents whose numerators are severally equal to unity. These equations and series may be multiplied to any extent, though such of them alone are useful in approximating to the circumference of a circle, as are restricted to a few terms and consist of tangents whose numerators are equal to unity.

The general formula which we have given from Euler is not the best adapted to the discovery of the formulæ which possess these properties ; investigations of this kind more properly belong to the solution of the following problem. Supposing

$$a\tan^{-1}\frac{1}{x}+a_1\tan^{-1}\frac{1}{x_1}+a_2\tan^{-1}\frac{1}{x_2}+\&c.=0,$$

to find the coefficients a, a_1, a_2, a_3 &c., when the denominators x, x_1, x_2, &c. are whole numbers : a solution of this problem, and a more elaborate discussion of the properties of series of inverse tangents will be found in the third and last part of this Work.

Before the discovery of these series, an approximation was made to the value of π, by calculating the lengths of the circumferences of an inscribed and circumscribed regular polygon, of the same number of sides, one being less and the other greater than the circumference of the circle. Thus Archimedes by means of polygons of 96 sides, the circumference of the inscribed polygon being

$$=96\surd\{2-\surd(2+\surd(2+\surd(2+\surd3)\},$$

and of the circumscribed

$$=\frac{192\surd\{2-\surd(2+\surd(2+\surd(2+\surd3)\}}{\surd\{2+\surd(2+\surd(2+\surd(2+\surd3)\}}$$

found $\pi=\frac{22}{7}$, a result which is too great by nearly $\frac{1}{800}$

of the diameter. Peter Metius by a similar process obtained $\frac{355}{113}$ for an approximate value of π, a remarkable result and which is accurate to 5 places of decimals. Vieta extended this approximation, by the same method, to 10 places of decimals, which number was increased to 35, by Ludolph van Ceulen; a labour of vast extent, when the means are considered, and of which he was so proud, that he directed, after the example of Archimedes, that the numbers should be engraved upon his tomb. De Lagny by an unknown process, but most probably by means of Machin's formula, found the value of π to 127 places of decimals, and some notion of the prodigious accuracy of this determination may be formed from the following hypothesis. If the diameter of the Universe be 100000000000 times the distance of the Sun from the Earth, and if a distance which is 100000000000 times this diameter be divided into parts, each of which is the 100000000000th part of an inch; if a circle be described whose diameter is 100000000000 times that distance repeated 100000000000 times as often as each of those parts of an inch is contained in it: then the error in the circumference of this circle as calculated from this approximation, will be less than the 100000000000th part of the 100000000000th part of an inch.

◆

On the Differentiation of Equations of two Variables, and the Elimination of their Constants.

Art. 38.—44. (1). Let the primitive equation be

$$y + ax + b = 0.$$

The differential equation of the first order, arising from the elimination of a, is

$$y - \frac{x\,dy}{dx} + b = 0.$$

The differential equation of the first order, which arises from the elimination of b, is

$$\frac{dy}{dx} + a = 0.$$

Lagrange. Calcul des Fonctions. Lec. 12.

(2). Let the primitive equation be

$$y + ax + c\sqrt{(1+a^2)} = 0.$$

The elimination of a, gives us

$$y - x\frac{dy}{dx} + c\sqrt{\left\{1 + \frac{dy^2}{dx^2}\right\}} = 0.$$

Id. . . . . .

(3). Let the primitive equation be

$$x^2 - 2ay - a^2 - b = 0. \qquad (\alpha).$$

The elimination of a, gives us

$$\frac{dy}{dx}\sqrt{(y^2 + x^2 - b)} - y\frac{dy}{dx} - x = 0. \qquad (\beta).$$

and that of b,

$$x - a\frac{dy}{dx} = 0. \qquad (\gamma).$$

Again, by eliminating b from (β), or a from (γ), we get the differential equation of the second order,

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0. \qquad (\delta).$$

Id. . . . . .

(4). Let the primitive equation be

$$2xy + ay^2 - bx^2 = 0.$$

By the elimination of a, we get

$$(xy - bx^2)p + bxy - y^2 = 0 \qquad (\beta).$$

where $$p = \frac{dy}{dx}.$$

The elimination of b, gives us

$$xy + ay^2 - (x^2 + axy)p = 0. \qquad (\gamma).$$

Again, by eliminating b from (β), or a from (γ), we get

$$x^3p^3 - 3yx^2p^2 + 3xy^2p - y^3 = 0$$

or, $$(xp - y)^3 = 0. \qquad (\delta).$$

The primitive equation being homogeneous, the values of y are mx and $m'x$, where m and m' are constant quantities: it is obvious therefore that $\frac{d^2y}{dx^2} = 0$, and that consequently the original equation (α) admits not of a derivative equation of the second order.

(5). Let the primitive equation be

$$ax^2y + bxy^2 + x^3 + y^3 = 0.$$

By eliminating a, we get

$$(bx^2y^2 + 2xy^3 - x^4)p - bxy^3 + x^3y - 2y^4 = 0. \qquad (\beta).$$

Also, by eliminating b, we get

$$(xy^3 - ayx^3 - 2x^4)p + ax^2y^2 + 2yx^3 - y^4 = 0. \qquad (\gamma).$$

If we eliminate b from (β), or a from (γ), we shall find after proper reductions

$$(xp - y)\{(x^3 + y^3)(xp - y)^2 + 3yx^3p(x - y)\} = 0. \qquad (\delta).$$

This equation is verified by making $xp - y = 0$, in the same manner as in the preceding example.

(6). Let the primitive equation be

$$(x-y)\,\epsilon^{\frac{x}{x-y}}=c \qquad \text{(Art. 255).}$$

The differentiation of this equation leads us to the homogeneous equation,

$$(x-2y)\,dx+y\,dy=0.$$

(7). Let the primitive equation be

$$(x+y)(c+\log x)=x\,\epsilon^{\frac{y}{x}}. \qquad \text{(Note } N\text{).}$$

By eliminating c, we get

$$2y\,dy-y^2\,dx=(x+y)^2\,\epsilon^{-\frac{y}{x}}\,dx.$$

(8). Let the primitive equation be

$$2cy+c^2-x^2=0. \qquad \text{(Art. 255).}$$

The elimination of c, gives us

$$x\,dy-y\,dx=dx\sqrt{(x^2+y^2)}.$$

(9). Let the primitive equation be

$$y=x\,\epsilon^{x\,\epsilon^{c}}. \qquad \text{(Note } N\text{).}$$

By a similar process we get

$$x\,dy-y\,dx=y(\log x-\log y)\,dx.$$

(10). Let the primitive equation be

$$y-c\epsilon^{-x^3}-a=0. \qquad \text{(Art. 257).}$$

The elimination of c, gives us

$$dy+3x^2y\,dx=3ax^2\,dx. \qquad (\beta).$$

Again, by eliminating a from (β), we get

$$x\,d^2y + (3x^3 - 2)\,dy\,dx = 0. \qquad (\gamma).$$

(11). Let the primitive equation be

$$(y - c)^2 - a^2x^2 = 0. \qquad \text{(Art. 268).}$$

By eliminating c, we get

$$dy^2 - a^2\,dx^2 = 0$$

(12). Let the primitive equation be

$$(x - c)^2 + (y - c')^2 - a^2 = 0.$$

By eliminating c, we get

$$(1 + p^2)(y - c')^2 - a^2 = 0. \qquad (\beta).$$

Again, by eliminating c' from (β), we find

$$\frac{(1 + p^2)^3}{q^2} - a^2 = 0; \qquad (\gamma).$$

where $$q = \frac{d^2y}{dx^2}.$$

◆

On the Use of Differentiation in eliminating Irrational and Transcendental Functions from Equations, and also in the Developement of Functions.

ART. 45. (1). Let $u = (a^2 + x^2)^{\frac{m}{n}}$; by differentiating this equation, we get

$$du = \frac{2mux\,dx}{n\{a^2 + x^2\}}:$$

an expression which is free from radical quantities.

(2). Let $u = \log v - \log v' + v''$: in this case,

$$d\,u = \frac{d\,v}{v} - \frac{d\,v'}{v'} + d\,v'' :$$

an equation in which no transcendental quantity appears.

(3). Let $u = \frac{\epsilon^x + \epsilon^{-x}}{\epsilon^x - \epsilon^{-x}}$: this gives us

$$\epsilon^{2x} = \frac{u+1}{u-1},\ 2\,x = \log\left\{\frac{u+1}{u-1}\right\}$$

and therefore

$$d\,x = \frac{d\,u}{1 - u^2}.$$

Euler. *Inst. Calc. Diff.* Pars. I. Art. 294.

(4). Let $u = \log\left\{\frac{\epsilon^x + \epsilon^{-x}}{2}\right\}$; therefore,

$$\frac{d\,u}{d\,x} = \frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}},$$

$$\epsilon^{2x} = \frac{d\,u + d\,x}{d\,u - d\,x},\ \text{and}\ 2\,x = \log\left\{\frac{d\,u + d\,x}{d\,u - d\,x}\right\}$$

from which we get, since $d\,x$ is constant,

$$d\,x^2 = d^2\,u + d\,u^2$$

Id......

(5). Let $u = a \sin^{-1}\frac{x}{a}$

$$d\,u = \frac{a\,d\,x}{\surd\{a^2 - x^2\}}.$$

(6). Let $u = \tan^{-1} x$

$$du = \frac{dx}{1+x^2}.$$

(7). Let $u = \cos \frac{u}{x}$: we hence get

$$x^2 du = (u\,dx - x\,du)\sqrt{\{1 - u^2\}}.$$

Id......

(8). Let $u = m \sin x + n \cos x$:

after two differentiations, we get

$$d^2 u + u\,dx^2 = 0.$$

Id......

(9). Let $u = \epsilon^v \cos v$:

by eliminating ϵ^v, we get

$$du = \{1 + \cot v\}\,dv:$$

and by differentiating again and eliminating the circular functions, we find

$$2\,dv^2 (u\,dv - du) + d^2 v\,(dv - du) = 0.$$

(10). Let $u = \frac{v^n}{v_1^{\ n_1}}$, where n and n_1 are any numbers whatever, rational or irrational.

By the same process as above, we find

$$v v^1 \frac{du}{dx} = u\left(n v_1 \frac{dv}{dx} - n_1 v \frac{dv_1}{dx}\right) \qquad (\beta).$$

This equation (β), is useful in the developement of some rather complicated functions.

Thus, suppose

$$u = \frac{\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. \}^{n}}{\{ \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \&c. \}^{n}{}_{1}} = \frac{v^n}{v^{n}{}_{1}}.$$

Assume

$$u = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$$

It is obvious that $A_0 = \frac{a_0^{\,n}}{\alpha^{n}{}_{1}}$; and by performing the operations indicated in the differential equation (β), we shall get the following equations for the determination of A_1, A_2, A_3, &c.

$$\left.\begin{matrix} a_0 \alpha_0 A_1 + n_1 a_0 \alpha_1 \\ - n a_1 \alpha_0 \end{matrix}\right\} A_0 = 0.$$

$${}_0\,\alpha_0 A_2 + \left.\begin{matrix} (n_1 + 1) a_0 \alpha_1 \\ -(n-1) a_1 \alpha_0 \end{matrix}\right\} A_1 + \left.\begin{matrix} 2 n_1 a_0 \alpha_2 \\ +(n_1 - n) a_1 \alpha_1 \\ - 2 n a_2 \alpha_0 \end{matrix}\right\} A_0 = 0,$$

$$\alpha_0 A_3 + \left.\begin{matrix} (n_1 + 2) a_0 \alpha_1 \\ -(n-2) a_1 \alpha_0 \end{matrix}\right\} A_2 + \left.\begin{matrix} (2n_1 + 1) a_0 \alpha_2 \\ +(n_1 - n + 1) a_1 \alpha_1 \\ -(2n-1) a_2 \alpha_0 \end{matrix}\right\} A_1 + \left.\begin{matrix} 3 n_1 a_0 \alpha_3 \\ +(2 n_1 - n) a_1 \alpha_2 \\ +(n_1 - 2n) a_2 \alpha_1 \\ - 3 n a_3 \alpha_0 \end{matrix}\right\} A_0 = 0, \ \&c.$$

Euler. *Inst. Calc. Diff.* Pars. II. Cap. viii. Art. 207.

(11). Let $u = \epsilon^{(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.)} = \epsilon^{v}$.

In this case, we have

$$\frac{du}{dx} = u \frac{dv}{dx}. \qquad (\beta)$$

Assume $\quad u = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.:$

It is obvious, that $A_0 = \epsilon^{a_0}$; also by performing the operations indicated in equation (β), we find

$$A_1 - a_1 A_0 = 0.$$
$$2 A_2 - a_1 A_1 - 2 a_2 A_0 = 0.$$
$$3 A_3 - a_1 A_2 - 2 a_2 A_1 - 3 a_3 A_0 = 0.$$
$$4 A_4 - a_1 A_3 - 2 a_2 A_2 - 3 a_3 A_1 - 4 a_4 A_0 = 0.$$
$$5 A_5 - a_1 A_4 - 2 a_2 A_3 - 3 a_3 A_2 - 4 a_4 A_1 - 5 a_5 A_0 = 0, \text{ \&c.}$$

(12). Let $u = \{ v + \sqrt{(v^2 - 1)} \}^n$: this gives us

$$\frac{du}{u} = \frac{n\,dv}{\sqrt{(v^2-1)}}, \text{ or}$$

$$(v^2 - 1)\,du^2 = n^2 u^2\,dv^2. \qquad (\beta).$$

By differentiating this equation (β), we shall obtain another equation, in which u and its differential coefficients do not exceed the first degree: or,

$$n^2 u - v\frac{du}{dv} - v^2\frac{d^2u}{dv^2} + \frac{d^2u}{dv^2} = 0. \qquad (\gamma).$$

Assume

$$u = a_0 + a_1 v + a_2 v^2 + a_3 v^3 + \text{\&c.} \qquad (\delta).$$

By making $v=0$, in u, we get $a_0 = \{ \sqrt{(-1)} \}^n$, and by making $v=0$, in $\frac{du}{dx}$, we also get

$$a_1 = \{ n\sqrt{(-1)} \}^{n-1}.$$

If we perform the operations indicated in equation (γ), we shall get

$$a_2 = -\frac{n^2 a_0}{2}$$

$$a_4 = \frac{n^2 (n^2 - 4)\,a_0}{2\,.\,3\,.\,4},$$

$$a_6 = -\frac{n^2(n^2-4)(n^2-16)a_0}{2.3.4.5.6}$$

$$a_8 = \frac{n^2(n^2-4)(n^2-16)(n^2-36)a_0}{2.3.4.5.6.7.8}, \text{ \&c.}$$

and

$$a_3 = \frac{-(n^2-1)a_1}{2.3}$$

$$a_5 = \frac{(n^2-1)(n^2-9)a_1}{2.3.4.5}$$

$$a_7 = -\frac{(n^2-1)(n^2-9)(n^2-25)a_1}{2.3.4.5.6.7}, \text{ \&c.}$$

(13). Let $u = \{v - \sqrt{(v^2-1)}\}^n$: by the same process as in the preceding example, we shall obtain a differential equation of the second order, which is identical with (γ).

Assume $u = a_0^1 + a_1^1 v + a_2^1 v^2 + a_3^1 v^3 + a_4^1 v^4 +$ &c.

In the same manner as before, we shall get

$$a_0^1 = \{-\sqrt{(-1)}\}^n, \quad \text{and} \quad a_1^1 = n\{-\sqrt{(-1)}\}^{n-1}$$

$$a_2^1 = \frac{-n^2}{2} . a_0^1 \qquad a_3^1 = \frac{-(n^2-1)}{2.3} . a_1^1$$

$$a_4^1 = \frac{n^2(n^2-4)}{2.3.4} . a_0^1 \qquad a_5^1 = \frac{(n^2-1)(n^2-9)}{2.3.4.5} . a_1^1,$$

&c. &c.

The two last examples enable us to investigate the series for cos nx in terms of cos x. Thus, suppose $v = \cos x$, we have, by Demoivre's formula

$$\cos nx = \{v + \sqrt{(v^2-1)}\}^n + \}v - \sqrt{(v^2-1)}\}^n$$

$$= \{ a_0 + a_0{}^1 \} \left\{ 1 - \frac{n^2}{2} v^2 + \frac{n^2 (n^2-4)}{2 \,.\, 3 \,.\, 4} v^4 - \right.$$

$$\left. \frac{n^2 (n^2-4)(n^2-16)}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} v^6 + \right\} + \{ a_1 + a_1{}^1 \} \left\{ v - \frac{(n^2-1)}{2 \,.\, 3} v^3 + \right.$$

$$\left. \frac{(n^2-1)(n^2-9)}{2 \,.\, 3 \,.\, 4 \,.\, 5} v^5 - \&c. \right\}$$

Now $a_0 + a_0{}^1 = \{ \sqrt{(-1)} \}^n + \{ -\sqrt{(-1)} \}^n =$

$$\{ \sqrt{(-1)} \}^n \{ 1 + (-1)^n \}, \text{ and}$$

$$a_1 + a_1{}^1 = n \{ \sqrt{(-1)} \}^{n-1} \{ 1 + (-1)^{n-1} \}.$$

If n be an odd number, $a_0 + a_0{}^1 = 0$.

If n be an even number, of the form $4m$, $a_0 + a_0{}^1 = 2$.

If n be an even number, of the form $4m+2$, $a_0 + a_0{}^1 = -2$.

If n be an odd number, of the form $4m+1$, $a_1 + a_1{}^1 = 2n$.

If n be an odd number, of the form $4m+3$, $a_1 + a_1{}^1 = -2n$.

If n be an even number, $a_1 + a_1{}^1 = 0$.

There are, consequently, six different series for $\cos nx$ in terms of $\cos x$, corresponding to different values of n: it is possible, however, to include them all under one general formula: thus, since

$$\{ \cos x \pm \sqrt{(-1)} \sin x \}^n = \cos nx \pm \sqrt{(-1)} \sin nx,$$

by making $x = \frac{\pi}{2}$, we get $\cos x = 0$, $\sin x = 1$,

and therefore

$$\{ \pm \sqrt{(-1)} \}^n = \cos \frac{n\pi}{2} \pm \sqrt{(-1)} \sin \frac{n\pi}{2}.$$

In the same manner,

$$\{ \pm \sqrt{(-1)} \}^{n-1} = \cos \left(\frac{n-1}{2} \right) \pi \pm \sqrt{(-1)} \sin \left(\frac{n-1}{2} \right) \pi.$$

consequently,

$$a_0 + a_0{}^1 = 2 \cos \frac{n\pi}{2}, \text{ and } a_1 + a_1{}^1 = 2n \cos \left(\frac{n-1}{2}\right) \pi :$$

and generally,

$$\cos nx = \cos \frac{n\pi}{2} \left\{ 1 - \frac{n^2 v^2}{2} + \frac{n^2 (n^2-4)}{2 \,.\, 3 \,.\, 4} v^4 - \&c. \right\}$$

$$+ n \cos \left(\frac{n-1}{2}\right) \pi \left\{ v - \frac{(n^2-1)}{2 \,.\, 3} v^3 + \right.$$

$$\left. \frac{(n^2-1)(n^2-9)}{2 \,.\, 3 \,.\, 4 \,.\, 5} v^5 - \&c. \right\}$$

By differentiating this result, we get

$$\sin nx = n \cos \frac{n\pi}{2} \sin x \left\{ \cos x - \frac{(n^2-4)}{2 \,.\, 3} (\cos x)^3 \right.$$

$$\left. + \frac{(n^2-4)(n^2-16)}{2 \,.\, 3 \,.\, 4 \,.\, 5} (\cos x)^5 - \&c. \right\}$$

$$+ \cos \frac{(n-1)\pi}{2} \sin x \left\{ 1 - \frac{(n^2-1)}{2} (\cos x)^2 + \right.$$

$$\frac{(n^2-1)(n^2-9)}{2 \,.\, 3 \,.\, 4} (\cos x)^4 - \frac{(n^2-1)(n^2-9)(n^2-25)}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} (\cos x)^6$$

$$\left. + \&c. \right\}$$

These formulæ readily furnish us with expressions for sin nx and cos nx, in terms of ascending powers of sin x: Thus

$$\cos nx = \left(\cos \frac{n\pi}{2}\right)^2 \left\{ 1 - \frac{n^2}{2} (\sin x)^2 + \right.$$

$$\left. \frac{n^2 (n^2-4)}{2 \,.\, 3 \,.\, 4} (\sin x)^4 - \&c. \right\}$$

$$+ n \left(\cos \frac{n-1}{2} . \pi\right)^2 \sin x \left\{ 1 - \frac{(n^2-1)}{2} (\sin x)^2 + \right.$$

$$\left. \frac{(n^2-1)(n^2-9)}{2 . 3 . 4} (\sin x)^4 - \&c. \right\}$$

and

$$\sin n x = n \left(\cos \frac{n \pi}{2}\right)^2 \cos x \left\{ \sin x - \frac{(n^2-4)}{2 . 3} (\sin x)^3 \right.$$

$$\left. + \frac{(n^2-4)(n^2-16)}{2 . 3 . 4 . 5} (\sin x)^5 - \&c. \right\}$$

$$+ \left(\cos \frac{(n-1) \pi}{2}\right)^2 \sin x \left\{ 1 - \frac{(n^2-1)}{2} (\sin x)^2 + \right.$$

$$\left. \frac{(n^2-1)(n^2-9)}{2 . 3 . 4} (\sin x)^4 - \&c. \right\}$$

The same differential equation (γ), may be applied to the developement of $\{ v + \surd(v^2-1) \}^n$, in a series of descending powers of v: thus assuming

$$u = a_n v^n + a_{n-1} v^{n-1} + a_{n-2} v^{n-2} + \&c.$$

and performing the requisite operations, we shall find

$$a_{n-2} = - \frac{n a_n}{2^2}$$

$$a_{n-4} = \frac{n (n-3) a_n}{2^4 . 1 . 2}$$

$$a_{n-6} = - \frac{n (n-4)(n-5) a_n}{2^6 . 1 . 2 . 3} \quad \&c. \&c.$$

the coefficients a_{n-1}, a_{n-3}, a_{n-5}, &c. being severally equal to nothing:

It yet remains to determine a_n: since

$$\{ v + \surd(v^2-1) \}^n = a_n v^n + a_{n-2} v^{n-2} + a_{n-4} v^{n-4} + \&c.,$$

we have, by dividing by v^n,

$$\left\{1+\sqrt{\left(1-\frac{1}{v^2}\right)}\right\}^n = a_n + a_{n-2}\frac{1}{v^2} + a_{n-4}\frac{1}{v^4} + \&c.:$$

making $\frac{1}{v} = 0$, we get

$$a_n = \{1+1\}^n = 2^n,$$

and the series becomes

$$\{v+\sqrt{(v^2-1)}\}^n = (2v)^n - n(2v)^{n-2} +$$

$$\frac{n(n-3)}{1.2}(2v)^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}(2v)^{n-6} + \&c.$$

Again, since

$$\{v-\sqrt{(v^2-1}\} = \frac{1}{\{v+\sqrt{(v^2-1)}\}},$$

we have

$$\{v-\sqrt{(v^2-1)}\}^n = \{v+\sqrt{(v^2-1)}\}^{-n}$$

$$= (2v)^{-n} + n(2v)^{-n-2} + \frac{n(n+3)}{1.2}(2v)^{-n-4}$$

$$+ \frac{n(n+4)(n+5)}{1.2.3}(2v)^{-n-6} + \&c.:$$

consequently, if $v = \cos x$, we have

$$2\cos nx = (2v)^n - n(2v)^{n-2} + \frac{n(n-3)}{1.2}(2v)^{n-4} - \&c.$$

$$+ \left\{(2v)^{-n} + n(2v)^{-n-2} + \frac{n(n+3)}{1.2}(2v)^{-n-4} + \&c.\right\}$$

If n be a whole number, all the terms of the first of these series disappear, which are between the $\left(\frac{n}{2}+1\right)^{\text{th}}$ and the $(n+1)^{\text{th}}$, when n is an even number, and between $\left(\frac{n+1}{2}\right)^{\text{th}}$

and the $(n+1)^{\text{th}}$, when n is an odd number: the remaining terms are severally equal to the terms of the second series, but with contrary signs, and must therefore mutually destroy each other. The series for $\cos nx$ will in this case consist of the positive powers of v only.

By differentiating the series for $\cos nx$, we get

$$\sin nx = \sin x \Big\{ (2v)^{n-1} - (n-2)(2v)^{n-3} + \frac{(n-3)(n-4)}{1 \,.\, 2}(2v)^{n-5} - \frac{(n-4)(n-5)(n-6)}{1 \,.\, 2 \,.\, 3}(2v)^{n-7} + \&c. \Big\}$$

This series is also finite and restricted to the positive powers of v, when n is a whole number.

The two last series express $\cos nx$ and $\sin nx$ in terms of descending powers of $\cos x$: the corresponding series in terms of the descending powers of $\sin x$ are

$$\cos nx = \frac{1}{2}\cos\frac{n\pi}{2}\Big\{(2\sin x)^n - n(2\sin x)^{n-2} + \frac{n(n-3)}{1 \,.\, 2}(2\sin x)^{n-4} - \&c.\Big\}$$
$$+ \sin\frac{n\pi}{2}\cos x\Big\{(2\sin x)^{n-1} - (n-2)(2\sin x)^{n-3} + \frac{(n-3)(n-4)}{1 \,.\, 2}(2\sin x)^{n-4} - \&c.\Big\}$$

and

$$\sin nx = \frac{1}{2}\sin\frac{n\pi}{2}\Big\{(2\sin x)^n - n(2\sin x)^{n-2} + \frac{n(n-3)}{1 \,.\, 2}(2\sin x)^{n-4} - \&c.\Big\}$$

$$+ \cos \frac{n\pi}{2} \cos x \left\{ (2 \sin x)^{n-1} - (n-2)\,(2 \sin x)^{n-3} \right.$$

$$\left. + \frac{(n-3)\,(n-4)}{1\,.\,2} (2 \sin x)^{n-5} - \&c. \right\}$$

The two last formulæ are restricted to positive and integral values of n.

The reader will find a complete analysis of these series in *Leçons* 10 and 11 of the *Calcul. des Fonctions* of Lagrange.

(15). Let $u = \epsilon^{\sin^{-1} x}$: the differential equation of the second order which arises from this, is,

$$(1 - x^2) \frac{d^2 u}{dx^2} - x \frac{du}{dx} - u = 0. \qquad (\gamma).$$

From which we shall find, by a process similar to that employed in Examples 13 and 14,

$$u = 1 + \frac{1}{1\,.\,2} \,.\, x^2 + \frac{1\,.\,5}{1\,.\,2\,.\,3\,.\,4} \,.\, x^4 + \frac{1\,.\,5\,.\,17}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} \,.\, x^6$$

$$+ \frac{1\,.\,5\,.\,17\,.\,37}{1\,.\,2\,..\,8} \,.\, x^8 + \&c.$$

In the same manner, if we suppose $u = a^{\sin^{-1} x}$, we shall find

$$u = 1 + \frac{(\log a)^2}{1\,.\,2} x^2 + \frac{(\log a)^2 \ \{(\log a)^2 + 4)\}}{1\,.\,2\,.\,3\,.\,4} x^4$$

$$+ \frac{(\log a)^2 \{(\log a)^2 + 4\} \ \{(\log a)^2 + 16\}}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} x^6 + \&c.$$

Cousin. *Calcul. Diff.* page 131.

Ex. (15). Let $u = (\cos x)^n$: this gives us

$$n u \sin x + \cos x \frac{du}{dx} = 0. \qquad (\beta).$$

Assume,

$$u = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \&c.:$$

if we perform the operations indicated in equation (β), and transform the products of sines and cosines, by means of the formula

$$\sin px \cos qx = \frac{1}{2} \sin (p+q)x + \frac{1}{2} \sin (p-q)x,$$

we shall find

$$(\cos x)^n = a_0 \left\{ 1 + \frac{n}{\frac{n}{2}+1} \cos 2x + \frac{n\left(\frac{n}{2}-1\right)}{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)} \cos 4x \right.$$

$$\left. + \frac{n\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)}{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)\left(\frac{n}{2}+3\right)} \cos 6x + \&c. \right\}$$

$$+ a_1 \left\{ \cos x + \frac{\left(\frac{n-1}{2}\right)}{\left(\frac{n+3}{2}\right)} \cos 3x + \right.$$

$$\left. \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)}{\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} \cos 5x + \&c. \right\}$$

If $x = 0$, we have

$$1 = a_0 \left\{ 1 + \frac{n}{\frac{n}{2}+1} + \frac{n\left(\frac{n}{2}-1\right)}{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)} + \&c. \right\}$$

$$+ a_1 \left\{ 1 + \frac{\left(\frac{n-1}{2}\right)}{\left(\frac{n+3}{2}\right)} + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)}{\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} + \&c. \right\}$$

$$= \frac{1 . 2 . 3 \dots . \frac{n}{2} . 2^n}{n(n-1) \dots . \left(\frac{n}{2}+1\right)} . a_0 + \frac{1 . 2 \dots . \left(\frac{n-1}{2}\right) 2^{n-1}}{n(n-1 \dots . \left(\frac{n+1}{2}\right)} . a_1$$

$$= A_0 a_0 + A_1 a_1.$$

Also, if $x = \pi$, we have

$$(-1)^n = \cos n\pi = a_0 \left\{ 1 + \frac{n}{\left(\frac{n}{2}+1\right)} + \frac{n\left(\frac{n}{2}-1\right)}{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)} + \&c. \right\} - a_1 \left\{ 1 + \frac{\left(\frac{n-1}{2}\right)}{\left(\frac{n+3}{2}\right)} + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)}{\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} + \&c. \right\}$$

$$= A_0 a_0 - A_1 a_1.$$

therefore $a_0 = \frac{(1 + \cos n\pi)}{2 A_0}$ and $a_1 = \frac{(1 - \cos n\pi)}{2 A_1}$

and $(\cos x)^n = \frac{n(n-1) \dots . \left(\frac{n}{2}+1\right)}{1 . 2 . 3 \dots . \frac{n}{2} . 2^{n+1}} \{ 1 + \cos n\pi \}$

$$\left\{ 1 + \frac{n}{\frac{n}{2} + 1} \cos 2x + \frac{n\left(\frac{n}{2} - 1\right)}{\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} + 2\right)} \cos 4x + \&c. \right\}$$

$$+ \frac{n(n-1) \ldots . \left(\frac{n+1}{2}\right)}{1 . 2 \ldots . \left(\frac{n-1}{2}\right) . 2^{n}} \left\{ 1 - \cos n\pi \right\}$$

$$\left\{ \cos x + \frac{\left(\frac{n-1}{2}\right)}{\left(\frac{n+3}{2}\right)} \cos 3x + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)}{\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} \right.$$

$$\left. \cos 5x + \&c. \right\}$$

This formula likewise gives us

$$(\sin x)^{n} = \frac{n(n-1) \ldots . \left(\frac{n}{2} - 1\right)}{1 . 2 . 3 \ldots\ldots \frac{n}{2} . 2^{n+1}} . \left\{ 1 + \cos n\pi \right\}$$

$$\left\{ 1 - \frac{n}{\frac{n}{2} + 1} \cos 2x + \frac{n\left(\frac{n}{2} - 1\right)}{\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} + 3\right)} \cos 4x - \&c. \right\}$$

$$+ \frac{n(n-1) \ldots . \left(\frac{n+1}{2}\right)}{1 . 2 \ldots . \left(\frac{n-1}{2}\right) . 2^{n}} \left\{ 1 - \cos n\pi \right\}$$

$$\left\{ \sin x - \frac{\left(\frac{n-1}{2}\right)}{\left(\frac{n+3}{2}\right)} \sin 3x + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)}{\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} \sin 5x - \&c. \right\}$$

Ex. (16). Let $u = (\cos x)^n \ (\sin x)^m$; to develope this expression in a series involving sines and cosines of x and its multiples.

Let $\theta = \cos x + \surd(-1) \sin x$, and therefore $\frac{1}{\theta} = \cos x - \surd(-1) \sin x$: consequently

$$u = \frac{1}{2^{m+n} \{\surd(-1)\}^m} \left\{\theta + \frac{1}{\theta}\right\}^n \left\{\theta - \frac{1}{\theta}\right\}^m$$

$$= \frac{1}{\{\surd(-1)\}^m} \left(\frac{\theta}{2}\right)^{m+n} \left\{1 + \frac{1}{\theta^2}\right\}^n \left\{1 - \frac{1}{\theta^2}\right\}^m.$$

Assume $z^2 = \frac{1}{\theta^2}$ and $u^1 = (1+z)^n (1-z)^m$: from which equation we deduce,

$$(1-z^2) \frac{du^1}{dz} = u^1 \{ n - m - (n+m) z \}$$

$$= u^1 (a - b z), \text{ if } a = n - m \text{ and } b = n + m.$$

If we apply this differential equation to the determination of the coefficients of the series

$$u^1 = 1 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + \&c.$$

we shall find

$$A_1 = a$$
$$2A_2 = aA_1 - b$$
$$3A_3 = aA_2 - (b-1) A_1$$
$$4A_4 = aA_3 - (b-2) A_2, \ \&c.$$

Also, it is evident from the form of the function u, that if we substitute in it $\frac{1}{\theta}$ in the place of θ, the resulting function will be $\pm u$, according as m is an even or an odd number.

In the first case, therefore

$$2u = \frac{1}{\{\sqrt{(-1)}\}^{m}\, 2^{b}}\Big\{\theta^{b} + \frac{1}{\theta^{b}} + A_1\left(\theta^{b-2} + \frac{1}{\theta^{b-2}}\right)$$

$$+ A_2\left(\theta^{b-4} + \frac{1}{\theta^{b-4}}\right) + \&c.\Big\}$$

$$\text{or,}\quad u = \frac{\cos\frac{m\pi}{2}}{2^{m+n}}\ \{\cos(m+n)\,x + A_1\cos(m+n-2)\,x$$

$$+ A_2\cos(m+n-4)\,x + \&c.\}$$

In the second case,

$$2u = \frac{1}{\{\sqrt{(-1)}\}^{m+1}\, 2^{b}}\Big\{\sqrt{(-1)}\left(\theta^{b} - \frac{1}{\theta^{b}}\right) +$$

$$A_1\sqrt{(-1)}\left(\theta^{b-2} - \frac{1}{\theta^{b-2}}\right) +$$

$$A_2\sqrt{(-1)}\left(\theta^{b-4} - \frac{1}{\theta^{b-4}}\right) + \&c.\Big\}$$

$$\text{or,}\quad u = \frac{\cos\frac{m+1}{2}\,.\,\pi}{2^{m+n}}\ \{\sin(m+n)\,x + A_1\sin(m+n-2)\,x$$

$$+ A_2\sin(m+n-4)\,x + \&c.\}$$

Thus, if $u = \tan x = \sin x\,(\cos x)^{-1}$: we have

$$a = 2,\ b = 0,\ A_1 = -2,\ A_2 = 2,\ A_3 = -2,\ \&c.$$

consequently,

$$\tan x = 2\{\sin 2x - \sin 4x + \sin 6x - \&c.\}$$

In the same manner, we shall find

$$\cot x = 2\{\sin 2x + \sin 4x + \sin 6x + \&c.\}$$

Euler, *Subsidium Calculi Sinuum, Nov. Comment. Petropol.* 1760.

Ex. (17). Let $u = \epsilon^{\cos x}$: this gives us

$$\frac{du}{dx} + u \sin x = 0. \qquad (\beta)$$

If we assume

$$u = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \&c.$$

we shall find, by means of equation (β),

$$a_2 = 2a_1 - 2a_0$$
$$a_3 = 4a_2 - a_1$$
$$a_4 = 6a_3 - a_2$$
$$a_5 = 8a_4 - a_3$$
$$a_6 = 10a_5 - a_4, \&c.$$

The coefficients a_0 and a_1 require a separate determination: we shall find

$$a_0 = 1 + \frac{1}{2^2} \cdot \frac{1}{1^2} + \frac{1}{2^4} \cdot \frac{1}{1^2 \cdot 2^2} + \frac{1}{2^6} \cdot \frac{1}{1^2 \cdot 2^2 \cdot 3^2} + \&c.$$

$$= 1.266065878680.$$

and,

$$a_1 = 1 + \frac{1}{2^2 \cdot 1^2 \cdot 2} + \frac{1}{2^4 \cdot 1^2 \cdot 2^2 \cdot 3} + \frac{1}{2^6} \cdot \frac{1}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} + \&c.$$

$$= 1.130318208170, \text{ very nearly.}$$

Ex. (18). Let $u = \epsilon^{\sin x}$: we shall find as before

$$u = a_0 + a_1 \sin x - a_2 \cos 2x - a_3 \sin 3x + a_4 \cos 4x$$
$$+ a_5 \sin 5x - a_6 \cos 6x - a_7 \sin 7x + \&c.$$

the coefficients being the same as in the series in the preceding example.

An example of the application of this method to the developement of a series of great importance in Physical Astronomy is given in Note *K.*

Lagrange's Theorem. (*Note* E.)

Ex. (1). Let $1-x+ax=0$: to develope ψx in terms of a, when ψx is any given function of the value of x in this equation.

If we compare this equation with $z-y+\phi(y)=0$, we find $z=1$, $y=x$ and $\phi(y)=ax$: by means of Lagrange's series, or formula (2), page 635, we shall find

(*a*). $\psi x = x^n = \dfrac{1}{(1-a)^n}$

$$= 1 + na + \frac{n(n+1)}{1.2}a^2 + \frac{n(n+1)(n+2)}{1.2.3}a^3 + \&c.$$

(*b*). $\psi x = x^{-n} = (1-a)^n$

$$= 1 - na + \frac{n(n-1)}{1.2}a^2 - \frac{n(n-1)(n-2)}{1.2.3}a^3 + \&c.$$

(*c*). $\psi x = \log x = -\log(1-a)$

$$= a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \&c.$$

(*d*). $\psi x = \epsilon^{1-\frac{1}{x}} = \epsilon^a$

$$= 1 + a + \frac{a^2}{1.2} + \frac{a^3}{1.2.3} + \frac{a^4}{1.2.3.4} + \&c.$$

(*e*). $$\psi x = \sin\left(1 - \frac{1}{x}\right) = \sin a$$

$$= a - \frac{a^3}{1 \cdot 2 \cdot 3} + \frac{a^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

(*f*). $$\psi x = \log(2x - 1) = \log\left(\frac{1+a}{1-a}\right)$$

$$= 2\left\{a + \frac{a^3}{3} + \frac{a^5}{5} + \frac{a^7}{7} + \&c.\right\}$$

and so on for any other form of ψx.

(2). Let $1 - x + \epsilon^x = 0$: the same formula gives us

(*a*). $$\psi x = x^n = 1 + n\epsilon + \frac{n(n+1)}{1 \cdot 2}\epsilon^2 + \frac{n(n^2 + 3n + 5)}{1 \cdot 2 \cdot 3}\epsilon^3$$

$$+ \frac{n(n^3 + 5n^2 + 28n + 44)}{1 \cdot 2 \cdot 3 \cdot 4}\epsilon^4 + \&c.$$

(*b*). $$\psi x = \epsilon^x = \epsilon\left\{1 + \frac{2\epsilon}{1 \cdot 2} + \frac{3^2\epsilon^2}{1 \cdot 2 \cdot 3} + \frac{4^3\epsilon^3}{1 \cdot 2 \cdot 3 \cdot 4}\right.$$

$$\left. + \frac{5^4\epsilon^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{6^5\epsilon^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c.\right\}$$

(3). Let $x^2 - 2x + 1 = 0$: from hence we deduce,

(*a*). $$\psi x = x^n = 1$$

$$= \frac{1}{2^n}\left\{1 + \frac{n}{1 \cdot 2^2} + \frac{n(n+3)}{1 \cdot 2 \cdot 2^4} + \frac{n(n+4)(n+5)}{1 \cdot 2 \cdot 3 \cdot 2^6}\right.$$

$$\left. + \frac{n(n+5)(n+6)(n+7)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^9} + \&c.\right\}$$

(*b*). $\psi x = x^{-n} = 1$

$$= 2^n \left\{ 1 - \frac{n}{1 \cdot 2^2} + \frac{n(n-3)}{1 \cdot 2 \cdot 2^4} - \frac{n(n-4)(n-5)}{1 \cdot 2 \cdot 3 \cdot 2^6} + \&c. \right\}$$

If this series be restricted $\frac{n}{2} + 1$, or $\frac{n+1}{2}$ terms, according as n is an even or an odd number, we shall have, by formula (4),

$$x^{-n} = 1^{-n} + 1^{-n} = 2 = \frac{1}{2^n} \left\{ 1 - \frac{n}{1 \cdot 2^2} + \frac{n(n-3)}{1 \cdot 2 \cdot 2^4} - \&c. \right\}$$

(*c*). $\psi x = \log x = \log 1 = 0$

$$= \log \frac{1}{2} + \frac{1}{2^2} + \frac{3}{1 \cdot 2 \cdot 2^4} + \frac{4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 2^6} + \&c.$$

or, $$\log 2 = \frac{1}{2^2} \left\{ 1 + \frac{3}{1 \cdot 2 \cdot 2^2} + \frac{4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 2^4} + \&c. \right\}$$

(*d*). $\psi x = \epsilon^x = \epsilon$

$$= \epsilon^{\frac{1}{2}} \left\{ 1 + \frac{1}{2^3} + \frac{9}{1 \cdot 2 \cdot 2^6} + \frac{145}{1 \cdot 2 \cdot 3 \cdot 2^9} + \frac{4241}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^{12}} + \&c. \right\}$$

or, $$\epsilon^{\frac{1}{2}} = 1 + \frac{1}{2^3} + \frac{9}{1 \cdot 2 \cdot 2^6} + \frac{145}{1 \cdot 2 \cdot 3 \cdot 2^9} + \&c.$$

(4). Let $x^2 - 5x + 6 = 0$: consequently

$$\psi x = x^n = \left(\frac{6}{5}\right)^n \left\{ 1 + n \cdot \frac{6}{5^2} + \frac{n(n+3)}{1 \cdot 2} \cdot \frac{6^2}{5^4} + \frac{n(n+4)(n+5)}{1 \cdot 2 \cdot 3} \cdot \frac{6^3}{5^6} + \&c. \right\}$$

$= 2^n$, since 2 is the least root of the equation.

(5). Let $a - bx + cx^2 = 0$: the same formula gives us

$$\psi x = x = \frac{a}{b} \left\{ 1 + \frac{ac}{b^2} + \frac{4}{1 \cdot 2} \cdot \frac{a^2 c^2}{b^4} + \frac{5 \cdot 6}{1 \cdot 2 \cdot 3} \cdot \frac{a^3 c^3}{b^6} + \&c. \right\}$$

a series which is identical with that arising from the developement of $\frac{b}{2c} - \frac{\sqrt{(b^2 - 4ac)}}{2c}$, which is the least of the two roots of the given equation.

We may deduce a series which will represent the other value which x admits of, by modifying the form of the equation: thus, if we put it under the form

$$\frac{b}{c} - x - \frac{a}{cx} = 0,$$

and make $z = \frac{b}{c}$ and $\phi(y) = -\frac{a}{cx}$, we shall find

$$\psi x = x = \frac{b}{c} - \frac{a}{b} \left\{ 1 + \frac{ac}{b^2} + \frac{4}{1 \cdot 2} \cdot \frac{a^2 c^2}{b^4} + \frac{5 \cdot 6}{1 \cdot 2 \cdot 3} \cdot \frac{a^3 c^3}{b^6} + \&c. \right\},$$

which is the developement of the second or greatest root of the equation.

It is possible to exhibit the values of both these roots in one series, involving the ascending powers of a quantity admitting of a positive and negative sign: in order to effect this, assume $u = x^2$, which reduces the given equation to

$$a + c u - b \sqrt{(u)} = 0 :$$

consequently, the general formula gives us, if $\delta = \pm \sqrt{\left(\frac{-a}{c}\right)}$, $\psi u = u^{\frac{1}{2}} = x$

$$= \delta - \delta^2 \left(\frac{b}{2a}\right) + \frac{1}{1.2}\, \delta^3 \left(\frac{b}{2a}\right)^2 - \frac{1.1.3}{1.2.3.4}\, \delta^5 \left(\frac{b}{2a}\right)^4$$

$$+ \frac{1.1.3.3.5}{1.2.3.4.5.6}\, \delta^7 \left(\frac{b}{2a}\right)^6 - \&c.$$

$$= - \delta^2 \left(\frac{b}{2a}\right) + \delta \left\{ 1 + \frac{1}{1.2}\, \delta^2 \left(\frac{b}{2a}\right)^2 - \frac{1.1.3}{1.2.3.4}\, \delta^4 \left(\frac{b}{2a}\right)^4 \right.$$

$$\left. + \&c. \right\}$$

$$= \frac{b}{2c} \pm \sqrt{\left(\frac{-a}{c}\right)} \left\{ 1 - \frac{1}{1.2} \cdot \frac{b^2}{2^2 . a c} - \frac{1.1.3}{1.2.3.4} \cdot \frac{b^4}{2^4 . a^2 c^2} \right.$$

$$\left. - \frac{1.1.3.3.5}{1.2.3.4.5.6} \cdot \frac{b^6}{2^6 . a^3 c^3} - \&c. \right\}$$

a series which represents the two roots of the equation and which may likewise be deduced from the developement of $\frac{b}{2c} \pm \frac{\sqrt{(b^2 - 4ac)}}{2c}$.

It ought to be remarked, that though there always exists an *algebraical* equality between the roots of the equation and each of the values of this series, yet these quantities can only, in some cases, be considered as *arithmetically* equal: for if a and c have the same sign, the two values of the series assume an imaginary form, though the roots themselves may be possible.

The converse of this, however, sometimes takes place in series deduced by the first method, or imaginary roots are represented by series all whose terms are real: thus supposing $x^2 - x + 1 = 0$, the formula gives us

$$\psi x = x = 1 + 1 + \frac{1 . 4}{1 . 2} + \frac{1 . 5 . 6}{1 . 2 . 3} +$$

$$\frac{1 . 6 . 7 . 8}{1 . 2 . 3 . 4} + \&c. = \frac{1 - \surd(1-4)}{2}:$$

it ought however to be observed, that this imaginary expression gives the same series with real terms, by the ordinary process of developement. The reader will find a complete discussion of this curious subject, in a paper by Lagrange in the "*Mémoires de l'Acad. de Berlin*" for 1770.

(6). Let $x^3 - p\,x + q = 0$: in this case $z = \frac{q}{p}$ and $\phi\,(y) = \frac{x^3}{p}$: consequently

$$\psi x = x = \frac{q}{p} \left\{ 1 + \frac{q^2}{p^3} + \frac{1 . 6}{1 . 2} . \frac{q^4}{p^6} + \frac{1 . 8 . 9}{1 . 2 . 3} . \frac{q^6}{p^9} \right.$$

$$\left. + \&c. \right\}$$

a series which represents the least of the three roots of the equation.

Again, assuming $u = x^3$, the equation becomes $u - p\,u^{\frac{1}{3}} + q = 0$, and the same formula gives us

$$\psi u = u^{\frac{1}{3}} = x$$

$$= (-q)^{\frac{1}{3}} \left\{ 1 + \frac{1}{3} . \frac{p}{(-q)^{\frac{2}{3}}} - \frac{2 . 1}{3^3 . 1 . 2 . 3} . \frac{p^3}{(-q)^2} \right.$$

$$+\frac{2\,.\,1\,.\,4}{3^4\,.\,1\,.\,2\,.\,3\,.\,4}\cdot\frac{p^4}{(-q)^{\frac{8}{3}}}-\frac{4\,.\,1\,.\,2\,.\,5\,.\,8}{3^6\,.\,1\,.\,2\,\ldots\,6}\cdot\frac{p^6}{(-q)^4}$$

$$+\&c.\Big\}$$

and since $(-q)^{\frac{1}{3}}$, admits of the three values, $-q^{\frac{1}{3}}$, $\left(\frac{1+\sqrt{(-3)}}{2}\right)q^{\frac{1}{3}}$ and $\left(\frac{1-\sqrt{(-3)}}{2}\right)q^{\frac{1}{3}}$, this series will furnish the developement of the three roots of the equation.

(7). Let $a-b\,x+c\,x^n=0$: from which we deduce

$$\psi\,x=x=\frac{a}{b}\Big\{1+\frac{c\,a^{n-1}}{b^n}+\frac{2\,.\,c^2\,a^{2n-2}}{1\,.\,2\,.\,b^{2n}}+\frac{2\,.\,3\,.\,c^3\,a^{3n-3}}{1\,.\,2\,.\,3\,.\,b^{3n}}$$

$$+\&c.\Big\}$$

Again, assuming $x^n=u$, the equation becomes

$$a-b\,u^{\frac{1}{n}}+c\,u=0;$$

from which we get, if $\delta=\left(\frac{a}{-c}\right)^{\frac{1}{n}}$,

$$\psi\,u=u^{\frac{1}{n}}=x=\delta\Big\{1-\frac{b}{n\,a}\,.\,\delta+$$

$$\frac{(2-n)}{1\,.\,2\,.\,n^2}\cdot\frac{b^2\,\delta^2}{a^2}-\frac{(3-n)\,(4-n)}{1\,.\,2\,.\,3\,.\,n^3}\cdot\frac{b^3\,\delta^3}{a^3}+\&c.\Big\}$$

But δ being equal to $\left(\cos\frac{2\,\lambda\,\pi}{n}+\sqrt{(-1)}\sin\frac{2\,\lambda\,\pi}{n}\right)$ $\sqrt[n]{\left(\frac{a}{-c}\right)}$, when we put for λ, the numbers $1, 2, 3 \,..\, n$, we shall have n different values of the series, corresponding to the n roots of the equation.

It is evident that the same method is applicable to the investigation of a series to represent the values of all the roots of any equation whatever: thus if the equation be

$$a_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots . b_n x^n = 0,$$

by assuming $u = x^n$, it is transformed into

$$a + b_1 u^{\frac{1}{n}} + b_2 u^{\frac{2}{n}} + b_3 u^{\frac{3}{n}} + \ldots \ldots . b_n u = 0.$$

If we assume $\delta = \left(\frac{a}{-b_n}\right)^{\frac{1}{n}}$, as before, and investigate the series for $u^{\frac{1}{n}}$ or x, the resulting expression will evidently admit of n values corresponding to the n roots of the given equation.

Lagrange, in the Memoir just cited, has explained a method of ascertaining the degree of convergency or divergency of these series for the roots of an equation: but the discussion of this subject is foreign to our present purpose, and the reader must be contented with this reference to sources where more complete information may be obtained.

(8). Let $x = \frac{e}{1 + \sqrt{(1-e^2)}}$: to develope ψx in terms of ascending powers of e.

The roots of the equation

$$u^2 - \frac{2}{e} u + 1 = 0,$$

are $\frac{e}{1 + \sqrt{(1-e^2)}}$ and $\frac{e}{1 - \sqrt{(1-e^2)}}$: consequently

$$\psi u = u^n = x^n = \left\{\frac{e}{1 + \sqrt{(1-e^2)}}\right\}^n$$

$$= \frac{e^n}{2^n}\left\{1 + n\left(\frac{e}{2}\right)^2 + \frac{n(n+3)}{1 . 2}\left(\frac{e}{2}\right)^4 + \&c.\right\}$$

(9). Let $x = \frac{a\,(1-e^2)}{1-e\cos v}$: consequently

$$a\,(1-e^2) - x + e\cos v \,.\, x = 0.$$

Making $z = a\,(1-e^2)$, and $e' = e\cos v$, we get

$$\psi x = x = a \,\{\, 1 + e' - (\tan v)^2 e'^2 - (\tan v)^2 e'^3 - (\tan v)^2 e'^4 - \&c. \,\{$$

$$\psi x = \log x = \log z + e' + \frac{e'^2}{2} + \frac{e'^3}{3} + \&c.$$

If it was required to express e, in terms of the other quantities of the equation, we must put it in the form

$$\frac{1}{\cos v} - e - \frac{a}{x\cos v} \,.\, (1-e^2) = 0:$$

from which we get by the ordinary formula

$$e = \frac{1}{\cos v} + \frac{a}{x} \cdot \frac{(\sin v)^2}{(\cos v)^3} + \frac{a^2}{1\,.\,2\,.\,x^2} \cdot 2^2 \cdot \frac{(\sin v)^2}{(\cos v)^5} +$$

$$\frac{a^3}{1\,.\,2\,.\,3\,.\,x^3} \cdot \frac{2\,.\,3\,.\,(\sin v)^2 \;\{\, 5\,(\cos v)^2 \,\}}{(\cos v)^7} + \&c.$$

(10). Let $u = nt + e\sin u$: comparing this expression with formula (1), we find $z = nt$, $x = e$, $y = u$ and $\phi(y) = \sin u$: consequently,

$$(a)\;\; u = z + e\sin z + \frac{e^2}{1\,.\,2} \cdot \frac{d\,(\sin z)^2}{dz} +$$

$$\frac{e^3}{1\,.\,2\,.\,3} \cdot \frac{d^2\,(\sin z)^3}{dz^2} + \&c.$$

$$= nt + e\sin nt + \frac{e^2}{1\,.\,2\,.\,2} \cdot 2\sin 2nt$$

$$+ \frac{e^3}{1\,.\,2\,.\,3\,.\,2^2}\,(3^2 \sin 3nt - 3\sin nt)$$

$$+\frac{e^4}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 2^3}(4^3 \sin 4\,n\,t - 4 \,.\, 2^3 \,.\, \sin 2\,n\,t)$$

$$+\frac{e^5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 2^4}\left(5^4 \sin 5\,n\,t - 5 \,.\, 3^4 \sin 3\,n\,t + \frac{5 \,.\, 4}{1 \,.\, 2} \sin n\,t\right)$$

$$+\&c.$$

If u represent the eccentric and $n\,t$ the mean anomaly of a planet, reckoning from the perihelion, and if e be the eccentricity of the orbit, this series converges with great rapidity for most of the planetary bodies, since e is a very small quantity. Laplace *Mécanique Céleste*, Lib. II. Art. 22.

$$(b). \quad \psi\, u = a\,(1 - e \cos u)$$

$$= a\left\{(1 - e \cos z) + e^2 (\sin z)^2 + \frac{e^3}{1 \,.\, 2} \,.\, \frac{d\,(\sin z)^3}{d\,z}\right.$$

$$\left. + \frac{e^4}{1 \,.\, 2 \,.\, 3} \,.\, \frac{d^2\,(\sin z)^4}{d\,z^2} + \&c.\right\}$$

$$= a\left\{1 + \frac{e^2}{2} - e \cos n\,t - \frac{e^2}{2} \cos 2\,n\,t\right.$$

$$- \frac{e^3}{1 \,.\, 2 \,.\, 2^2}(3 \cos 3\,n\,t - 3 \cos n\,t)$$

$$- \frac{e^4}{1 \,.\, 2 \,.\, 3 \,.\, 2^3}(4^2 \,.\, \cos 4\,n\,t - 4 \,.\, 2^2 \,.\, \cos 2\,n\,t)$$

$$- \frac{e^5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 2^4}\left(5^3 \,.\, \cos 5\,n\,t - 5 \,.\, 3^3 \,.\, \cos 3\,n\,t\right.$$

$$\left. + \frac{5 \,.\, 4}{1 \,.\, 6} \,.\, \cos n\,t\right)$$

$$- \frac{e^6}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 2^5}\left(6^4 \,.\, \cos 6\,n\,t - 6 \,.\, 4^4 \,.\, \cos 4\,n\,t\right.$$

$$\left.\left. + \frac{6 \,.\, 5}{1 \,.\, 2} \,.\, 2^4 \,.\, \cos 2\,n\,t\right) - \&c.\right\}$$

This series expresses the radius vector of a planet's orbit, in terms of the mean anomaly, reckoned from the perihelion. Laplace. *Id.*

(11). Let $u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1 \,.\, 2} + \frac{d^3u}{dx^3} \cdot \frac{h^3}{1 \,.\, 2 \,.\, 3} + \text{\&c.} = 0$; to find an expression for h in terms of u and its differential coefficients.

If we put p, q, r, s, &c. for the differential coefficients of u, and compare the equation with $z - y + \phi(y) = 0$, we find $z = -\frac{u}{p}$ and $\phi(y) = -\frac{1}{p}\left\{q \cdot \frac{h^2}{1 \,.\, 2} + r \cdot \frac{h^3}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \text{\&c.}\right\}$; consequently, by a process similar to that employed in the Example given in Note *E*, p. 646, we get

$$h = -\left\{\frac{u}{p} + \frac{q}{p^3} \cdot \frac{u^2}{1 \,.\, 2} + \left(\frac{3\,q^2 - p\,r}{p^5}\right)\frac{u^3}{1 \,.\, 2 \,.\, 3}\right.$$
$$\left. + \left(\frac{3 \,.\, 5 \,.\, q^3 - 2 \,.\, 5\,p\,q\,r + p^2 s}{p^7}\right)\frac{u^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \text{\&c.}\right\}$$

See *Lagrange Résolution des Equations Numériques*, Note 11, and Euler. *Inst. Calc. Diff.* Pars. II. Art. 232, where the uses of this reverted series in the resolution of numerical equations are explained.

Paoli in his *Elementi d' Algebra*, Vol. II. page 44, has put this series under a very elegant and remarkable form:

If $A = -\frac{d\,x}{d\,u}$, then

$$h = A\,u + \frac{A\,d\,A}{d\,x} \cdot \frac{u^2}{1 \,.\, 2} + \frac{A\,d\,(A\,d\,A)}{d\,x^2}\;\frac{u^3}{1 \,.\, 2 \,.\, 3}$$
$$+ \frac{A\,d\,\{A\,d\,(A\,d\,A)\}}{d\,x^3} \cdot \frac{u^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \text{\&c.}$$

a result which may be very easily verified. Thus if $u = x^4 - 2\,x^2 + 4\,x - 8$, we shall find by means of this series,

$$h = -\left\{\frac{1}{2^2(x^3 - x + 1)}\cdot\frac{u}{1} + \frac{(3x^2 - 1)}{2^4(x^3 - x + 1)^3}\cdot\frac{u^2}{1.2}\right.$$

$$\left. + \frac{(21x^4 - 12x^2 - 6x + 3)}{2^6(x^3 - x + 1)^5}\cdot\frac{u^3}{1.2.3} + \&c.\right\}$$

If a be a root of the equation $u=0$, and x be an approximate value of a, we shall find

$$a=x+h=x-\frac{1}{2^2(x^3-x+1)}\cdot\frac{u}{1}-\frac{(3x^2-1)}{2^4(x^3-x+1)^3}\cdot\frac{u^2}{1.2}-\&c.$$

Thus if $x=\frac{3}{2}$ and therefore $u=-\frac{23}{16}$, we shall get

$$a=\frac{3}{2}+\frac{1}{8}-\frac{1}{64}+\frac{391}{135424}-\&c.=1.61, \text{ nearly.}$$

If we substitute this new value of x, and confine ourselves to four terms of the series, as before, we shall find, $a=1.6117662$: and by a continuation of this process, the approximation may be carried to any required degree of accuracy.

Again, let $u=x\log.x-\log.100$: this series gives us, if a be a root of the equation $u=0$, $a=x+h$

$$=x-\frac{u}{1+\log.x}-\frac{u^2}{1.2.x(1+\log.x)^3}$$

$$-\frac{(4+\log.x)u^3}{1.2.3.x^2(1+\log.x)^5}$$

$$-\frac{\{27+14\log.x+2(\log.x)^2\}u^4}{1.2.3.4.x^3(1+\log.x)^7}$$

$$-\&c.$$

If we make $x=3.5$, and therefore $u=-.220499796$, we hall get, by confining ourselves to the three first terms,

$x = 3.5972722$, a result which is accurate to four places of decimals. Euler. *Inst. Calc. Diff.* Pars. II. Art. 241, 242.

(12.) To find the general term of the developement of

$$\frac{P + Qx}{1 - 2x\cos.\theta + x^2}.$$

It appears from the investigation of formula (4), that the coefficient of x^n, in the developement of $\frac{fx}{z - x + \phi x}$, is

$$t_n = \frac{fz}{z^{n+1}} + \frac{d}{1.dz}\cdot\left(\frac{fz \times \phi z}{z^{n+1}}\right) + \frac{d^2}{1.2.dz^2}\cdot\left(\frac{fx(\phi z)^2}{z^{n+1}}\right)$$

$$+ \frac{d^3}{1.2.3.dz^3}\cdot\left(\frac{fz(\phi z)^3}{z^{n+1}}\right) + \&c.$$

the series being restricted to those terms which involve negative powers of z.

Now, in this case, $fz = \frac{P + Qz}{2\cos\theta}$, $z = \frac{1}{2\cos\theta}$ and $\phi z = \frac{z^2}{2\cos\theta}$: consequently

$$t_n = P\left\{(2\cos\theta)^n - (n-1)(2\cos\theta)^{n-2} + \frac{(n-2)(n-3)}{1.2}(2\cos\theta)^{n-4} - \&c.\right\}$$

$$+ Q\left\{(2\cos\theta)^{n-1} - (n-2)(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5} - \&c.\right\}$$

$$= \frac{P\sin(n+1)\theta + Q\sin n\theta}{\sin\theta} \qquad \text{(Ex. 13. p. 82.)}$$

It from hence appears that

$$\frac{P + Q x}{1 - 2 x \cos \theta + x^2} = \frac{1}{\sin \theta} \Big\{ P \sin \theta + (P \sin 2 \theta + Q \sin \theta) x$$

$$+ (P \sin 3 \theta + Q \sin 2 \theta) x^2 + \&c. \Big\}$$

The general term of the developement of $\frac{f x}{(z - x + \phi x)^m}$ may be easily deduced from that of $\frac{f x}{z - x + \phi x}$; for if we make

$$u = \frac{f x}{z - x + \phi x}, \text{ we find}$$

$$- \frac{d u}{d z} = \frac{f x}{(z - x + \phi x)^2}$$

$$\frac{d^2 u}{1 . 2 . d z^2} = \frac{f x}{(z - x - \phi x)^3}$$

and

$$\frac{(-1)^{m-1} d^{m-1} u}{1 . 2 \ldots (m - 1) d z^{m-1}} = \frac{f x}{(z - x + \phi x)^m}.$$

In the same manner, if we put v for the function of z, which is the coefficient of x^n in the developement of u, we shall find $- \frac{d v}{d z}$ and $\frac{(-1)^{m-1} d^{m-1} v}{1 . 2 \ldots (m-1) d z^{m-1}}$, for the respective coefficients of the corresponding terms in the developements of $\frac{f x}{(z - x + \phi x)^2}$, and $\frac{f x}{(z - x + \phi x)^m}$.

Thus, if we substitute $\frac{1}{2 \cos \theta}$ for z, we shall find

$$- \frac{d v}{d z} = P \Big\{ (n + 1) (2 \cos \theta)^{n+1} - n (n - 1) (2 \cos \theta)^{n-1}$$

$$+ \frac{(n - 1) (n - 2) (n - 3)}{1 . 2} (2 \cos \theta)^{n-3} - \&c. \Big\}$$

$$+ Q \left\{ n \, (2 \cos \theta)^n - (n-1)\,(n-2)\,(2 \cos \theta)^{n-2} \right.$$

$$\left. + \frac{(n-2)\,(n-3)\,(n-4)}{1\,.\,2\,.\,3} (2 \cos \theta)^{n-4} - \&c. \right\}$$

the series being restricted, as before, to such terms as involve positive powers of $2 \cos \theta$.

In the same manner, the general term of the developement of

$$\frac{P + Q\,x}{(1 - 2\,x \cos \theta + x^2)^m} = \frac{(-1)^{m-1}\, d^{m-1}\, v}{1\,.\,2 \ldots . (m-1)\, d^{m-1}\, z}$$

$$= P \left\{ (n+1)\,(n+2) \ldots . (n+m-1)\,(2 \cos \theta)^{n+m-1} \right.$$

$$- (n-1)\, n \ldots . (n+m-2)\,(2 \cos \theta)^{n+m-3}$$

$$\left. + \frac{(n-3)\,(n-2) \ldots . (n+m-3)}{1\,.\,2} (2 \cos \theta)^{n+m-5} - \&c. \right\}$$

$$+ Q \left\{ n\,(n+1) \ldots . (n+m-2)\,(2 \cos \theta)^{n+m-2} \right.$$

$$- (n-2)\,(n-1) \ldots . (n+m-3)\,(2 \cos \theta)^{n+m-4}$$

$$+ \frac{(n-4)\,(n-3) \ldots \ldots . (n+m-4)}{1\,.\,2} (2 \cos \theta)^{n+m-6}$$

$$\left. - \&c. \right\}$$

Lagrange, *Résolution des Equations Numériques.* Note XI.

On the Maxima and Minima of Functions of one Variable.

ART. 47—51. (Ex. 1). Let $u = x^2 + 3x + 2$: to find when u is a *maximum* or a *minimum*.

If $x = -\frac{3}{2}$, $u = -\frac{1}{4}$, a *minimum*.

Euler. *Inst. Calc. Diff.* Pars II. Art. 261.

(2). Let $u = x^3 - 15x^2 + 56x - 60$.

If $x = \frac{15 + \sqrt{(57)}}{3}$, $u = -230 - \frac{38}{9}\sqrt{(57)}$, a *minimum*.

If $x = \frac{15 - \sqrt{(57)}}{3}$, $u = -230 + \frac{38}{9}\sqrt{(57)}$, a *maximum*.

(3). Let $u = x^3 - ax^2 + bx - c$.

If $x = \frac{a + \sqrt{(a^2 - 3b)}}{3}$, $u = -\frac{2a^3}{27} + \frac{ab}{3} - c - \frac{2(a^2 - 3b)^{\frac{3}{2}}}{27}$, a *minimum*.

If $x = \frac{a - \sqrt{(a^2 - 3b)}}{3}$, $u = -\frac{2a^3}{27} + \frac{ab}{3} - c + 2\frac{(a^2 - 3b)^{\frac{3}{2}}}{27}$, a *maximum*.

If $a^2 = 3b$, the two values of x are equal, and the function is neither a *maximum*, nor a *minimum*. Euler. *Id.*

(4). Let $u = x^4 - 8x^3 + 22x^2 - 24x + 12$.

If $x = 3$, $u = 3$, a *minimum*.

If $x = 2$, $u = 4$, a *maximum*.

If $x = 1$, $u = 3$, a *minimum*.

Euler, *Id.*

(5). Let $u = x^5 - 5x^4 + 5x^3 + 1$.

If $x = 3$, $u = -27$, a *minimum*.

If $x = 1$, $u = 2$, a *maximum*.

If $x = 0$, $u = 1$, which is neither a *maximum*, nor a *minimum*, since *two* roots of the equation $\frac{du}{dx} = 0$, are equal to nothing.

If the function had been $x^6 - 5x^5 + 5x^4 + 1$, then *three* values of x in the equation $\frac{du}{dx} = 0$, would have been equal to nothing, and $u = 1$, would have been a *minimum* value of the function. Euler. *Id.*

(6). Let $u = (x - a)^n$.

If $x = a$, then $u = 0$, a *minimum*, when n is even; but neither a *maximum*, nor a *minimum*, when n is odd. Euler. *Id.*

(7). Let $u = 3x^4 - 28ax^3 + 84a^2x^2 - 96a^3x + 48b^4$.

If $x = 4a$, $u = 136a^4 + 48b^4$, a *minimum*.

If $x = 2a$, $u = -32a^4 + 48b^4$, a *maximum*.

If $x = a$, $u = -37a^4 + 48b^4$, a *minimum*.

(8). Let $u = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$; the roots of $du = 0$, are $1, 0, 0, \surd(-1), -\surd(-1)$.

If $x = 1$, $u = 13$, a *minimum*.

If $x = 0$, u is neither a *maximum*, nor a *minimum*.

If $x = \pm \surd(-1)$, u is neither a *maximum*, nor a *minimum*, since the result of the substitution of either of those roots in $\frac{d^2 u}{d x^2}$ is an imaginary quantity, which can neither be considered as positive or negative.

If the roots a_1, a_2, $a_3, \ldots a_n$ of the equation $\frac{du}{dx} = 0$, arranged in the order of their magnitudes, be substituted in the equation $\frac{d^2 u}{d x^2} = 0$, which is a limiting equation to the former, they will give results, which are alternately positive and negative, and will consequently make u alternately a *minimum* and a *maximum*, beginning with the first. It appears, therefore, that if there are m unequal and real roots in the equation $\frac{du}{dx} = 0$, there are always m *minimum* and *maximum* values of u corresponding to them.

If in the same equation there be p roots equal to a, q roots equal to b, r roots equal to c, and so on, then there will be one *minimum* value of u corresponding to each of these classes of equal roots, when p, q, or r, is an odd number, and no such values whatever, when these numbers are even. Also, if any such values of u exist, they will be necessarily equal to nothing, or to c, when c is a constant quantity connected with u by the sign + or −.

It is hardly necessary to remark, that no *maximum* or *minimum* value of u, can ever arise from an imaginary root of $\frac{du}{dx} = 0$.

(9). Let $u = \frac{x}{x^2 + 1}$.

If $x = 1$, $u = \frac{1}{2}$, a *maximum*.

If $x = -1$, $u = -\frac{1}{2}$, a *minimum.*

It is very evident, that if u be a *maximum*, $\frac{1}{u}$ must be a *minimum*, and *versâ vice:* this consideration will sometimes enable us to shorten the necessary operations; as, in the case before us, the differential coefficients of $\frac{1}{u}$ or $x + \frac{1}{x}$, are more readily found than those of u. Euler. *Id.* Art. 265.

(10). Let $u = \frac{2 - 3x + x^2}{2 + 3x + x^2}$.

If $x = \sqrt{2}$, $u = 12\sqrt{2} - 17$, a *minimum.*

If $x = -\sqrt{2}$, $u = -12\sqrt{2} - 17$, a *maximum.*

This example shews that the *maximum* or *minimum* values of a function, do not depend upon the relation which they bear to each other, but to the values of the function, which immediately precede and follow them. Euler. *Id.*

(11). Let $u = \frac{x^2 - x + 1}{x^2 + x - 1}$.

If $x = 0$, $u = -1$, a *maximum.*

If $x = 2$, $u = \frac{3}{5}$, a *minimum.*

Euler. *Id.*

(12). Let $u = \frac{x^3 + x}{x^4 - x^2 + 1}$.

If $x = 1$, $u = 2$, a *maximum.*

If $x = -1$, $u = -2$, a *minimum.*

The other values of x, in the equation $\frac{du}{dx} = 0$, are

$\pm\sqrt{\left\{\frac{-5\pm\sqrt{21}}{2}\right\}}$, which are all of them imaginary. Euler. *Id.*

(13). Let $u = \dfrac{x^3 - x}{x^4 - x^2 + 1}$.

If $x = \dfrac{1+\sqrt{5}}{2}$, $u = \dfrac{1}{2}$, a *maximum.*

If $x = \dfrac{1-\sqrt{5}}{2}$, $u = \dfrac{1}{2}$, a *maximum.*

If $x = \dfrac{-1+\sqrt{5}}{2}$, $u = -\dfrac{1}{2}$, a *minimum.*

If $x = \dfrac{-1-\sqrt{5}}{2}$, $u = -\dfrac{1}{2}$, a *minimum.*

Euler. *Id.*

In finding the *maximum* and *minimum* values of functions under a fractional form, it will frequently be useful to remark;

1st, That if $u = \dfrac{v}{v'}$, we have $d\,u = \dfrac{v}{v'}\left\{\dfrac{d\,v}{v} - \dfrac{d\,v'}{v'}\right\}$: consequently, when $\dfrac{d\,u}{d\,x} = 0$, we must have either $v = 0$, $\dfrac{d\,v}{v} = \dfrac{d\,v'}{v'}$, or $v' = \infty$.

2dly, Also, $d^2 u = d\,u\left\{\dfrac{d\,v}{v} - \dfrac{dv'}{v'}\right\} + u\left\{\dfrac{d^2 v}{v} - \dfrac{d^2 v'}{v'}\right\}$ $= \dfrac{v'\,d^2 v - v\,d^2 v'}{v'^2}$, when $d\,u = 0$; and since v'^2 is essentially positive, the sign of $\dfrac{d^2 u}{dx^2}$ will depend upon that $v'd^2v - vd^2v'$, when we substitute for x, the roots of $\dfrac{d\,u}{d\,x} = 0$.

3dly, If $u = \frac{v^n}{v'^{n'}}$, we have $d\,u = \frac{v^n}{v'^{n'}}\left\{\frac{n\,d\,v}{v} - \frac{n'\,d\,v'}{v'}\right\}$, and therefore $\frac{d\,u}{d\,x} = 0$, when $v^n = 0$, $\frac{n\,d\,v}{v} = \frac{n'\,d\,v'}{v'}$, or $v'^n = \infty$.

4thly, Also, when $\frac{d\,u}{d\,x} = 0$, we find

$$d^2\,u = u\left\{\frac{n\,d^2\,v}{v} - \frac{n'\,d^2\,v'}{v'} - \frac{n\,d\,v}{v}\left(\frac{d\,v}{v} - \frac{d\,v'}{v'}\right)\right\}$$

from which the sign of $d^2\,u$ may be easily determined.

(14). Let $u = \frac{(x+3)^3}{(x+2)^2}$.

$$d\,u = \frac{(x+3)^2}{(x+2)^3} \cdot (x) = 0.$$

If $x = 0$, $u = \frac{27}{4}$, a *minimum.*

If $x = -3$, u is neither a *maximum*, nor a *minimum.*

This function becomes a *maximum*, when $x = -2$, a case in which u as well as its differential coefficients become infinitely great; this case can hardly be considered as an exception to the general proposition in Art. 48, the truth of which entirely depends upon the differential coefficients of the function being finite or evanescent, upon the substitution of a particular value of x. It may be explained as follows: let $u = \frac{v}{(x-a)^n\,Q}$, and therefore $\frac{1}{u} = \frac{(x-a)^n\,Q}{v}$, a quantity which may be readily shewn to become a *minimum*, when $x = a$, if n be an even number; and since the *minimum* values of $\frac{1}{u}$, are the *maximum* values of u, it is obvious that in this case, we must reckon $u = \infty$, among the *maximum* values of the function.

(15). Let $u = \frac{(x+3)^4}{(x+2)^6}$

$$\frac{du}{dx} = \frac{-(x+3)^3}{(x+2)^7} \cdot (2x+10) = 0.$$

If $x = -5$, $u = \frac{16}{729}$, a *maximum.*

If $x = -3$, $u = 0$, a *minimum.*

If $x = -2$, $u = \infty$, a *maximum.*

(16). Let $u = \frac{(x-1)^2}{(x+1)^3}$

$$\frac{du}{dx} = \frac{(x-1)}{(x+1)^4} \cdot (5-x) = 0.$$

If $x = 5$, $u = \frac{2}{27}$, a *maximum.*

If $x = 1$, $u = 0$, a *minimum.*

If $x = -1$, $u = \infty$, which is neither a *maximum,* nor a *minimum.*

(17). Let $u = \frac{(1+x)^3}{(1+x^2)^3}$

$$\frac{du}{dx} = \frac{(1+x)^2}{(1+x^2)^4} (3 - 4x - x^2).$$

If $x = -2 + \sqrt{7}$, $u = \frac{17 + 7\sqrt{7}}{16}$, a *maximum.*

If $x = -2 - \sqrt{7}$, $u = \frac{17 - 7\sqrt{7}}{16}$, a *minimum.*

There is no *maximum* or *minimum,* corresponding to $x = -1$, or $x = \pm \sqrt{-1}$. Euler. *Id.* Art. 267.

(18). Let $u = \sqrt{(a^2 + x^2)} - x$.

If $x = \infty$, $u = 0$, a *minimum*.

Euler. *Id. Art.* 271.

(19). Let $u = \sqrt[4]{(1 + 8nx^4)} + \sqrt[4]{(1 - nx^4)}$.

If $x = \sqrt[4]{\left(\frac{5}{8n}\right)}$, $u = \frac{3\sqrt[4]{6}}{2}$, a *maximum*.

(20). Let $u = \left\{\frac{a + x}{a + x}\right\} \left\{\frac{b + x}{b - x}\right\}$.

If $x = \sqrt{(ab)}$, $u = \frac{-(\sqrt{a} + \sqrt{b})^2}{(\sqrt{a} - \sqrt{b})^2}$, a *minimum*.

If $x = -\sqrt{(ab)}$, $u = \frac{-(\sqrt{a} - \sqrt{b})^2}{(\sqrt{a} + \sqrt{b})^2}$, a *maximum*.

If the function had been $\sqrt{\left\{\frac{a + x}{a - x}\right\} \cdot \left\{\frac{b + x}{b - x}\right\}}$, it would have admitted neither of a *maximum* nor a *minimum*.

In general, if any function u be a *maximum*, or a *minimum*, we may likewise consider $u^{\frac{m}{n}}$ as a *maximum*, or a *minimum*, unless m be an odd and n an even number; in which case it is necessary that the *maximum* and *minimum* values of u should have a positive sign.

In finding, however, the *maximum* and *minimum* values of u from those of u^m, there may exist *minimum* values of the second function which have no such values corresponding to them in the first; for the *maxima* and *minima* of u^m are determined equally from the equation $u^{m-1} = 0$, and $\frac{du}{dx} = 0$; though no such value of u^m arises from the first of these equations, unless m be an even number.

But this statement requires some modification; for if $V = u^m$, it does not always follow that $\frac{dV}{dx} = 0$, when $u^{m-1} = 0$, or $\frac{du}{dx} = 0$, since the same value of x which verifies the first equation, may make $\frac{du}{dx} = \infty$, and *versâ vice*; it appears therefore, that $\frac{dV}{dx}$ may have a finite value under these circumstances, which is inconsistent with the condition of a *maximum* or *minimum* value of V.

(21). Let $u = x\sqrt{(ax - x^2)}$.

If $x = \frac{3a}{4}$, $u = \frac{3\sqrt{3}}{16} a^2$, a *maximum*.

Assume $V = u^2$, from which we get

$$\frac{dV}{dx} = 2u\frac{du}{dx} = 3ax^2 - 4x^3,$$

the roots of which equation are $\frac{3a}{4}$, 0, 0; the first of which makes V a *maximum*; but $u = 0$, when $x = a$, a value which makes $\frac{du}{dx} = \infty$, and $\frac{dV}{dx}$ a finite quantity.

(22). Let $u = x(x - a)^3$.

If $x = \frac{a}{4}$, $u = -\frac{3^3}{4^4} a^4$, a *maximum*.

If we make $V = u^2 = x^2 (x - a)^6$, we find

$$\frac{dV}{dx} = 2u\frac{du}{dx} = 2x(x - a)^5 (4x - a) = 0:$$

consequently, if $x = \frac{a}{4}$, $V = \frac{3^6}{4^6} a^8$, a *maximum*.

If $x = a$, $V = 0$, a *minimum*.

If $x = 0$, $V = 0$, a *minimum*.

In this case, u^2 admits of two *minimum* values which have no existence in u.

(23). Let $u = b + c\,(x - a)^{\frac{2}{3}}$: it is obvious, that when $x = a$, this function is a *minimum* or a *maximum*, according as c is positive or negative; a case in which we have

$$\frac{d\,u}{d\,x} = \frac{2\,c}{3}\,(x - a)^{-\frac{1}{3}} = \infty.$$

Let $V = (u - b)^3 = c^3\,(x - a)^2$, an expression which is a *minimum* or a *maximum*, at the same time with u: making therefore

$$\frac{dV}{d\,x} = 2\,c^3\,(x - a) = 0,$$

we find, when $x = a$, $V = 0$ or $u = b$, a *minimum* or a *maximum*, according to the sign of c.

(24). Art. 51. To find the *maximum* and *minimum* values of u in the implicit function,

$$u^2 + m\,x\,u + a^2 + b\,x + n\,x^2 = 0.$$

$$\frac{d\,u}{d\,x} = \frac{P}{Q} = \frac{m\,u + b + 2\,n\,x}{2\,u + m\,x} = 0: \text{ consequently,}$$

$P = 0$, or $Q = \infty$: in the first case,

$u = -\dfrac{(b + 2\,n\,x)}{m}$; from the substitution of which in the original equation, we get

$$x = \frac{2\,n\,b \pm m\sqrt{\{n\,b^2 + n\,(m^2 - 4\,n)\,a^2\}}}{n\,(m^2 - 4\,n)}$$

$$\text{and}\quad u = \frac{-m\,b \mp 2\sqrt{\{n\,b^2 + n\,(m^2 - 4\,n)\,a^2\}}}{m^2 - 4\,n}$$

the negative sign prevailing when u is a *minimum*, and the positive when u is a *maximum*.

In order to distinguish the *maximum* and *minimum* values of u from each other, we must examine the sign of $\frac{d^2 u}{d x^2}$ which arises from the substitution of those values of x or u, which make $\frac{d u}{d x} = 0$: but, in general, since, in this case, $\frac{d u}{d x} = \frac{P}{Q} = 0$, we have

$$P\,d x + Q\,d u = 0.$$

By differentiating a second time, we find

$$d P\,d x + d Q\,d u + Q\,d^2 u = 0 \qquad (\beta)$$

since $d x$ is constant: but P being a function of x and u, we have

$$d P = p\,d x + q\,d u,$$

where p is the differential coefficient of P, considering x alone as variable: substituting this value of $d P$ in equation (β) and making $d u = 0$, we obtain

$$p\,d x^2 + Q\,d^2 u = 0;$$

or $\frac{d^2 u}{d x^2} = -\frac{p}{Q}$, which, in the case we are considering, becomes $\pm \frac{2 n^2}{\sqrt{\{n b^2 + n (m^2 - 4 n) a^2\}}}$, according as the first or the second value of x is substituted.

If $x = 0$, u admits of neither a *maximum*, nor a *minimum*.

If $m^2 = 4 n$, then u admits but of one value, which is $\frac{m^2 a^2 - b^2}{m b}$, a *minimum*.

Euler. *Inst. Calc. Diff.* Pars II. Art. 277.

(25). Let $u^2 - x^2 u + x - x^3 = 0$.

$$\frac{du}{dx} = \frac{1 - 3x^2 - 2xu}{2u - x^2} = 0:$$

from which we get the equation to determine values of x, or

$$1 - 6x^2 + 2x^3 + 9x^4 + 2x^5 = 0.$$

If $x = -1$, $u = 1$, a *maximum*, or $u = 0$, which is neither a *maximum*, nor a *minimum*. Euler. *Id.*

(26). Let $x^3 + u^3 = a^2 x$.

If $x = \frac{a}{\sqrt{3}}$, $u = -a\sqrt[3]{\frac{2}{3\sqrt{3}}}$, a *minimum.*

If $x = -\frac{a}{\sqrt{3}}$, $u = a\sqrt[3]{\frac{2}{3\sqrt{3}}}$, a *maximum.*

(27). Let $x^2 u - a u^2 - b^2 x = 0$.

If $x = -\sqrt[3]{\frac{(ab^2)}{2}}$, $u = -\sqrt[3]{\frac{(b^4)}{4a}}$, a *maximum.*

(28). Let $u^2 + 2x^2 u + 4x - 3 = 0$.

If $x = -\frac{1}{2}$, $u = 2$, a *maximum.*

If $x = 1$, $u = -1$, which is neither a *maximum*, nor a *minimum*. Euler. *Id.*

(29). Let $u^3 + x^3 - 3axu = 0$.

The equation to determine the values of x is

$$x^6 - 2a^3 x^3 = 0.$$

If $x = a\sqrt[3]{2}$, $u = a\sqrt[3]{4}$, a *maximum.*

If $x = 0$, $u = 0$, a value which requires a particular examination, since in this case $\frac{du}{dx}$ as well as $\frac{d^2u}{dx^2}$ becomes $\frac{0}{0}$.

By the process explained in Note *F*, page 650, after one differentiation of the equation $P\,dx + Q\,du = 0$, we shall find $\frac{du}{dx} = 0$, or $\frac{du}{dx} = \frac{a}{u} = \infty$, since $u = 0$. Also, by differentiating the same equation a second time, and making $x = 0$, $u = 0$ and $\frac{du}{dx} = 0$, we shall find $\frac{d^2u}{dx^2} = \frac{2}{3a}$, which indicates a *minimum* value of u.

If we suppose $\frac{du}{dx} = \infty$, all the other differential coefficients become infinite under the same circumstances, which shews that Taylor's series fails to give the developement of u' or the succeeding value of u when $x = 0$: we must have recourse, therefore, in this case, to other methods.

Thus, if in the place of $x = 0$, we put $x = 0 + h$, or h, we have, if $u = k$,

$$k^3 + y^3 - 3akh = 0:$$

and if we suppose h so small that all the terms of the series for k in terms of h, after the first, may be neglected, we shall find

$$k = \frac{h^2}{3a}, \; + \sqrt{(3ah)}, \; - \sqrt{(3ah)}.$$

It appears, therefore, that $u = 0$, is a *minimum*, since this value of u or k is less than those corresponding to $x = h$ and $x = -h$: the other values of k, however, give no minimum value of u, since they become imaginary when h is negative. Euler. *Ib.*

(30). Let $u^4 - 4a^2xu + x^4 = 0$:

The equation which gives the values of x is

$$x^{12} - 3a^8x^4 = 0:$$

If $x = a\sqrt[8]{3}$, $u = a\sqrt[8]{27}$, a *maximum.*

If $x = -a\sqrt[8]{3}$, $u = -a\sqrt[8]{27}$, a *minimum.*

(31). Let $u = \dfrac{x}{\log x}$:

If $\log x = 1$, or if $x = \epsilon$, $u = \epsilon$, a *minimum.* This is the solution of the following question. "To find the number which bears the least ratio to its logarithm." Euler. *Ib.* Art. 272.

(32). To find the number x, whose x^{th} root is the greatest possible: or let $u = x^{\frac{1}{x}}$.

If $x = \epsilon$, $u = \epsilon^{\frac{1}{\epsilon}}$, a *maximum.*

(33). To divide a number a into such a number of equal parts, that their continued product may be a maximum; or let $u = \left(\dfrac{a}{x}\right)^x$.

If $x = \dfrac{a}{\epsilon}$, $u = \epsilon^{\frac{a}{\epsilon}}$, a *maximum.*

(34). A debt A£ accumulating at compound interest is to be liquidated by payments of a£ at the end of the first year, $2a$£ at the end of the second, $3a$£ at the end of the third, and so on: if r be the rate of interest of 1£ for one year, after what number of years will the debt be a *maximum?* or let $u = A(1+r)^x - \dfrac{a}{r^2}(1+r)^{x+1} + \dfrac{a}{r^2}(1+r+rx)$.

$$\text{If } x = \frac{\log ar - \log\{[a(1+r) - r^2A]\log(1+r)\}}{\log(1+r)},$$

u is a *maximum.*

If A be greater than $\frac{(1+r)a}{r^2}$, the debt increases continually, and therefore admits not of a *maximum* value.

(35). Let $u = (\sin x)^m . \{\sin(a - x)\}^n$.

If $\sin(a - 2x) = \left(\frac{n - m}{n + m}\right) \sin a$, u is a *maximum*: it is evident that $a - 2x$, and therefore x may be determined by means of trigonometrical tables.

(36). Let $u = (\tan x)^m \{\tan(a - x)\}^n$.

If $\tan(a - 2x) = \left(\frac{n - m}{n + m}\right) \tan a$, u is a *maximum*.

(37). Let $u = \sin x \cos(a - x)$.

If $x = \frac{a}{2} + \frac{\pi}{4}$, $u = \frac{1 + \sin a}{2}$, a *maximum*.

If $x = \frac{a}{2} - \frac{\pi}{4}$, $u = -\frac{(1 - \sin a)}{2}$, a *minimum*.

In this, as well as in the two last Examples, there are an infinite number of values of x, which make u a *maximum*, or a *minimum*.

(38). Let $u = x \sin x$.

The values of x must be derived from the solution of the equation $x = -\tan x$; one of these values which gives u a *maximum* is $x = 116°.14'.21''.20'''$. Euler, *Ib.* Art. 272.

(39). Let $u = \frac{\epsilon^x}{\sin(a - x)}$.

If $x = a + \frac{\pi}{4}$, $u = \sqrt{2}\,\epsilon^{a + \frac{\pi}{4}}$, a *minimum*.

(40). Let $u = \epsilon^x \sin(a - x)$.

If $x = a - \frac{\pi}{4}$, $u = \frac{\epsilon^{a - \frac{\pi}{4}}}{\sqrt{2}}$, a *maximum*.

We shall now proceed to the solution of problems in Geometry, which involve the consideration of *maxima* and *minima*, an application of this method, which is at once interesting and important: and though in many instances, the analytical solutions of questions of this nature, are inferior both in brevity and elegance, to those which are effected by the ordinary processes of Geometry, yet for the most part they have the advantage of superior generality.

We shall give no examples which suppose the reader to be acquainted with Mechanics and the other branches of Natural Philosophy, the study of which ought properly to follow, not precede, that of the Differential and Integral Calculus.

We take this opportunity of referring our readers to Mr. Cresswell's *Treatise* on *Maxima* and *Minima*, a work written in the most rigorous spirit of the ancient Geometry, and which likewise contains the analytical solutions of several very interesting problems.

(41). Through the given point P, to draw the shortest line MM' terminated by the two given straight lines AB and AC. Fig. 1.

If we draw PQ and PQ' respectively parallel to AB and AC, and assume $MQ=x$, $PQ=a$, $PQ'=b$, and the angle $BAC=\theta$: then the equation which determines x or MQ, when MM' is the least possible, is

$$x^3 - a\cos\theta \,.\, x^2 + ab\cos\theta \,.\, x - a^2 b = 0.$$

If $\theta = 90°$, this equation becomes

$$x^2 - a^2 b = 0, \text{ from which we find}$$

$$u = MM' = \{ a^{\frac{3}{2}} + b^{\frac{3}{2}} \}^{\frac{3}{2}}.$$

Vince's *Fluxions*, p. 35.

(42). The same things being given, to draw MM' so that the sum of the lines AM and AM' may be a *minimum*. Fig. 1.

If $x = \sqrt{(ab)}$, then $u = AM + AM' = \{\sqrt{a} + \sqrt{b}\}^2$, a *minimum*. Simpson's *Fluxions*, Vol. I. p. 23.

(43). The same things being given, to draw MM' in such a manner, that the triangle AMM' may be a *minimum*. Fig. 1.

If $x = b$, or if $MP = MP'$, then $u = 2ab \sin\theta$, a *minimum*.

(44). To bisect a triangle by the shortest line. Fig. 2.

Let $AB = c$, $AC = b$, $BC = a$ and $AM = x$: then if $x = \sqrt{\left(\frac{bc}{2}\right)}$ and therefore $AM = AM'$, we have

$$u = MM' = \sqrt{\left\{\frac{(a+b-c)(a+c-b)}{2}\right\}}, \text{ a minimum.}$$

(45). Given the base and altitude of a triangle, to describe it so, that the vertical angle may be a *maximum*. Fig. 3.

Let $CD = e$, $AB = c$ and $AD = x$: then

$$u = \tan ACB = \frac{bc}{e^2 - bx + x^2},$$

which is a *maximum*, when $x = \frac{b}{2}$, or when the triangle is isosceles.

(46). To find the least isosceles triangle which can circumscribe a given circle. Fig. 4.

Assume $CQ = r$ and $PQ = \theta$: then

$$u = r^2 \left\{\tan\frac{\theta}{2} - \frac{\tan\theta}{2}\right\}, \text{ is a minimum,}$$

when $\theta = 120^\circ$, $u = \frac{3\sqrt{3}}{2} r^2$, and the triangle is equilateral.

(47). To inscribe the greatest rectangle in a triangle. Fig. 5.

Assume $AD = e$, $BC = a$, and $AP = x$: then $u = MM'N'N$ is a *maximum*, when $x = \frac{e}{2}$ or when the sides of the triangle are bisected.

(48). To find the point D, in the line CE, from which AB subtends the greatest angle. Fig. 6.

Let $AC = a$, $BC = b$, $CD = x$ and the angle $ACD = \theta$: if $x = \sqrt{(a\,b)}$, we have

$$u = \tan ADB = \frac{(a-b)\sqrt{(a\,b)}\,.\sin\theta}{2\sqrt{(a\,b)}-(a+b)\cos\theta}, \text{ a } maximum.$$

If $x = -\sqrt{(a\,b)}$, $u = \dfrac{(a-b)\sqrt{(a\,b)}\,.\sin\theta}{2\sqrt{(a\,b)}+(a+b)\cos\theta}$, which is likewise a *maximum*, corresponding to a point D' in DC produced.

(49). To find the point P in the circumference of a circle from which the given line AB subtends the greatest angle. Fig. 7.

Draw PD and Cd perpendicular to AB, and assume $Ad = a$, $Bd = b$, $Cd = c$, $CP = r$ and $cp = x$: then the equation which determines the values of x is

$$(r^2 - a\,b - c^2)\sqrt{(r^2 - x^2)} = (a-b)(r^2 - c\,x).$$

If $c = 0$, or if A and B be two points taken in a line passing through the center of the circle, then if

$$x = \frac{r\sqrt{\{(r^2-a^2)(r^2-b^2)\}}}{r^2 - a\,b}, \text{ we have}$$

$$u = \tan APB = \frac{(a+b)\,r}{\sqrt{\{(a^2-r^2)(b^2-r^2)\}}}, \text{ a } maximum.$$

If $a=b$, or if the two points be equidistant from the center of the circle and upon opposite sides of it, then $x=r$ and $u=\dfrac{2ar}{a^2-r^2}$.

(50). In the trapezoid $ADCB$, the two equal sides AD, CB and one of the parallel sides DC, are given, to find AB, so that the area of the figure may be a *maximum.* Fig. 8, 9.

Let $AD=b$, $DC=a$ and $AP=x$: then if

$$x=-\frac{a}{4}+\frac{\sqrt{(a^2+8b^2)}}{4}, \text{ we have}$$

$$u=\frac{\{3a+\sqrt{(a^2+8b^2)}\}}{8}\sqrt{\left\{2b^2-\frac{a^2}{2}+\frac{a}{2}\sqrt{(a^2+8b^2)}\right\}},$$

a *maximum.*

Again, if $x=-\dfrac{a}{4}-\dfrac{\sqrt{(a^2+8b^2)}}{4}$, we find

$$u=\frac{\{3a-\sqrt{(a^2+8b^2)}\}}{4}\sqrt{\left\{2b^2-\frac{a^2}{2}-\frac{a}{2}\sqrt{(a^2+8b^2)}\right\}},$$

a *minimum*; it is evident that in this case, x is negative and greater than $\dfrac{a}{2}$: consequently the trapezoid assumes the form of the double triangle represented in Fig. 9. and u is the difference of the two triangles.

If $a=b$, $x=-a$ and $u=0$, or the triangles are in this case equal.

If $a>b$, the expression for u becomes imaginary, or the figure decreases continually and admits not of a *minimum.*

If $a=0$, the trapezoid becomes a triangle, which is a *maximum*, when $x=\dfrac{b}{\sqrt{2}}$, or when the two equal sides are at right angles to each other.

If we make $\frac{du}{dx} = \infty$, we find $x = \pm b$, which give values of u, such as are denominated by Euler, *minima* of the second species*: the first value makes $AB = a + 2b$, and $u = 0$, which may be considered as a *minimum* value of the trapezoids which can be described on each side of AB, but not in the sense of the definition: the same remarks are applicable to the value of u which results from the second value of x. Maupertuis, *Mémoires de l'Acad. des Sciences*. 1728.

(51). To find the greatest space which can be included by four given lines. Fig. 10.

Assume $AB = a$, $BC = b$, $CD = c$, $DA = d$, the angle $BAD = \phi$, and the angle $BCD = \theta$: the area of the trapezium

$$\text{or} \quad u = ad \sin\phi + bc \sin\theta.$$

Also, since $ad \cos\phi - bc \cos\theta = \frac{a^2 + d^2 - b^2 - c^2}{2}$, by differentiating this equation and making $\frac{du}{dx} = 0$, we shall find

$$-\frac{\cos\theta}{\sin\theta} = \frac{\cos\phi}{\sin\phi}, \text{ and therefore } \theta + \phi = \pi,$$

or the trapezium is such as may be inscribed in a circle; if therefore $s = a + b + c + d$, we shall get the *maximum* area

$$\text{or} \quad u = \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}.$$

Simpson's *Fluxions*, Vol. I. p. 39.

(52). To draw the chord PP' parallel to the diameter AB, so that the triangle $PP'Q$ may be the greatest possible. Fig. 11.

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* Inst. Calc. Diff. Pars II. Art. 279.

If $AP = \theta$, we shall find $\cos\theta = .781$ nearly, or $\theta =$ $38^\circ\ 40'$, when u is a *maximum.*

(53). To inscribe the greatest rectangle in a segment of a circle. Fig. 12.

Let $CD = a$, $AD = b$, $CA = r$ and $MP = x$: if

$$x = \left\{\frac{\sqrt{(9r^2 - b^2)} - 3a}{4}\right\}, \text{ then}$$

$$u = \left\{\frac{\sqrt{(9r^2 - b^2)} - 3a}{4}\right\} \sqrt{\left\{2r^2 + \frac{a}{2}\{\sqrt{(9r^2 - b^2)} - a\}\right\}},$$

a *maximum.*

If $a = 0$, and therefore $b = r$, we find

$$x = \frac{r}{\sqrt{2}} \text{ and } u = r^2.$$

(54). Given the length of a circular arc, to find what portion of a circle it must be, so that the corresponding segment may be a *maximum.* Fig. 13.

Let $a =$ arc QDQ' and $CD = x$: the segment QDQ', or

$$u = ax - x^2 \sin\frac{a}{x} \cos\frac{a}{x}.$$

If $x = \dfrac{2a}{\pi}$, or if the segment be a semi-circle, then

$$u = \frac{2a^2}{\pi}, \text{ a } maximum.$$

If $x = \infty$, $u = 0$, a *minimum.*

A solution of this problem is given in Lib. 5. Prop. 16. of the Mathematical Collections of Pappus Alexandrinus.

In the same manner, we should find that a hemisphere includes the greatest segment of a sphere, corresponding to a given spherical surface.

(55). To find the point in the line joining the centers of two spheres, from which the greatest portion of spherical surface is visible.

Let r and r' be the radii of the spheres A and A', d the distance of their centers, and x the distance of the required point from the center of A: then if

$$x = \frac{r^{\frac{3}{2}} d}{r^{\frac{3}{2}} + r'^{\frac{3}{2}}},$$

$$u = 2\pi \left\{ r^2 + r'^2 - \frac{(r^{\frac{3}{2}} + r'^{\frac{3}{2}})}{d} \right\}^2, \text{ a } maximum.$$

(56). To find the point P in the line CD, so that if A and B be given points, we shall have $m \,.\, AP + n \,.\, BP$ a *minimum*. Fig. 14.

If $AQ = a$, $BQ' = b$, $QQ' = c$ and $QP = x$: then the equation to determine the values of x, is

$$x^4 - c x^3 + \left(c^2 + \frac{m^2 b^2 - n^2 a^2}{m^2 - n^2}\right) x^2 - \frac{2 n^2 a^2 c}{m^2 - n^2} x + \frac{n^2 a^2 c^2}{m^2 - n^2} = 0.$$

If $m = n$, this equation becomes

$$x^2 + \frac{2 a^2 c}{a^2 - b^2} x + \frac{a^2 c}{a^2 - b^2} = 0.$$

Cresswell's *Maxima* and *Minima*, p. 21 and 222.

In a paper by Leibnitz in the *Acta Eruditorum* of Leipsic for 1684, which contained the first *published* explanation of the principles of the Differential Calculus, a solution is given of the following problem. "If A and B be two points situated in two different and uniform media, which are separated by the line CD and whose densities are in the ratio of m to n; to find the path, in which light must move from A to B in the shortest possible time, its velocity being proportional to the density of the medium." This problem is evidently identical with the one, whose solution we have given, and is remarkable as involving the principle of the "*least action*," which afterwards became so celebrated in the hands of Maupertuis and Euler.

(57). Of all cones, whose convex surface is given, to find that whose solid content is a *maximum*.

If s = convex surface of cone, x = radius of base, then if

$$x = \sqrt{\left(\frac{s}{\pi\sqrt{3}}\right)}, \quad u = \frac{s}{3}\sqrt{\left(\frac{2s}{3\pi}\right)}, \text{ a } maximum.$$

Consequently the *maximum* cone is generated by the revolution of a right-angled triangle, whose vertical angle is 35° 16′, nearly

(58). To inscribe the greatest cone in a sphere. Fig. 15.

If $AC = r$, and $CP = x$: then if $x = \frac{r}{3}$, $u = \frac{32\pi r^3}{81}$, a *maximum*.

(59). To cut the greatest ellipse from a given cone. Fig. 16.

Let $AC = a$, $CD = b$, $CP = x$: if

$$x = \frac{2b(a^2 - b^2) \pm b\sqrt{(a^4 - 14a^2b^2 + b^4)}}{3(a^2 + b^2)},$$

then u will be a *minimum* or a *maximum*, according as the upper or lower sign prevails.

If $a^4 - 14a^2b^2 + b^4$ be negative, or if $\frac{a}{b}$ be less than $2 + \sqrt{3}$, or if the angle CAD be greater than 15°, the values of x are imaginary, and the area of the section increases continually from the vertex to the base of the cone and admits not of a *maximum* or a *minimum*.

The *analytical maximum* of the sections of the cone, determined in virtue of the definition, may be less than the absolute *maximum*, a circumstance which we have had occasion to notice on other occasions: thus the base of the cone will be the greatest section, whenever the vertical angle CAD exceeds 11° 57′. Simpson's *Fluxions*, Vol. I. p. 38.

(60). From a given point within or without a parabola, to draw the shortest line to the curve. Fig. 17.

Let B be the given point, and assume $AC=a$, $CB=b$, the latus rectum $=l$ and $Bq=x$: the equation which determines the values of x is

$$(2x-l)\sqrt{(a-x)}+b\sqrt{l}=0;$$

from which expression, it is easily shewn that the shortest line is the normal passing through B.

If $b=0$, the values of x which are given by this equation are $\frac{l}{2}$ and a, the first of which only, makes u a *minimum* in the sense of the definition. The other value gives a *minimum* of the second species, such as was remarked in Ex. 50.

(61). To find the least parabola which can circumscribe a given circle. Fig. 18.

Let $BC=r$, and $CP=x$: if $x=\frac{r}{2}$, or if the latus rectum of the parabola be equal to the radius of the circle, then $u=\frac{9r^2}{4}$, is a *minimum*.

(62). To inscribe the greatest parabola, in a given triangle. Fig. 19.

Let $DB=DC=b$, $AD=a$, the angle $ADC=\theta$, and

$AP = x$: if $x = \frac{a}{2}$ or if the point of contact bisect each side of the triangle, then $u = \frac{1}{4} a\, b \sin\theta \sqrt{3}$, a *maximum*. This gives also the solution of the converse problem: "To describe the least triangle round a given segment of a parabola."

(63). In a given triangle, to inscribe the greatest ellipse. Fig. 20.

Let $BD = DC = b$, $AD = a$, the angle $ADC = \theta$, and $Da = x$: if

$$x = \frac{2a}{3},\quad u = \frac{\pi\, a\, b \sin\theta}{3\sqrt{3}},\ \text{a maximum.}$$

The sides of the triangle are bisected in the points of contact M and M'.

If a' and b' be the semi-axis major and semi-axis minor of the inscribed ellipse, we shall find

$$a' = \frac{1}{6}\sqrt{\{a^2 + 2\sqrt{3}\, a\, b \sin\theta + 3b^2\}}$$

$$+ \frac{1}{6}\sqrt{\{a^2 - 2\sqrt{3}\, a\, b \sin\theta + 3b^2\}}$$

$$b' = \frac{1}{6}\sqrt{\{a^2 + 2\sqrt{3}\, a\, b \sin\theta + 3b^2\}}$$

$$- \frac{1}{6}\sqrt{\{a^2 - 2\sqrt{3}\, a\, b \sin\theta + 3b^2\}}.$$

(64). To determine the least triangle which can circumscribe any segment of an ellipse. Fig. 21.

Let APB be the elliptic segment; assume $CP = a$, CD the conjugate diameter $= b$, $CE = c$, and $CQ = x$: if

$$x = \frac{c}{4} + \frac{\sqrt{(c^2 + 8a^2)}}{4},$$

$$u = \frac{b \sin\theta\, \{4a^2 - c^2 - c\sqrt{(c^2 + 8a^2)}\}^{\frac{3}{2}}}{\sqrt{2}\, a\, \{c + \sqrt{(c^2 + 8a^2)}\}},\ \text{a minimum.}$$

The other value of x, which is $\frac{c}{4} - \frac{\sqrt{(c^2 + 8a^2)}}{4}$, corresponds to the minimum value of the triangle which can be circumscribed round the other segment of the ellipse. If $c = 0$, $x = \pm \frac{a}{\sqrt{2}}$, which determines the least triangle which can circumscribe a semi-ellipse, cut off by any diameter.

(65). To find the point in an ellipse, where the portion of the tangent, intercepted by a perpendicular from the center, is a *maximum*. Fig. 22.

Let $CA = a$, $CB = b$, and $CD = x$: if $x = \sqrt{(ab)}$,

$$u = a - b, \text{ a } maximum.$$

(66) To find the conjugate diameters of an ellipse, which make the least angle with each other.

Let a and b be the axes of the ellipse, θ the angle which the conjugate diameters make with each other, and x one of these diameters: then if $x = \sqrt{\left(\frac{a^2 + b^2}{2}\right)}$ or if the diameters be equal to each other,

$$u = \sin\theta = \frac{2ab}{a^2 + b^2}, \text{ is a } minimum.$$

(67). From a given point in the periphery of an ellipse, to draw the longest line to the periphery again. Fig. 23.

Let $AC = a$, $CB = b$, $a^2 - b^2 = a^2 e^2$, $CM = c$, $PM = d$, and $CM = x$: the equation, from which the values of x are determined, which make u a *maximum* or a *minimum*, is $e^2 x^4 - 2ce^2 x^3 + (c^2 + d^2 - d^2 e^2 - a^2 e^4) x^2 - 2a^2 c e^2 x - a^2 c^2 = 0.$

If $c = 0$, and therefore P be in the extremity of the axis-minor, we shall find

$$x = \pm \frac{\sqrt{(2b^2 - a^2)}}{e}.$$

from which the two points on each side of the center of the ellipse are determined, through which the longest lines are to be drawn.

If $2b^2 = a^2$, the longest line is the semi-axis minor; and when $2b^2 < a^2$, the values of x become imaginary, and the line admits not of a *maximum* in the sense of the definition.

(68). To inscribe the greatest ellipse in a semi-circle. Fig. 23.

Let $AC = r$, $CP = x$: if $x = \frac{r}{\sqrt{2}}$,

$$u = \frac{2\pi r^2}{3\sqrt{3}}, \text{ a } maximum.$$

(69). To find the least ellipse which can circumscribe a given trapezoid. Fig. 24.

Let AB be parallel to DC, and draw CP bisecting AB and CK conjugate diameters. Assume $AQ = a$, $DQ' = a'$, $QQ' = c$, the angle $PCK = \theta$, $CP = y$, $CK = y'$, and $CQ = x$.

The general property of the ellipse gives us

$$a^2 = \frac{y'^2}{y^2}(y^2 - x^2),\ a'^2 = \frac{y'^2}{y^2}\{y^2 - (c-x)^2\};$$

and by making $\frac{a'^2 - a^2}{c} = e$, and $\frac{y^2}{y'^2} = z$, we shall have

$$x = \frac{ez + c}{2}: \text{ from whence we find}$$

$$y' = \frac{\sqrt{\{c^2 + e^2z^2 + z(4a^2 + 2ec)\}}}{2\sqrt{z}}$$

and $$y = \frac{\sqrt{\{c^2 + e^2z^2 + z(4a^2 + 2ec)\}}}{2}.$$

Consequently, if

$$z = \frac{\{ -(a'^2+a^2)+2\sqrt{(a'^4-a'^2a^2+a^4)} \} c^2}{3(a'^2-a^2)^2},$$

or if

$$x = \frac{\{ a'^2-2a^2+\sqrt{(a'^4-a'^2a^2+a^4)} \} c}{3(a'^2-a^2)}$$

then $u = \pi \sin\theta\, y y'$

$$= \frac{2\pi \sin\theta\, c \{ a'^4-4a'^2a^2+a^4+(a'^2+a^2)\sqrt{(a'^4-a'^2a^2+a^4)} \}}{\sqrt{3}(a'^2-a^2)\sqrt{\{ -(a'^2+a^2)+2\sqrt{(a'^4-a'^2a^2+a^4)} \}}},$$

a *minimum.*

It is not necessary to notice the negative value of the radical in the expression for z, since in that case z or $\frac{y^2}{y'^2}$ would be negative.

If $a' = a$, or if the trapezoid become a parallelogram, the expression for u given above, becomes $= \frac{0}{0}$: it will be necessary therefore to have recourse to the method explained in Art. 56., in order to determine its value, by which we shall find $u = \pi \sin\theta\, a c$. Bossut. *Calcul. Diff.* p. 190.

(70). To describe about a given trapezoid, an ellipse which is the least different from a circle. Fig. 24.

Making the same suppositions as in the preceding example, and assuming α and β to represent the two semi-axes of the ellipse, we shall find

$$u = \frac{\alpha}{\beta} = \frac{\sqrt{(1+z+2\sin\theta\sqrt{z})}+\sqrt{(1+z-2\sin\theta\sqrt{z})}}{\sqrt{(1+z+2\sin\theta\sqrt{z})}-\sqrt{(1+z-2\sin\theta\sqrt{z})}}$$

which is a *minimum*, when $z=1$, or when the conjugate diameters y and y' are equal to each other. Bossut. *Id.* p. 192.

Bossut, in the work just referred to, has indicated the steps of the process of solution of the two last problems, when the ellipses are to be described round any quadrilateral figure whatever: the reader may likewise consult a very elegant solution of the first of these problems, which is given by Euler in the *Petersburg Acts* for 1780.

On the Values of Functions which under certain circumstances become $\frac{0}{0}$.

ART. 32--53. (1). Let $u = \frac{a - \sqrt{\{a^2 - x^2\}}}{x^2}$: to find its value, when $x = 0$.

After two differentiations of the numerator and denominator of this fraction, we find, when $x = 0$, $u = \frac{1}{2a}$.

Euler. *Inst. Calc. Diff.* Pars II. Art. 358.

(2). Let $u = \frac{\sqrt{\{a^2 + ax + x^2\}} - \sqrt{\{a^2 - ax + x^2\}}}{\sqrt{\{a + x\}} - \sqrt{\{a - x\}}}$:

when $x = 0$, $u = \sqrt{a}$.

Euler. *Id.*

(3). Let $u = \frac{x^5 - 4x^4 + 8x^3 - 16x^2 + 16x}{x^4 - 6x^3 + 13x^2 - 12x + 4}$:

when $x = 2$, $u = 16$.

(4). Let $u = \frac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{(2ax - x^2)}}{x^2 - 2ax - a^2 + 2a\sqrt{(2ax - x^2)}}$:

when $x=a$, we find, after four differentiations of the numerator and denominator, $u=-5a$. Euler. *Id.*

(5). Let $u = \frac{a\sqrt{(ax)}-x^2}{a-\sqrt{(ax)}}$:

$$\text{when} \quad x=a, \quad u=\frac{0}{0}=3a.$$

Joh. Bernouilli. *Acta Eruditorum*, 1704.

(6). Let $u = \frac{\sqrt{(2a^3x-x^4)}-a\sqrt[3]{(a^2x)}}{a-\sqrt[4]{(ax^3)}}$:

$$\text{when} \quad x=a, \quad u=\frac{0}{0}=\frac{16a}{9}.$$

Joh. Bernouilli. *Id.*

(7). Let $u = \frac{a\sqrt[3]{(4a^3+4x^3)}-ax-a^2}{\sqrt{\{2a^2+2x^2\}}-x-a}$:

$$\text{when} \quad x=a, \quad u=\frac{0}{0}=2a.$$

Joh. Bernouilli. *Id.*

(8). Let $u = \frac{a+\sqrt{(2a^2-2ax)}-\sqrt{(2ax-x^2)}}{a-x+\sqrt{(a^2-x^2)}}$.

$$\text{when} \quad x=a, \quad u=\frac{0}{0}=1.$$

Euler. *Id.*

(9). Let $u = \frac{x^4+ax^3-9a^2x^2+11a^3x-4a^4}{x^4-ax^3-3a^2x^2+5a^3x-2a^4}$:

$$\text{when} \quad x=a, \quad u=\frac{0}{0}=\frac{5}{3}.$$

(10). Let $u = \frac{x\sqrt{(3a^3x-2x^4)} - ax\sqrt[5]{(a^4x)}}{a-\sqrt[3]{(ax^2)}}$:

when $x = a$, $u = \frac{0}{0} = \frac{81}{20}a^2$.

(11). Let $u = \frac{x - x^{2n+1}}{1-x^2}$:

when $x = 1$, $u = \frac{0}{0} = n$.

Euler. *Id.* Art. 360.

(12). Let $u = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$:

when $x = 1$, $u = \frac{0}{0} = \frac{n(n+1)}{1.2}$.

Euler. *Id.*

(13). Let $u = \frac{x+x^3-(2n+1)x^{2n+1}+(2n-1)x^{2n+3}}{(1-x^2)^2}$:

when $x = 1$, $u = \frac{0}{0} = n^2$.

Euler. *Id.*

4.) Let $u = \frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3}$:

when $x = 1$, $u = \frac{0}{0} = \frac{n(n+1)(2n+1)}{1.2.3}$.

This is the sum of the series $1^2+2^2+3^2+\dots+n^2$.

Euler. *Id.*

(15). Let $u = \frac{x^m - x^{m+n}}{1-x^{2p}}$:

when $x = 1$, $u = \frac{0}{0} = \frac{n}{2p}$.

Euler. *Id.*

(16). Let $u = \dfrac{a^n - x^n}{\log a - \log x}$:

when $x = a$, $u = \dfrac{0}{0} = n a^n$.

Euler. *Id.*

(17). Let $u = \dfrac{\log x}{\sqrt{(1 - x)}}$:

when $x = 1$, $u = \dfrac{0}{0} = 0$.

Euler. *Id.*

(18). Let $u = \dfrac{\epsilon^x - \epsilon^{-x}}{\log (1 + x)}$:

when $x = 0$, $u = \dfrac{0}{0} = 2$.

Euler. *Id.*

(19). Let $u = \dfrac{\epsilon^x - 1 + \log (1 + x)}{x^2}$:

when $x = 0$, $u = \dfrac{0}{0} = 1$.

Euler. *Id.*

(20). Let $u = \dfrac{x\,\epsilon^{2x} + x\,\epsilon^x - 2\,\epsilon^{2x} + 2\,\epsilon^x}{(\epsilon^x - 1)^3}$:

when $x = 0$, $u = \dfrac{0}{0} = \dfrac{1}{6}$.

(21). Let $u = \left(1 - \dfrac{2x}{\pi}\right) \tan x = \dfrac{1 - \dfrac{2x}{\pi}}{\cot x}$:

when $x = \dfrac{\pi}{2}$, $u = \dfrac{0}{0} = \dfrac{2}{\pi}$.

(22). Let $u = \frac{x}{x-1} - \frac{1}{\log x} = \frac{x \log x - x + 1}{(x-1) \log x}$:

$$\text{when} \quad x = 1, \quad u = \frac{0}{0} = \frac{1}{2}.$$

Euler. *Id.* Art. 365.

(23). Let $u = \frac{1}{1-x} - \frac{2}{1-x^2}$:

$$\text{when} \quad x = 1, \quad u = \infty - \infty = \frac{0}{0} = -\frac{1}{2}.$$

Euler. *Id.*

Whenever the numerator and denominator of a fraction admit not of developement according to integral and positive values of the increment of x or the base of the function, which is frequently the case when a particular value of x is assumed (Note *F*, and Art. 55), we must have recourse to the direct method which is explained in Art. 56: a process, which in many other cases will enable us to determine the values of fractions under the circumstances now under consideration, much more readily than by means of the successive differentiation of the numerator and denominator.

(24). Let $u = \frac{(x^2 - a^2)^{\frac{5}{2}}}{(x-a)^{\frac{5}{2}}}$:

$$\text{when} \quad x = a, \quad u = \frac{0}{0} = 4 \sqrt{2}\, a^{\frac{5}{2}}.$$

(25). Let $u = \frac{(x^2 + a^2)(x^2 - a^2)^{\frac{3}{2}}}{(3x^3 - a^2 x)(x^3 - a^3)^{\frac{3}{2}}}$:

$$\text{when} \quad x = a, \quad u = \frac{0}{0} = \frac{2\sqrt{2}}{3\sqrt{3}\, a^{\frac{5}{2}}}.$$

(26). Let $u = \dfrac{(a^2-x^2)^{\frac{1}{3}}+(a-x)^{\frac{2}{3}}}{(a-x)^{\frac{1}{3}}-(a^3-x^3)^{\frac{1}{3}}}$:

when $x = a$, $u = \dfrac{0}{0} = \dfrac{\sqrt[3]{(2\,a)}}{1-\sqrt[3]{(3\,a^2)}}$.

(27). Let $u = \dfrac{\pi\,x-1}{2\,x^2} + \dfrac{\pi}{x(\epsilon^{2\pi x}-1)}$, which expresses the sum of the series,

$$\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\text{\&c. } \textit{in infinitum.}$$

If we reduce u to the form

$$\frac{\pi\,x\,\epsilon^{2\pi x}-\epsilon^{2\pi x}+\pi\,x+1}{2\,x^2\,\epsilon^{2\pi x}-2\,x^2},$$

we shall find, when $x=0$, $u = \dfrac{0}{0} = \dfrac{d^3\,v}{d^3\,v'} = \dfrac{\pi^2}{6}$, a result which may be more readily deduced by the method given in Art. 56. Euler. *Id.*

(28). Let $u = \dfrac{1}{2\,x^2} - \dfrac{\pi}{2\,x\tan\pi\,x}$, which is the sum of the series

$$\frac{1}{1^2-x^2}+\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}+\text{\&c. } \textit{in infinitum.}$$

When $x=0$, $u=\infty-\infty=\dfrac{0}{0}=\dfrac{\pi^2}{6}$, the same result as in the last example.

If we make $u = \dfrac{1}{2\,x^2} - \dfrac{\pi}{2\,x\tan\pi\,x} - \dfrac{1}{1-x^2}$, which is the sum of the same series, without the first term, we shall find, when $x = 1$,

$$u = \frac{1}{2} + \infty - \infty = \frac{1}{2} + \frac{0}{0} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

which is the sum of the series

$$\frac{1}{1\,.\,3} + \frac{1}{2\,.\,4} + \frac{1}{3\,.\,5} + \frac{1}{4\,.\,6} + \text{\&c. } \textit{in infinitum.}$$

In the same manner, if we make

$$u = \frac{1}{2\,x^2} - \frac{\pi}{2\,x \tan \pi\, x} - \frac{1}{n^2 - x^2},$$

which is a sum of the same series, omitting the term $\frac{1}{n^2 - x^2}$, we shall find,

$$\text{when} \quad x = n,\ u = \frac{1}{2\,n^2} + \infty - \infty = \frac{1}{2\,n^2} + \frac{1}{4\,n^2} = \frac{3}{4\,n^2}.$$

Euler. *Id.*

(29). Let $u = \frac{\pi}{4\,x} + \frac{\pi}{2\,x\,(\epsilon^{\pi x} - 1)}$, which is the sum of the series

$$\frac{1}{1^2 + x^2} + \frac{1}{3^2 + x^2} + \frac{1}{5^2 + x^2} + \text{\&c. } \textit{in infinitum}:$$

$$\text{when} \quad x = 0, \quad u = \infty - \infty = \frac{0}{0} = \frac{\pi^2}{8}.$$

(30). Let $u = \dfrac{\pi \sin \frac{\pi\, x}{2}}{4\,x \cos \frac{\pi\, x}{2}}$, which is the sum of the series

$$\frac{1}{1^2 - x^2} + \frac{1}{3^2 - x^2} + \frac{1}{5^2 - x^2} + \text{\&c. } \textit{in infinitum}:$$

$$\text{when} \quad x=0, \quad u = \frac{0}{0} = \frac{\pi^2}{8},$$

the same result as in the last example.

If we make $u = \dfrac{\pi \sin \dfrac{\pi x}{2}}{4\, x \cos \dfrac{\pi x}{2}} - \dfrac{1}{1-x^2}$, which is the sum of the same series, without its first term, we shall find, when $x=1$, $u = \infty - \infty = \frac{1}{4}$, which is the sum of the series

$$\frac{1}{2\,.\,4} + \frac{1}{4\,.\,6} + \frac{1}{6\,.\,8} + \&\text{c. } \textit{in infinitum.}$$

Euler. *Id.*

On the Tangents and Asymptotes of Curve Lines.

(1). Let the curve be the hyperbola, referred to its asymptote, whose equation is

$$y = \frac{a\,b}{x}.$$

(*a*). Sub-tangent $PT = -x$, or the abscissa and sub-tangent are equal to each other, but on different sides of the ordinate.

(*b*). Equation to the tangent is (Art. 67.)

$$y' = -\frac{a\,b}{x^2}\, x',$$

where x' and y' are co-ordinates to the tangent, reckoned from the same point and referred to the same axis, as the co-ordinates of the curve.

(*c*). Equation to the normal is (Art. 67.)

$$a\,b\,x\,y'' = x^3\,x'' + a^2\,b^2 - x^4,$$

where x'' and y'' are co-ordinates to the normal.

(2). Let the curve be the Cissoid of Diocles, (Wood's *Algebra*, Art. 496), whose equation is

$$y^2 = \frac{x^3}{a - x}.$$

(*a*). $PT = \dfrac{2\,x\,(a - x)}{3\,a - 2\,x}.$

(*b*). Equation to the tangent

$$y' = \frac{x^{\frac{1}{2}}}{2\,(a - x)^{\frac{3}{2}}}\{(3\,a - 2\,x)\,x' - a\,x\}.$$

(*c*). If $x = a$, y' becomes infinite and coincides with the tangent, or in other words is an asymptote to the curve. This may likewise be shewn by the method given in Art. 70.

(3). Let the curve be the Conchoid of Nicomedes (Wood's *Algebra*, Art. 497), whose equation is

$$(a + x)^2\,(b^2 - x^2) = x^2\,y^2.$$

(*a*). $PT = -\dfrac{x\,(a + x)\,(b^2 - x^2)}{x^3 + a\,b^2}.$

(*b*). Equation to the tangent

$$y' = -\frac{1}{x^2\sqrt{(b^2 - x^2)}}\{(x^3 + a\,b^2)\,x' - 2\,a\,b^2\,x - b^2\,x^2 + a\,x^3\}.$$

If $x=0$, y' becomes infinite and coincides with the tangent, and is consequently an asymptote to the curve.

(4). Let the curve be the *Witch* (Agnesi's *Analytical Institutions*, Vol. I. Art. 242.), whose equation is

$$y = \frac{a\sqrt{(a x - x^2)}}{x}.$$

(*a*). $PT = -\dfrac{2(a x - x^2)}{a}.$

(*b*). Equation to the tangent

$$y' = -\frac{a}{2 x\sqrt{(a x - x^2)}}\{a x' + 2 x^2 - 3 a x\}.$$

(*c*). Equation to the normal

$$y'' = \frac{2 x \sqrt{(a x - x^2)}}{a^2}\left\{x'' - x + \frac{a^3}{2 x^2}\right\}.$$

(*d*). A line passing through the origin of the abscissæ, at right angles to the axis, is an asymptote to the curve.

(5). Let the equation of the curve be

$$y^3 + x^2 y + a y^2 - a x^2 = 0.$$

(*a*). $PT = \dfrac{y(3y^2 + 2 a y + x^2)}{2x(a - y)}$: in this case; $\dfrac{dy}{dx}$ and also $\dfrac{y\,dx}{dy}$, becomes $\dfrac{0}{0}$, when $x=0$, and $y=0$, or at the origin of the co-ordinates: this indicates a *multiple point* of the curve, the meaning and nature of which will be more fully explained by the examples given in the next section, on the "*singular or remarkable points of curve lines.*"

(6). Let the equation of the curve be

$$y^3 = a x^2 + x^3.$$

(*a*). A line whose equation is,

$$y' = x' + \frac{a}{3},$$

is an asymptote to the curve: this may be constructed as follows: make $AR = AE = \frac{a}{3}$: then the line passing through R and E is the asymptote required. Vince's *Fluxions*, p. 52.

(7). Let the curve be the hyperbola referred to its axis, whose equation is

$$y^2 = \frac{b^2}{a^2} (2 a x + x^2).$$

(*a*). The linear equation

$$y' = \pm \frac{b}{a} \{ x' + a)$$

determines the position of the two asymptotes of the curve.

(8). Let the equation of the curve be

$$a x^4 - b y^4 + c^3 x y = 0.$$

(*a*). This curve admits of an asymptote, whose equation is

$$y' = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x'.$$ Vince's *Fluxions*, p. 53.

If we investigate series for y in terms of x, by means of the theorem of Lagrange, we shall find one of them, involving descending powers of x, which is

$$y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x + \frac{c^3}{4\, a^{\frac{1}{2}}\, b^{\frac{1}{2}}\, x} - \frac{c^6}{4^2 . 1 . 2 . b^{\frac{3}{4}}\, a^{\frac{5}{4}}\, x^3} + \&c.$$

If we confine ourselves to the first term, we shall have a linear equation, which determines the position of the rectilinear asymptote of the curve. If we include two terms of the series, we shall have the equation of a *curve*, which may be considered as an asymptote to the original curve, and which approximates more nearly to it than the rectilinear asymptote which has just been determined. The same is true, of the curves which arise from including three, four, or a greater number of terms of the series, in the equation of the asymptotic curve. This method, which admits of general application, was given by Stirling, in a Treatise in illustration of Newton's classification of lines of the third order, which was published in 1717.

(9). Let the equation of the curve be

$$a x^3 + x^3 y - a y^3 = 0.$$

In this case

$$y = -a - \frac{a^4}{x^3} - \frac{3\, a^7}{x^6} - \&c.$$

and $y = -a$, is the equation to the rectilinear asymptote of the curve: it is not necessary to explain its construction.

(10). Let the equation of the curve be

$$y^3 - 2 x y^2 + x^2 y - a^3 = 0. \quad \text{Fig. 22.}$$

The series which involve descending powers of x are

$$(1). \quad y = x + \frac{a^2}{\sqrt{(a x)}} - \frac{a^3}{2 . x^2} + \frac{7 a^4}{2^2 . 1 . 2 . x^2 \sqrt{(a x)}} - \&c.$$

$$(2).\quad y = x - \frac{a^2}{\sqrt{(a\,x)}} - \frac{a^3}{2\,x^2} - \frac{7\,a^4}{2^2\,.\,1\,.\,2\,.\,x^2\sqrt{(a\,x)}} - \&c.$$

$$(3).\quad y = \frac{a}{x^2} + \frac{2\,a^6}{x^5} + \frac{7\,a^3}{x^8} + \&c.$$

Consequently $y = x$, and $y = 0$, are the equations to two asymptotes, one of which AD passes through the origin A of the co-ordinates, making an angle of 45° with the axis, and the other is co-incident with the axis. Stirling, *Lineæ Tertii Ordinis Newtonianæ*. p. 29, 51, 128.

(11). Let $x\,y^2 - e\,y = a\,x^3 + b\,x^2 + c\,x + d$, be the equation of the curve. Fig. 23.

The series for y are

$$(1).\quad y = x\sqrt{a} + \frac{b}{2\sqrt{a}} + \frac{4\,a\,c - b^2 + 4\,a\,c\sqrt{a}}{8\,a\,x\sqrt{a}} + \&c.$$

$$(2).\quad y = -\,x\sqrt{a} - \frac{b}{2\sqrt{a}} + \frac{4\,a\,c - b^2 - 4\,a\,c\sqrt{a}}{8\,a\,x\sqrt{a}} - \&c.$$

$$(3).\quad y = \frac{e}{x} + \frac{d}{e} + \frac{c\,x}{e} + \frac{b\,x^2}{e} + \&c.$$

$$(4).\quad y = -\frac{d}{e} - \frac{c\,x}{e} - \frac{b\,x^2}{e} - \&c.$$

The equations $y = x\sqrt{a} + \frac{b}{2\sqrt{a}}$, and $y = -\,x\sqrt{a} - \frac{b}{2\sqrt{a}}$ represent two asymptotes, which cut the axis at a distance $AD = \frac{b}{2\,a}$, from the origin of the co-ordinates and make angles with the axis whose trigonometrical tangents are $\sqrt{a}$ and $-\sqrt{a}$. From the third series, it appears that y is infinite, when $x = 0$, and is likewise an asymptote to the curve at that point. These three asymptotes form a triangle $D\,d\,\delta$ of given species and magnitude: if $b = 0$, this triangle vanishes in a point A, and if $a = 0$, it coincides with the line AD.

If a be negative, the curve has only one asymptote, the equations for the two first involving imaginary quantities.

If $a=0$, and $b=0$, the two first series for y become

$$(1).\quad y = \sqrt{c} + \frac{e\sqrt{c}+d}{2x\sqrt{c}} + \&c.$$

$$(2).\quad y = -\sqrt{c} + \frac{e\sqrt{c}-d}{2x\sqrt{c}} - \&c.$$

and the equations for the two asymptotes, are $y=\sqrt{c}$, and $y=-\sqrt{c}$, which correspond to two lines drawn parallel to the axis, at a distance $=\sqrt{c}$, above and below it.

If $c=0$, these two asymptotes coincide with the axis, which is therefore an asymptote to the curve. Stirling, *Lin. Tert. Ord. Newt.* p. 87.

(12). Let $xy = ax^3+bx^2+cx+d$.

The ordinate passing through the origin of the coordinates, is an asymptote to the curve.

(13). Let $y^4-2x^2y^2-x^4+2ax^2y^2-5ax^3=0$, be the equation of the curve. Fig. 24.

The series for y are

$$(1).\quad y = \sqrt{(1+\sqrt{2})} + \frac{a(3\sqrt{2}-4)}{8\sqrt{(1+\sqrt{2})}} + \frac{A}{x} + \&c.$$

$$(2).\quad y = -\sqrt{(1+\sqrt{2})} - \frac{a(3\sqrt{2}-4)}{8\sqrt{(1+\sqrt{2})}} + \frac{A}{x} + \&c.$$

which furnish the equations to the two asymptotes of the curve.

(14). Let $y = \dfrac{ax^3+bx^3+cx^2+dx+e}{fx^3+gx^2+hx+k}$: Fig. 25.

The equation to the asymptote, determined by this method is

$$y = \frac{a x}{f} + \frac{b f - a g}{f^2}.$$

There are, however, in this case, other asymptotes whose equations cannot be determined by this developement: for if we make x equal to a root of the equation

$$f x^3 + g x^2 + h x + k = 0,$$

y becomes infinite and is an asymptote to the curve: and if all the roots of the equation be possible, there is an asymptote corresponding to each of them. Stirling, *Lin. Tert. Ord. Newt.* p. 53.

Newton, in his *Enumeratio Linearum Tertii Ordinis*, has shewn that all curve lines of the third order are comprehended under some one of these four equations;

(1). $x y^2 - e y = a x^3 + b x^2 + c x + d.$

(2). $x y = a x^3 + b x^2 + c x + d.$

(3). $y^2 = a x^3 + b x^2 + c x + d.$

(4). $y = a x^3 + b x^2 + c x + d,$

in which a, b, c, d, e may be positive, negative, or evanescent, excepting those cases in which the equation would thus become one of an inferior order of curves.

He distinguished 65 different species of curves comprehended under the first equation, which constitute eleven distinct classes: four new species were subsequently discovered by his commentator Stirling, and four more by De Gua*.

The first class contains 9 species, the second 14, the

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[*] *Usage de l'Analyse de Descartes.* 1740.

third 4, and the fourth 9: The curves comprehended in these four classes are named *redundant* hyperbolas, from their possessing six hyperbolic branches, a number greater than in the conical hyperbola. They admit of three asymptotes, which in the fourth class only, pass through the same point. In all these classes a is positive; in the second $e = 0$; in the third $e = 0$ and $b^2 = 4\,a\,c$, and in the fourth also $b = 0$.

The fifth class consists of 6 species, and the sixth of 7; each admitting of one asymptote; they are named *defective* hyperbolas from their possessing only two hyperbolic branches. In both these classes, a is negative, and in the second $e = 0$: amongst these is comprehended the Cissoid of Diocles.

The seventh class includes 7 species and the eighth 4 species, each possessing one asymptote and two hyperbolic and two parabolic branches, and therefore called *parabolic* hyperbolas: in both these classes $a = 0$, and in the second $e = 0$.

The ninth class includes 4 species, which are called *hyperbolisms* of the hyperbola, possessing six hyperbolic branches and three asymptotes, two of which are parallel and the third perpendicular to the axis: the tenth class consists of three species, called *hyperbolisms* of the ellipse, possessing one asymptote and two hyperbolic branches: the eleventh class consists of 2 species, called *hyperbolisms* of the parabola, each curve possessing two asymptotes and four hyperbolic branches: in these three classes $a = 0$, and $b = 0$, and c is positive in the first, negative in the second, and evanescent in the third.

The curves comprehended in this equation have one diameter when $e = 0$, and three diameters when $e = 0$ and $b^2 = 4\,a\,c$.

The second equation comprehends only one species of curves, to which Newton has given the name of *Trident*, possessing one asymptote, two hyperbolic and two parabolic branches. The third includes 5 species, each possessing two parabolic branches: amongst these, is the *semi-cubical parabola*, remarkable as the first curve whose rectification was effected. The fourth equation comprehends only one species of curves, commonly called the *cubical parabola*.

The Reader who is familiar with the description of curves from the properties of their algebraical equations and the determination of their asymptotes and *singular* points, will find little difficulty in distinguishing most of the different species of curves comprehended in these classes, particularly if assisted by Newton's Enumeration and the figures which accompany it: this enquiry, however, is sometimes attended with considerable difficulties, since some of the species escaped the scrutinizing accuracy of Newton, and some of them are probably yet unknown.

A similar classification of curves of the fourth order would comprehend more than 5000 species, a work of such immense labour and difficulty that no analyst has hitherto had the courage to undertake it.

Cramer* has given a classification of curves of different orders, by the consideration of the number and nature of their infinite branches, a division which is very simple and natural, but too general to be of much service: he has thus made five classes of curves of the third order, nine of curves of the fourth order and eleven of curves of the fifth order: the distribution of the curves comprehended in these classes, into genera would be attended with the same difficulties as Newton's enumeration of species.

This enumeration, however, though not destitute of interest, is of little real utility: for the properties of every curve are most readily deduced by direct methods from the particular equation, by which it is expressed, without reference to any general class of curves or any general equation. Our attention will be thus directed, as it ought to be, to curves which are remarkable for curious and instructive properties, or which result from the application of mathematics to physical problems.

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* *Introduction a l'Analyse des Lignes Curves Algebriques.* 1750.

On the Singular or Remarkable Points of Curve Lines.

Art. 77—93. (1). Let the equation of the curve be

$$a^2 y = x^3.$$

This curve, which is a cubic parabola, has a point of *inflexion* or *contrary flexure*, at the origin of the co-ordinates. Fig. 26.

(2). Let $y^2 = \frac{a^2 (a x - x^2)}{x^2}$, (Ex. 5. p. 144.)

If $x = \frac{3a}{4}$, and therefore $y = \pm \frac{a}{\sqrt{3}}$, there is a point of inflexion corresponding to each of these points in the two similar branches of the curve.

(3). Let $x y^2 = c x - d$.

If $x = \frac{3d}{4c}$, and $y = \pm \sqrt{\frac{-c}{5}}$, there is a point of inflexion corresponding to each of these points. Fig. 27.

This curve is termed an *hyperbolism* of an ellipse, and constitutes the 63d of Newton's and the 67th of Stirling's enumeration of lines of the third order.

(4). Let $x^3 - axy - b^2y = 0$.

A point of *inflexion*, when $x=0$ and $y=0$. Fig. 28.

This curve belongs to the class of curves, to which Newton has given the name of *Trident*.

(5). Let $ax^3+by^3+c^4=0$. Fig. 29.

Two points of *inflexion*, one corresponding to $x=0$, and $y = -c\sqrt[3]{\frac{c}{b}}$, and the other to $x = -c\sqrt[3]{\frac{c}{a}}$ and $y = 0$. Cramer. *Lignes Curves*. Art. 198.

(6). Let $xy^2 + ax^2 + b^3 = 0$.

If a and b have the same sign, there are two points of *inflexion*, corresponding to $x = \pm b\sqrt{\left\{(3+\sqrt{12})\frac{b}{a}\right\}}$; and if a and b have different signs, these two points correspond to $x = \pm b\sqrt{\left\{(3 - \sqrt{12})\frac{b}{a}\right\}}$. Cramer. *Id.* Art. 199. Newton *Enum. Lin. Tert. Ord. Species* 53, 56.

(7). Let $x^4 - a^2x^2 + a^3y = 0$. Fig. 30.

There is a point of *inflexion* corresponding to each of the points determined by making $x = \pm\frac{a}{\sqrt{6}}$ and therefore $y = \frac{5a}{36}$. Cramer. *Id.* Art. 199.

(8). Let $a^3y = x^4$.

There is a point of *double inflexion* at the origin of the co-ordinates, which is likewise termed a point of *serpentement* or of *undulation* and may be considered as arising from the union of four points of the curve. Such points as these are points of inflexion, in an analytical rather than a geometrical sense, since the curve itself presents no visible character of a change in the nature of its curvature.

It will aid the student in conceiving the nature of this point, to consider the curve whose equation is

$$a^3 y = x^4 - (b^2 + c^2) x^2 + b^2 c^2 :$$

if we make b and c equal to nothing, the equation will coincide with that given above, and the four points corresponding to $x = b$, $x = -b$, $x = c$ and $x = -c$, will unite into one, which is a point of *undulation*. Fig. 31.

(9). Let $a^4 y = x^5$. Fig. 32.

If $x = 0$ and $y = 0$, there is a point of *triple inflexion*, which is coincident, in its geometrical characters, with a point of *simple inflexion*: it obviously arises from the union of five points of the curve.

(10). The corresponding point in the curve, whose equation is

$$a^5 y = x^6,$$

is one of *double undulation*. It is unnecessary to discuss the nature of this point in similar parabolic curves of higher orders.

(11). Let $(y^2 + x^2)^2 - 4 a y (x^2 + y^2 - 2 a y + 2 a^2) = 0$. Fig. 33.

A point of *double inflexion* or of *undulation*, at the origin of the co-ordinates. Cramer *Id.* Art. 186.

(12). Let there be a series of curves whose equations are

(1). $y = b + (x - a)^{\frac{3}{5}}$.

(2). $y = b + (x - a)^{\frac{5}{3}}$.

(3). $y = x + (x-a)^{\frac{5}{3}}$.

(4). $y = x + (x-a)^{\frac{7}{3}}$.

There is a point of *inflexion* in all these curves, corresponding to $x=a$.

In the first, the tangent at the point of *inflexion* A, is perpendicular to the axis. Fig. 34.

In the second, the tangent at A, is parallel to the axis. Fig. 35.

In the third and fourth, the tangent at A is inclined at an angle of 45° to the axis. Fig. 36.

(13). Let $a y^2 - x^3 - b x^2 = 0$. Fig. 37.

At the origin of the co-ordinates, we find $\frac{dy}{dx} = \pm \sqrt{\frac{b}{a}}$, which indicates a *node* or *double point.*

If $b=0$, the curve becomes the *semi-cubical parabola;* in this case, the two tangents coincide with each other and the *node* is changed into a *point* of *rebroussement* or cusp of the first kind.

If b be negative, the values of $\frac{dy}{dx}$ become imaginary, and this point is an insulated or *conjugate* point. Newton. *Enum. Lin. Tert. Ord. Species* 68, 70, 69.

(14). Let $y^4 - 8y^3 - 12xy^2 + 16y^2 + 48xy + 4x^2 - 64x = 0$. Fig. 38.

If $x=2$ and $y=24$, we find $\frac{dy}{dx} = \pm \sqrt{8}$, which indicates a *double point* arising from the intersection of two of the four branches of the curve.

This example was appealed to by Rolle, one of the earliest adversaries of the Differential Calculus, to prove that its processes sometimes lead to erroneous results: he made no distinction between the whole curve denoted by the given equation and that which corresponds to one of its factors $y-2-\sqrt{(4x)}-\sqrt{(4+2x)}=0$, and supposed that a fraction whose numerator and denominator are evanescent, must itself be evanescent. A full explanation of these difficulties was afterwards given by Saurin*.

(15). Let $x^4 - ayx^2 + by^3 = 0$. Fig. 39.

A *triple* point at the origin of the co-ordinates, arising from the intersection of three branches of the curve: the values of $\frac{dy}{dx}$ at this point, are $\pm\sqrt{\frac{b}{a}}$ and 0.

This is another of Rolle's instances, which was likewise explained by Saurin.

(16). Let $x^4 - 2\sqrt{2}\,ax^3 + 2a^2x^2 - ay^3 - a^2y^2 = 0$. Fig. 40.

If $x=0$, there is a *double point*, where $\frac{dy}{dx} = \pm\sqrt{2}$.

If $x = a\sqrt{2}$ and $y = 0$, there is a double point in every respect similar to the former. At both these points, there is an *inflexion* in that branch, whose tangent is determined by making $\frac{dy}{dx} = -\sqrt{2}$. Cramer. *Lignes Curves*. Art. 186.

(17). Let $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$. Fig. 41.

Three *double* points corresponding to $x = a$ and $y = 0$, $x = -a$ and $y = 0$, and $x = 0$ and $y = -a$: for the first two points $\frac{dy}{dx} = \pm\sqrt{\frac{2}{3}}$, and for the third $\frac{dy}{dx} = \pm\sqrt{\frac{4}{3}}$.

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* *Mémoires de l'Académie*. 1716, 1723.

(18). Let $x^4 + y^4 - 2ay^3 + 2bx^2y = 0$. Fig. 42.

A *triple point* or *double node* at the origin of the co-ordinates.

If $b = 0$, the inferior ovals disappear and vanish in a point A, which may be considered as an *invisible triple point.*

If $a = 0$, the superior oval disappears and the point A is a *triple point*, arising from the coincidence of two tangents and a third tangent coincident with the axis.

If the equation be $y^4 - x^4 + 2bxy^2 = 0$, the point A will be a *triple point* of a similar nature, though the form of the curve will be different. Fig. 43.

(19). Let $a^3y^2 = x^5$.

A *cusp* of the first kind, when $x = 0$ and $y = 0$.

(20). Let $(xy + 1)^2 + (x - 1)^3(x - 2) = 0$. Fig. 44.

If $x = 1$ and $y = -1$, there is a cusp of the first kind Newton. *Lin. Tert. Ord. Species* 35.

(21). Let $y = b + (x - a)^{\frac{5}{2}}$. Fig. 43.

If $x = a$ and $y = b$, there is a cusp of the first kind.

(22). Let $y = b + cx^2 + (x - a)^{\frac{5}{2}}$. Fig. 46.

If $x = a$ and $y = b + ca^2$, there is a cusp of the second kind.

(23). Let $y = \pm (x - a)^2 \sqrt{(x - b)} + c$. Fig. 47.

If $x = a$ and $y = c$, the two branches have a common tangent parallel to the axis, but no cusp; there is an oval between $x = a$, and $x = b$.

(24). Let $y = \pm(x-a)^3 \sqrt{(x-b)} + c$. Fig. 48.

If $x = a$, and $y = c$, the two branches have a common tangent and equal and similar curvature at their point of intersection, but no cusp; there is likewise an oval between $x=a$ and $x=b$.

In this, (as well as in the last example), we must suppose $a > b$; if $a < b$, the *double point* in question is, in both cases, a *conjugate point*.

(25). Let $y^5 + a x^4 - b^2 x y^2 = 0$. Fig. 49.

At the point A, the origin of the co-ordinates, there is an *inflection* of the infinite branch CAD and a *cusp* of the first kind: the point A is properly a *triple point*. Cramer. *Id.* Art. 221.

(26). Let $a^3 y^2 - 2 a b x^2 y - x^5 = 0$. Fig. 50.

At the origin of the co-ordinates, we find what is called an *oscu-inflexion*, or two branches of the curve touch each other at A, one of which experiences an *inflexion* at that point. Cramer. *Id.* Art. 220.

(27). Let $x^4 - a x^2 y - a x y^2 + \frac{1}{4} a^2 y^2 = 0$. Fig. 51.

At the origin of the co-ordinates, we find a *cusp* of the second species. Cramer. *Id.* Art. 220.

(28). Let $(x^2 + y^2)^3 - 4 a^2 x^2 y^2 = 0$. Fig. 52.

A *quadruple* point at the origin of the co-ordinates.

This curve is thus constructed: If a circle be described whose radius $AB = a$, and if AM be always taken equal to $a \sin 2\,BAM$, the point M will trace out the curve. Cramer. *Id.* Art. 170.

(29). Let $x y^2 + x^3 - (2 b + a) x^2 + (b^2 + 2 a b) x - a b^2 = 0$. Fig. 53.

A double point corresponding to $x = b$ and $y = 0$.

The construction of this curve is as follows: Take C any point in the diameter AB of a circle: draw the ordinate PN, join AN and draw CM parallel to AN: the point M traces out the curve. Cramer. *Ib.* Art. 174.

(30). Let $y^4+x^4-2a^2x^2-2a^2y^2+a^4=0$. Fig. 54.

Four *double points* corresponding to $x=a$ and $y=0$, $x=-a$ and $y=0$, $x=0$, and $y=a$, and $x=0$ and $y=-a$.

This curve arises from the intersections of two ovals, whose equations are

$$y^2+xy\sqrt{2}+x^2-a^2=0, \text{ and}$$

$$y^2-xy\sqrt{2}+x^2-a^2=0.$$

Cramer. *Ib.* 173.

The description of curves of higher orders than the fourth and the determination of their singular points, becomes, in most cases, exceedingly difficult, from the extreme complication of their forms and the difficulties attending the solution of equations of higher orders than the fourth, to which these investigations commonly lead. Curves of this kind, however, seldom occur in physical enquiries, and their discussion is therefore of little importance, unless as a subject of analytical curiosity.

We shall have occasion to notice other instances of singular points, in enumerating the principal properties of some particular curves which have become celebrated from historical or other circumstances. The reader who wishes for further information upon this subject, will find his curiosity amply gratified by the complete discussion of a great number of excellent examples, in the very elaborate work of Cramer to which we have so often referred.

On the Curvature of Curve Lines.

Art. 87—100. Let $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If we make $a^2 - b^2 = a^2 e^2$, we find the radius of the circle of curvature, or

$$\gamma = \frac{(a^2 - e^2 x^2)^{\frac{3}{2}}}{a b}.$$

If α and β be the co-ordinates of the center of the circle of curvature, we have

$$\alpha = \frac{e^2 x^3}{a^2}, \text{ and } \beta = -\frac{e^2}{a b}(a^2 - x^2)^{\frac{3}{2}};$$

and by eliminating α and β from these equations, we get

$$a^{\frac{2}{3}} \alpha^{\frac{2}{3}} + b^{\frac{2}{3}} \beta^{\frac{2}{3}} = a^{\frac{4}{3}} e^{\frac{4}{3}} = (a^2 - b^2)^{\frac{2}{3}};$$

$$\text{or making } \frac{a^2 - b^2}{a} = a', \quad \frac{a^2 - b^2}{b} = b',$$

$\alpha = x'$ and $\beta = y'$, the equation of the evolute becomes

$$\left(\frac{x'}{a'}\right)^{\frac{2}{3}} + \left(\frac{y'}{b'}\right)^{\frac{2}{3}} = 1.$$

The evolute has a cusp at E; also $CE = \frac{a^2 - b^2}{b}$,

$CQ = CQ' = \frac{a^2 - b^2}{a}$ and the arc $EQ = \frac{a^3 - b^3}{a b}$. Fig. 55.

In the case of the hyperbola the equation of the evolute is,

$$\left(\frac{x'}{a'}\right)^{\frac{2}{3}} - \left(\frac{y'}{b'}\right)^{\frac{2}{3}} = 1,$$

where $a' = \frac{a^2 + b^2}{a}$, and $b' = \frac{a^2 + b^2}{b}$.

(2). Let $x y = a^2$, or let the curve be the rectangular hyperbola referred to its asymptotes.

$$\gamma = \frac{-x^3}{2a^2}\left(1 + \frac{a^4}{x^4}\right)^{\frac{3}{2}},$$

$$\alpha = \frac{3x}{2} + \frac{a^4}{2x^3}, \quad \beta = \frac{3a^2}{2x} + \frac{x^3}{2a^2};$$

In order to determine the equation of the evolute, we find

$$\alpha + \beta = \frac{a}{2}\left\{\frac{a^3}{x^3} + \frac{x^3}{a^3} + 3\left(\frac{a}{x} + \frac{x}{a}\right)\right\} = \frac{a}{2}\left(\frac{a}{x} + \frac{x}{a}\right)^3,$$

$$\alpha - \beta = \frac{a}{2}\left\{\frac{a^3}{x^3} - \frac{x^3}{a^3} - 3\left(\frac{a}{x} - \frac{x}{a}\right)\right\} = \frac{a}{2}\left(\frac{a}{x} - \frac{x}{a}\right)^3;$$

consequently

$$\frac{2a}{x}\sqrt[3]{\frac{a}{2}} = \sqrt[3]{(\alpha + \beta)} + \sqrt[3]{(\alpha - \beta)}$$

$$\frac{2x}{a}\sqrt[3]{\frac{a}{2}} = \sqrt[3]{(\alpha + \beta)} - \sqrt[3]{(\alpha - \beta)},$$

and therefore

$$(4a)^{\frac{2}{3}} = (\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}}.$$

The evolute has a cusp at E; and two infinite branches EQ and EQ'. Fig. 56.

(3). Let $a^2 y = x^3$, the equation of the cubical parabola,

$$\gamma = -\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^2 x}.$$

This expression is a *minimum*, or the curvature is the greatest when

$$x = \frac{a}{\sqrt[4]{45}} \text{ and } y = \frac{a}{\sqrt[4]{91125}}.$$

Also $\alpha = \dfrac{x}{2} - \dfrac{9}{2}\cdot\dfrac{x^5}{a^4}, \quad \beta = \dfrac{a^2}{6x} + \dfrac{5x^3}{2a^2};$

The elimination of x from these equations, leads to a very complicated expression for the equation of the evolute, which does not seem to admit of reduction.

(4). Let $a y^2 = x^3$, which is the equation to the semi-cubical parabola:

$$\gamma = - \frac{4 a^{\frac{1}{2}} x^{\frac{1}{2}}}{3} \left(1 + \frac{9 x}{4 a}\right)^{\frac{3}{2}}.$$

If $x = 0$, $\gamma = 0$, or the curvature at the vertex is infinite.

$$\alpha = - x - \frac{9 x^2}{2 a},$$

$$\beta = 4 \left(x + \frac{a}{3}\right) \sqrt{\frac{x}{a}}.$$

The equation of the evolute is

$$\frac{6561}{256} \beta^4 + \frac{243}{4} a \alpha \beta^2 + \frac{9}{8} a^2 \beta^2$$
$$+ 72 a \alpha^3 + 24 a^2 \alpha^2 + 2 a^3 \alpha = 0.$$

(5). Let $a^{m-1} y = x^m$;

$$\gamma = - \frac{a^{m-1} \left\{ 1 + \frac{m^2 x^{2m-2}}{a^{2m-2}} \right\}^{\frac{3}{2}}}{m (m - 1) x^{m-2}}.$$

If $m > 2$, γ is infinite when $x = 0$, as in Example 3; the curve experiences an inflexion at this point.

If $m < 2$, $\gamma = 0$, when $x = 0$, and there is a cusp at that point.

(6). Let $y = \frac{a \sqrt{(a x - x^2)}}{x}$. (Ex. 4. p. 144.)

$$\gamma = - \frac{\{ a^4 + 4 a x^3 - 4 x^4 \}^{\frac{3}{2}}}{2 a^2 x^2 (3 a - 4 x)}.$$

If $x = 0$, $\gamma = \infty$: at this point $y = \infty$, and the curve coincides with its asymptote.

If $x = \frac{3a}{4}$, $\gamma = \infty$, which corresponds to a point of inflexion,

$$\alpha = \frac{a^4 + 10\,a x^3 - 12\,x^4}{2\,x^2\,(3a - 4x)};$$

$$\beta = \frac{4\sqrt{(ax - x^2)}}{a^2\,(3a - 4x)}\ \{ a^4 - a^3 x + a x^3 - x^4 \}.$$

The only difficulty attending the determination of the equation of the evolute of an algebraical curve, arises from the elimination of x from the expressions for α and β, which generally requires the resolution of equations of higher orders.

On the Conchoid of Nicomedes, the Cissoid of Diocles and other remarkable Curves, both Algebraical and Transcendental.

(1). On the Conchoid of Nicomedes. Fig. 57, 58.

If a line be drawn from the point P, cutting the indefinite straight line CD in R, and if RM be taken equal to a given line; the curve EBe traced out by M, in the different positions of PM, is called the Conchoid of Nicomedes.

If Rm be taken equal to RM, the curve Fbf traced out by m, is called the *inferior* conchoid.

The indefinite line CD is called the *rule*, and the given line RM, the *modulus* of the conchoid.

If a line PAB at right angles to CD, be considered as the axis of the abscissæ, and if we make AB or $RM = a$, $AP = b$, $AH = x$, and $HM = y$, we shall find

$$y^2 = \frac{(b + x)^2\,(a^2 - x^2)}{x^2},$$

or $$x^4 + 2bx^3 + (b^2 - a^2 + y^2)x^2 - 2a^2bx - a^2b^2 = 0,$$

which is the equation to a curve of the fourth order.

The determination of the points of inflexion of this curve, depends upon the solution of the equation

$$x^3 + 3bx^2 - 2ba^2 = 0.$$

If $a = b$, or if the inferior conchoid pass through the pole P, the roots of this equation are $a(\sqrt{3} - 1)$, $-a$ and $-a(\sqrt{3} + 1)$; the first of which corresponds to two points of inflexion in the similar branches EB and eB, and the second indicates a cusp of the first kind at the pole of the conchoid: the third root gives imaginary values of y and therefore corresponds to no point of the curve.

If b be greater than a, there are four points of inflexion, two in the superior, and two in the inferior conchoid: the pole P is in this case a conjugate point.

If b be less than a, there are two points of inflexion in the superior conchoid: the pole P is a double point and there is an oval included between P and b. Fig. 58.

The polar equation of this curve is

$$u = \frac{b}{\cos\theta} + \sqrt{a^2},$$

where $u = PM$ and $\theta =$ angle MPB.

If its rectangular co-ordinates be expressed by trigonometrical quantities, the conchoid, from the nature of the resulting equation, may be considered as a cycloidal curve:

$$\text{thus,} \quad y = b\tan\theta + \sqrt{a^2} \,.\, \sin\theta, \quad \text{and}$$

$$x = \sqrt{a^2} \,.\, \cos\theta.$$

The whole area included by the curve BE and its asymptote AD is infinite: but the solid generated by the revolution of the same area round AD is finite and equal to a hemisphere whose radius is a, together with a cylinder whose base is πa^2 and whose altitude is $\frac{\pi b}{2}$. This property was discovered by Wallis*.

Nicomedes, who appears to have flourished about two centuries before the Christian æra, made use of this curve in the practical solution of the two problems so celebrated among the geometers of antiquity, namely the trisection of an angle and the insertion of two mean proportionals between two given lines, upon which problem the duplication of a cube depended: it was peculiarly adapted for this purpose, in consequence of its admitting of a very simple mechanical description †.

A method nearly similar may be made use of to describe the ellipse and hyperbola: for if we make RQ constantly equal to a and draw QM at right angles to AD, the point M will trace out an hyperbola, whose asymptotes are KD and KL, the point K being determined by making $AK=a$, and KL being parallel to AB.

Again, if PRM be drawn from different points of AP in such a manner that the part PR may be constantly equal to b and RM to a, then the curve traced out by M will be an ellipse whose semi-axes are $a+b$ and a and whose centre is A. Also, if m be taken in RP or RP produced, the point m will describe an ellipse, whose semi-axes are a and $b-a$, if a be less than b; and a and $a-b$ when a is greater than b.

It is obvious that all these motions may be executed mechanically.

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* Wallisii Opera. Tom. I. p. 550.

† Montucla. *Histoire des Mathematiques,* Tom. I. p. 236. and Newton. *Appendix ad Arith. Universalem.* Art. 5.

(2). On the Cissoid of Diocles. Fig. 59.

If PQ and pq be two ordinates, drawn at equal distances from A and B, the extremities of the diameter of a circle; and if we join Aq cutting PQ or PQ produced in M, the curve traced out by the point M is the Cissoid of Diocles.

If we make $AB=a$, $AP=x$ and $PM=y$, we find

$$y^2 = \frac{x^3}{a-x}.$$

The curve consists of two infinite hyperbolic branches AE and Ae, with a cusp at A, and an asymptote passing through B at right angles to AB.

The area APM of the Cissoid, is equal to three times the circular segment AQA diminished by the triangle AQP, and consequently the whole area of the curve between its infinite branches and asymptote, is equal to three times the area of the generating circle: the solid generated by the revolution of this area round AB is infinite.

The polar equation of the curve $u = \dfrac{2a \sin^2 \theta}{\cos \theta}$.

This curve may be generated mechanically as follows. Produce CA to E making $AE=AC$; and let the ordinate CD be produced indefinitely; let the indefinite line KF and the line FH which is equal to AB and bisected in M, be placed at right angles to each other: then if KF pass constantly through the point E and the point H move along CD or CD produced, the point M will trace out the Cissoid*. Fig. 60.

Diocles the inventor of this curve flourished in the sixth century after Christ, and was one of the latest geometers of the school of Alexandria: if we take for granted the geometrical description of the cissoid, it will afford an easy solution of the same problems which suggested the invention of the conchoid.

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* Newton. *App. ad Arith. Universalem* Sect. 46.

If we suppose AB to be the axis or a portion of the axis of any other curve, an infinite number of cissoidal curves may be described, by a similar construction. Thus suppose the locus of q to be the side $a\,b$ of the rectangle $A\,a\,b\,B$, Fig. 61, the point M will trace out a curve, whose equation is

$$x\,b = (a - x)\,y,$$

which, by making $a-x=x'$ and $y+b=y'$, may be transformed into

$$x'\,y' = a\,b,$$

the equation to the rectangular hyperbola. Any other hyperbola may be described, supposing PQ and $p\,q$ to make any given angle with AB.

If we suppose the generating curve to be the cissoid, the secondary cissoid will be the generating circle of the first: but if we reverse the position of the generating cissoid, this secondary curve will have for its equation

$$y^2 = \frac{x^5}{(a-x)^3};$$

and the continuation of this process, will give us a series of curves whose equations are

$$y^2 = \frac{x^7}{(a-x)^5},\ y^2 = \frac{x^9}{(a-x)^7},\ \&c.$$

And generally, if the equation $y = \psi\,x$, of the generating curve be required, so that the cissoidal curve corresponding, may be any given curve whose equation is $y=\phi\,x$; we shall find

$$y = \psi\,x = \frac{x\,\phi\,(a-x)}{a-x},$$

(3). On the Lemniscata of James Bernoulli.

If from the center C of an equilateral hyperbola AQ, a perpendicular be let fall upon the tangent QTM, the curve traced out by the point M in different positions of the tangent, is called the *lemniscata*. Fig. 62.

If $AC = a$, $CP = x$ and $PM = y$, the equation to the curve is

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2),$$

which is therefore of the fourth order.

It consists of two oval figures forming a double point at C, with an inflexion of each branch at that point, the tangents making an angle of 45° with the axis.

This curve consisting of two ovals which intersect each other, admits of complete quadrature and its whole area $= a^2$.

Its polar equation is

$$u = a \,.\, \cos 2\,\theta,$$

where $u = CM$ and $\theta =$ the angle ACM.

If we take s to represent the arc AM, we shall find

$$d s = \frac{a^2 \, d u}{\sqrt{(a^4 - u^4)}},$$

or, if $a = 1$,

$$d s = \frac{d u}{\sqrt{(1 - u^4)}}.$$

This curve was first considered by James Bernoulli *, who shewed that its arcs were equal to those of the Elastic curve, whose differential equation is

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* Jacobi Bernoullii, *Opera*. Tom. I. p. 609.

$$dy = \frac{x^2\,dx}{\sqrt{(a^4 - x^4)}},$$

the abscissa x corresponding to the chord u of the arc of the Lemniscata.

He also shewed its use in the construction of the curve, whose differential equation is

$$(x\,dx + y\,dy)\sqrt{y} = (y\,dx - x\,dy)\sqrt{a},$$

upon which a body descending by the action of gravity, will recede uniformly from the point from which the motion begins.

Fagnani, in the same work in which he demonstrated that it was possible to determine, in an infinite number of ways, two arcs of an ellipse or hyperbola, whose difference is assignable geometrically, also exhibited analogous and still more remarkable properties in the Lemniscata: thus if the chord $CM = u$, and $CN = \sqrt{\left\{\frac{1-u^2}{1+u^2}\right\}}$, supposing $a=1$, then the arc CM is equal to the arc AN: and again, if we take $CN' = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$, then the arc CN' is equal to twice the arc CM. The second of these properties was generalized by Euler*, who shewed that if we take the chord of $CM = u$ and the chord CN of n times that arc $= z$, then the chord of an arc, which is $(n+1)$ times CN, is

$$\frac{u\sqrt{\left\{\frac{1-z^2}{1+z^2}\right\}} + z\sqrt{\left\{\frac{1-u^2}{1+u^2}\right\}}}{1 - uz\sqrt{\left\{\frac{(1-u^2)(1-z^2)}{(1+u^2)(1+z^2)}\right\}}}.$$

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* *Nov. Comment. Petrop.* 1761.

Euler in the same Memoir has mentioned several other properties of the arcs of this curve, which are deducible from the preceding; and the same subject is discussed with great elegance and generality by Legendre in his work upon Elliptic Transcendents*.

This is not the only curve to which the name of Lemniscata has been given: if we describe a circle, with a centre C and radius CA, and make PM or y constantly equal to $a \,.\, \sin\theta \,.\, \cos\theta$, where θ is the angle ACQ, we shall find a curve whose equation is

$$x^4 - a^2 x^2 + a^2 y^2 = 0,$$

which in its form and singular points is precisely similar to the former, though its arcs present no analogous properties.

Fig. 63.

(4). On the Semi-cubical Parabola.

The equation of this curve is

$$a y^2 = x^3.$$

This curve is the evolute of the conical parabola †; and its arc, reckoned from the origin of the co-ordinates is equal to

$$\frac{8a}{27}\left\{\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} - 1\right\} ‡.$$

(5). On the Logarithmic Curve.

This curve was imagined by James Gregory §: its most important properties were discovered by Huygens ||.

If $AP = x$, and $PM = y$, Fig. (64.) its equation is

$$y = a^x.$$

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* *Exercises du Calcul Integral*, p. 39.

† Huygens *Horologium Oscillatorium*.

‡ This curve was rectified by Neil in 1657, in consequence of a remark made by Wallis in his *Arithmetica Infinitorum* in a Scholium to the 38th *Proposition*. Wallisii Opera. Tom. I. p. 551.

§ *Geometriæ Pars Universalis. Padua* 1668.

|| In his treatise *De Causa Gravitatis*, *Opera Reliqua*, Tom. II. p. 149.

The ordinate $AB = 1$, and is therefore independent of a, or the base of the system of logarithms which are represented by the abscissæ of the curve, the corresponding ordinates representing the numbers.

The subtangent $PT = \frac{1}{\log a}$, or the modulus of the system of logarithms.

The area $ABMP = PT(PM - AB)$, and the whole area included between the curve, the axis or asymptote CP, and the ordinate PM, is equal to twice the triangle PTM.

The solid generated by the revolution of the same area round the axis CP, is equal to one half the cylinder, whose altitude is PT and the radius of whose base is PM.

(6). On the Quadratrix of Dinostratus, &c.

If $CA = 1$, $AP = x$, $PM = y$, Fig. 65, then

$$y = (1 - x) \tan \frac{\pi x}{2}$$

is the equation of the curve.

The following method of determining the point B is given by John Bernoulli*.

Join AD and draw CE perpendicular to it; with centre C and radius CE describe the circular arc EF, cutting CD in F; join EF and draw CE' perpendicular to it; describe the circular arc $E'F'$, with centre C and radius CE', and proceed as before: the point F approximates nearer and nearer to the point B after each operation, and ultimately coincides with it. The points E, E', E'' &c. are also points in the curve; and by drawing the chord of any other arc AD' of the quadrant, and repeating the same operations as before, any number of points whatever of the curve may be determined.

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* *Commercium Epistolicum Leibnitzii et Bernoullii, Tom.* II. p. 177.

The distance $CB = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}$, &c. *in infinitum* $= \frac{2}{\pi}$, and is consequently a third proportional to the quadrantal arc and the radius.

It is obvious from the mode of describing this curve, that if it be continued without the circle, it will consist of a series of infinite hyperbolic branches, cutting the axis ACI produced in points which are separated from each other by a distance equal to the diameter of the circle.

The tangent at B is parallel to the axis.

If we assume $CM = u$ and the angle $BCM = \theta$, we find

$$u = \frac{2\theta}{\pi \sin\theta},$$

which is the polar equation of the curve.

If we suppose the radius AC of the circle to move parallel to itself and uniformly through the quadrant AD, whilst the ordinate PM moves uniformly through AC, their common intersection will trace out a curve whose equation is

$$y = \sin\frac{\pi x}{2}$$

and which is called the Quadratrix of Tschirnhausen. Fig. 66.

This curve has likewise an infinite course, consisting of an infinite number of curves similar and equal to that included in the circle and referred to the same axis indefinitely produced. This curve is the projection of a curve of double curvature, called the *helix*, upon a plane coinciding with the axis of the cylinder, upon whose surface it is described.

Its polar equation is

$$u \sin \theta = \cos \frac{\pi u \cos \theta}{2}.$$

The area ACD of the curve is a third proportional to the square described upon the radius and the area of the semicircle.

If we divide any abscissa AP, in either of these curves in any ratio whatever, the radii in one case and the sines in the other, which pass through the extremities of the corresponding ordinates, will divide the arc AQ in the same ratio: the mechanical description of these curves, therefore, would furnish us with the multisection of an arc.

The following curve is called the *Trisectrix* from its characteristic property *.

With centres A and B, Fig. (67.) and the same radius AB, describe two circles: draw any line AQ meeting the second circle in Q, and in AQ and AQ produced, make $QP = QP' = AB$: the points P and P' are points of the curve.

Its polar equation is

$$u = 2 \cos \theta \pm 1,$$

where $u = AP$ or AP' and $\theta =$ the angle PAB.

Its equation, when referred to rectangular co-ordinates, is

$$\sqrt{\{x^2 + y^2\}} = \frac{2x}{\sqrt{\{x^2 + y^2\}}} \pm 1,$$

$$\text{or } y^4 + (2x^2 - 4x + 1)y^2 + x^2(x^2 - 4x + 3) = 0.$$

If we draw any line BEF cutting the curve in E and the circle in F, and then join AE and produce it to meet the circle in f, the arc BF is always triple of the arc Bf. The description of this curve, would therefore enable us to trisect any arc from 0 to 360°.

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* *Trisection de l'Angle,* par *Azemar et Garnier,* Paris 1809.

(7.) On the Tractory, Syntractory, &c.

Let BE (Fig. 68.) be a curve, referred to the axis AC, whose tangent MT is always equal to the given line AB: if we make $AB=a$, $AP=x$, and $PM=y$, the differential equation of the curve is

$$dx = -\frac{dy}{y}\sqrt{(a^2 - y^2)},$$

which becomes by integration

$$x = a \log\left\{\frac{a + \sqrt{(a^2 - y^2)}}{y}\right\} - \sqrt{(a^2 - y^2)}.$$

Huygens and subsequent writers on the properties of this curve, have considered it as described mechanically, by attaching one extremity of a string, whose length is AB, to a weight at B, whilst the other is drawn along the line or *directrix* AC: the weight B was supposed to trace out the curve.

This conclusion, however, is erroneous, unless we suppose the friction of the plane of traction to be infinitely great, so that the momentum of the weight which is generated by its motion, may be instantly destroyed: the mechanical tractory of a straight line upon a perfectly smooth plane is an inverted semicycloid, the extremity of whose base is in B and whose axis is equal and parallel to AB: the motion of the weight B being supposed to commence from B*.

The arc $BM = a \log \frac{a}{y}$; consequently, if we describe a logarithmic curve BF, whose axis is CA produced and whose subtangent is AB; and draw Mpm parallel to CA, meeting the logarithmic curve in m; then the line pm is equal to the arc BM.

If with centre A and radius AB, we describe the quadrant $Bm'b$; then the area of the quadrant is equal to the whole area included between AB, the curve, and the asymptote or axis AC.

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* Euler *Nova Comm. Petrop.* 1784.

The solid generated by the revolution of the same area round *AC* is equal to one-half the hemisphere whose radius is *AB*; and its surface is equal to that of the same hemisphere.

The line $Mpm' = AT = a \log \left\{ \frac{a + \sqrt{(a^2 - y^2)}}{y} \right\}$ $= a \log \cot \frac{\theta}{2}$, if θ be the angle $m'Ab$. Consequently, if θ be the co-latitude of a place upon the surface of the earth, *Mpm'* or *AT* will represent the length of a degree of latitude upon Wright or Mercator's Projection of the sphere, the length of a degree of longitude being represented by *AB**.

If from the points *M* and *T*, we draw perpendiculars to *MT* and *AT* respectively, meeting each other in *O*, the line *MO* is the radius of curvature of the curve.

The locus of the point *O* or the *evolute* of the curve is the common *catenary*, whose vertex is *B* and the constant in whose equation is *AB*.

If a point *m* (Fig. 69.) be taken in *TM* or in *TM* or *MT* produced, so that *TM* may be constantly equal to *b*, a given line, then the curve traced out by *m* has been termed by Riccati, the Syntractory †.

If $Ap = x$, and $pm = y$, its equation is

$$dx = \frac{-(ab - y^2)\,dy}{y\sqrt{(b^2 - y^2)}}$$

or

$$x = a \log \left\{ \frac{b + \sqrt{(b^2 - y^2)}}{y} \right\} - \sqrt{(b^2 - y^2)}.$$

If $b < a$, the syntractory has a point of inflexion corresponding to $y = \frac{b\sqrt{a}}{\sqrt{(2a - b)}}$.

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* Perks, *Philosophical Transactions*, 1715.

† Vincenzio Riccati, *Comment. Bononensia*, Tom. III. 1755.

If $b > a$, the curve has no point of *inflexion*: but when $y = \sqrt{a b}$, the tangent is at right angles to the axis. (Fig. 70.)

If b be negative, there is a point of *inflexion* when $y = \dfrac{-b \sqrt{a}}{\sqrt{(2a + b)}}$

In all three cases, the tangent corresponding to $x = 0$ and $y = b$, is parallel to the axis.

The investigation of the equation of the *tractory*, when the *directrix*, along which one extremity of the string is supposed to be drawn, is no longer a straight line, but any assigned curve, becomes a problem of great and in most cases of almost insuperable difficulty: Euler has solved the problem, when the *directrix* is a circle and the curve of traction is either within or without the circle*.

The following curve is equitangential in a certain sense, and possesses some curious properties. To find the nature of the curve, the portion of whose tangent intercepted by a perpendicular from a given point, is always equal to a given line.

Let A (Fig. 71.) be the given point, ME the curve, $a = MY$, the portion of the tangent MF intercepted by the perpendicular AT, $x = AP$ and $y = MP$. The differential equation of the curve is

$$x = -y \frac{dy}{dx} + a \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)};$$

a particular solution of which equation gives us, making $dx = \dfrac{y\, dv}{a}$

$$x = \frac{a v}{\sqrt{(a^2 + v^2)}} - \frac{a^2}{\sqrt{(a^2 + v^2)}} \tan^{-1} \frac{v}{a},$$

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* Euler, *Id.*

$$y = \frac{a^2}{\sqrt{(a^2+v^2)}} + \frac{a\,v}{\sqrt{(a^2+v^2)}}\,.\,\tan^{-1}\frac{v}{a}\,.$$

If we make $AM = u = \sqrt{(x^2+y^2)}$, these equations give us

$$\tan^{-1}\frac{v}{a} = \phi = \frac{\sqrt{(u^2-a^2)}}{a},$$

which is a polar equation of the curve.

But this result may be more simply obtained, by investigating the polar equation of the curve in the first instance: we shall thus find

$$\theta + \sec^{-1}\frac{u}{a} = \phi = \frac{\sqrt{(u^2-a^2)}}{a};$$

the construction of which is as follows: with centre A and radius $AB = a$, describe a circle BEQ and from M draw MQ a tangent to the circle; then MQ is constantly equal to the circular arc BEQ. Fig. 72.

The curve BM is obviously the *involute* of the circle.

At the point B, the tangent to the curve coincides with the radius AB.

The area $BAM = \frac{(\text{arc } QB)^3}{6\,AB}$; and the sector $BAQ =$ the triangle MAQ.

The arc of the curve BM is a third proportional to the diameter of the circle and the radius vector AM.

The locus of the extremity of the perpendicular AY upon the tangent MY, is the spiral of Archimedes, whose polar equation is $a\,\theta' = u'$, where u' is the radius vector, and θ' the angle described by it from the origin of the curve.

The point from which θ' is reckoned, will be determined by drawing Ab at right angles to AB; we thus have $a\,\theta' =$ arc $b\,e =$ arc $QEB = AY$.

The involute of the circle, therefore, is the curve of traction, where the spiral of Archimedes is the base or *directrix*.

Amongst curves, related to this, is the following: To find the curve, the portion of whose tangent intercepted by the spiral subtangent, or by a perpendicular to the radius vector, is always equal to the same given line.

Let $AM = u$ (Fig. 73.) and $MT = a$, the portion of the tangent intercepted by AT the subtangent; then the polar equation of the curve is

$$\theta = \frac{\sqrt{(a^2 - u^2)}}{u} - \cos^{-1}\frac{u}{a},$$

where $\theta = 0$, when $u = a$.

In order to construct the curve, we must describe a circle, with centre A and radius $AB = a$, and from M draw ME perpendicular to AM, meeting the circle in E: draw also Em a tangent to the circle at E, meeting AM produced in m; the point B is determined, by making the arc BE equal to the tangent Em.

It is evident, that the point m is in the *involute* of the circle, commencing from the point B. The lines Am, AB and AM are in continued proportion*.

This curve is called the *Complicated Tractrix.*

The arc $BM = a \log \frac{a}{u}$, which becomes infinite when $u = 0$.

The area BAM is equal to one half the circular area EDM; and the whole area, between $u = a$ and $u = 0$, is equal to one eighth of the whole area of the circle.

The locus of the point T is expressed by the equation

$$\theta' = \frac{u'}{\sqrt{(a^2 - u'^2)}} - \sin^{-1}\frac{u'}{a},$$

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* Cotes; *Harmonia Mensurarum.* p. 84.

where θ' is reckoned from the point b, determined by making $B\,b$ equal to a quadrant.

The circle is an asymptote to this curve.

This is the curve, which is the *directrix* of the *Complicated Tractrix,* considered as a curve of Traction.

(8). On the Catenary.

This is the curve formed by a perfectly flexible and uniform chain, suspended from two given points.

The reader will find an elaborate discussion of the equations and properties of this curve and of others related to it, in Vol. I. p. 169, of a Treatise on Mechanics, by William Whewell, M. A. Fellow of Trinity College, Cambridge: 1819.

(9). On the spiral of Archimedes, the Hyperbolic spiral, the Lituus, &c.

The equation of the spiral of Archimedes is

$$u = a\,\theta$$

where u and θ are the polar co-ordinates.

If $u = r$, when $\theta = 2\,\pi$, then $a = \frac{r}{2\,\pi}$, and the equation becomes

$$u = \frac{r}{2\,\pi} \cdot \theta.$$

The equation between the perpendicular p upon the tangent and the radius vector u, is

$$p = \frac{u^2\,d\,\theta}{\sqrt{(d\,u^2 + u^2\,d\,\theta^2)}} = \frac{u^2}{\sqrt{(a^2 + u^2)}}.$$

The area of the spiral AEM (Fig. 74.) $= \frac{u^3}{6\,a} = \frac{\pi\,u^3}{3\,r}$; and if $u = r$, the whole area BEA, included within the circle

BDB, whose radius $AB = r$, is one third of the area of that circle.

The arc of the spiral included between its origin and the radius vector u, is equal to that of a parabola, whose latus rectum is $2a$, included between the vertex and an ordinate $= u$; and the corresponding area of the spiral is equal to one half the corresponding area of the parabola.

If $\phi =$ the angle AMT, between AM and the tangent MT, then

$$\tan \phi = \frac{u}{a} = \theta.$$

The subtangent $AT = \dfrac{u^2}{a}$.

The locus of the point T is a curve, whose equation is

$$u = a\,\theta^2,$$

where u and θ are the polar co-ordinates, θ being reckoned from a point b, determined by making $B\,b = 90°$.

In the same manner, if T' be the extremity of the subtangent of the curve which is the locus of T, T'' of that traced out by T', T''' of that traced out by T'', and so on, the polar equations of these curves will be

$$u = \frac{a}{2}\,\theta^3,\; u = \frac{a}{2\,.\,3}\,.\,\theta^4,\; u = \frac{a}{1\,.\,2\,.\,3\,.\,4}\,\theta^5,\; u = \frac{a}{2\,.\,3\,.\,4\,.\,5}\,\theta^6, \text{ \&c.}$$

We thus find a series of parabolic spirals of ascending orders, in which the polar co-ordinates θ are reckoned from points which are 90° distant from each other.

Amongst the spirals comprehended in the general polar equation

$$u = a\,\theta^{-n},$$

the most remarkable is that, in which $n = 1$, which is termed the *hyperbolic* spiral, from the analogy subsisting between its polar equation, and the equation to rectangular co-ordinates of the conical hyperbola, when referred to its asymptotes.

If we describe a circle with centre A (fig. 75.), and radius $AB = a$, and if we suppose the curve to cut the circle in D, the point B, from which θ is reckoned, is determined by making the arc $BD = a$; and whatever be the radius of the circle whose centre is A, the arc intercepted between AB or AB produced and the curve, is constantly equal to a.

If ϕ = the angle AMT, then $\tan \phi = \theta$. The subtangent $AT = a$.

If we draw AE perpendicular to AB and the indefinite line EX parallel to it, the line EX is an asymptote to the curve.

The area of the spiral from its origin to the point $M = \frac{au}{2}$ = the triangle TAM.

The radius vector makes an infinite number of revolutions, before the point M reaches the centre A; but the arc ACM is finite, being

$$= a \log. \left\{ \frac{u}{\sqrt{(a^2 + u^2)} + a} \right\} + \sqrt{(a^2 + u^2)} - a.$$

The equation between p and u is

$$p = \frac{au}{\sqrt{(a^2 + u^2)}}.$$

It is sometimes convenient to transform equations of spiral curves, expressed in terms of p and u, into others, which are expressed by polar co-ordinates.

Thus, let

$$p = \frac{bu}{\sqrt{(a^2 - u^2)}};$$

the polar equation is

$$\theta = \frac{b}{c} \log \left\{ \frac{c + \sqrt{(c^2 - u^2)}}{u} \right\},$$

where $c = \sqrt{(a^2 - b^2)}$.

If a circle be described with centre A (Fig. 76.), and radius $AB = c$, then θ is measured from the point B, where the spiral touches the circle, and out of which it can never pass.

Again, let

$$p = \frac{b\,u}{\sqrt{(a^2 + u^2)}}:$$

this equation represents different curves, when $a = b$, $a < b$, and $a > b$.

In the first case, the curve is the *hyperbolic* spiral, which we have considered before.

In the second,

$$\theta = \frac{b}{c} \log \left\{ \frac{\sqrt{(u^2 + c^2)} - c}{u} \right\},$$

where $c = \sqrt{(a^2 - b^2)}$.

With centre A (Fig. 77.), and radius $AB = c$, describe a circle, cutting the curve in C; θ is measured from the point B, determined by making the arc $BC = b \log (\sqrt{2} - 1)$.

If $u = 0$, $\theta = \infty$, and if $u = \infty$, $\theta = 0$.

In the third case,

$$\theta = \frac{b}{c} \sec^{-1} \frac{u}{c},$$

$$\text{or } u = c \sec \frac{c\,\theta}{b};$$

where $c = \sqrt{(b^2 - a^2)}$.

With centre A (Fig. 78.), and radius $AB = c$, describe a circle; θ is measured from B, where the spiral touches the circle.

An indefinite line EX parallel to AB and at a distance $=b$, is, in both these cases, an asymptote to the curve.

Amongst these, may be comprehended the more simple case, where

$$p = \frac{b\,u}{a},$$

which gives

$$\theta = \frac{b}{\sqrt{(a^2 - b^2)}} \log \frac{u}{a},$$

the polar equation of the *logarithmic* spiral, which will be more particularly considered hereafter.

These five curves, which are known by the name of Cotes's spirals[*], comprehend all the trajectories described by a body projected in any direction, and with any velocity, when attracted to a given point, by a force varying inversely as the cube of the distance.

The following curve, whose polar equation is

$$u^2 = a^2\,\theta^{-1},$$

is called the *Lituus*[†], from its form.

If a circle be described with centre A (Fig. 79.) and radius $AB = a$, cutting the curve in C, then the θ is reckoned from the point B, determined by making the arc $CB = a$.

The radius AB produced, is an asymptote to the curve.

Wherever the point M is taken, the sector AME is equal to the sector ABC.

If $\phi =$ the angle AMT, then $\tan \phi = 2\,\theta$.

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[*] Cotes. *Harmonia Mensurarum.* Pars II. p. 31.

[†] Id. *Ib.* Pars III. p. 85.

The triangle $AMT = 2$ sector ABC, and is therefore a constant quantity.

The area $CAM = a^2 \log \frac{u}{a}$; and the area CAM is constantly equal to the area $CMEB$.

It is unnecessary to mention the properties of other spiral curves, included in the more general equation

$$u^n = a^n \theta^m,$$

few of which are sufficiently remarkable to merit particular attention.

(10). On the Logarithmic Spiral.

The characteristic property of this curve is, that the angle between the radius vector and the curve is constantly the same; it has hence been termed the *Equiangular Spiral*.

If ϕ represent this constant angle, the equation of the curve expressed in terms of p and u is

$$p = u \sin \phi,$$

which being transformed into an equation of polar co-ordinates, gives

$$\theta = \tan \phi \,.\, \log \frac{u}{r}.$$

With centre A (Fig. 80.) and radius $AB = r$, describe a circle cutting the curve in B; then θ is measured from the point B, and the arc BC is equal to the logarithm of $\frac{AM}{AB}$ to modulus $r \tan \phi$, or the subtangent of the curve at B; it is hence called the *Logarithmic Spiral*.

The triangle AMT is constantly similar to itself.

The whole arc of the curve, between A and M, is equal to the tangent MT; the corresponding area is equal to one half the area of the triangle AMT.

Those logarithmic spirals are similar and *equal,* in which the angle AMT or ϕ is the same.

The radius of the circle of curvature $MO = \frac{u\,du}{dp} = \frac{u}{\sin\phi}$, and its chord passing through the centre $A = \frac{2p\,du}{dp} = 2u$.

The *evolute* of this curve is a logarithmic spiral *similar* to the original, the distance from B of the point from which θ is measured, being $= r \tan\phi \log\tan\phi$.

The *involute* of this curve is the locus of the point T, which is also a spiral similar to the original: and in general, the loci of all points, which are the extremities of lines making constantly the same angle with the radius vector AM and which bear a constant ratio to it, are logarithmic spirals similar and therefore equal to the original.

Consequently, its *caustic*, by *reflected* rays, the focus of incidence being A, which is traced out by the point a, determined by making $Ma = MA$, and the angle $OMa =$ the angle OMA; as well as its *caustic*, by *refracted* rays, traced out by the point b, determined by making the angle AMb equal to the angle of deviation and drawing Ob at right angles to Mb; are logarithmic spirals similar to the original.

The properties of this curve appeared so remarkable to James Bernoulli, that he usually denominated it *spira mirabilis:* and in a paper in the Leipsic Acts[*], in which he noticed its singular property of reproducing itself in its evolute, caustic, &c., he concludes with this quaint and singular paragraph.

"Cum autem ob proprietatem tam singularem tamque admirabilem mire mihi placeat Spira hæc mirabilis, sic ut ejus contemplatione satiari vix queam; cogitavi, illam ad varias res symbolice repræsentandas non inconcinne adhiberi posse. Quoniam enim semper sibi similem et eandem spiram gignit, utcunque volvatur, evolvatur, radiet; hinc poterit esse vel sobolis parentibus per omnia similis Emblema; *Simillima Filia Màtri.* Vel (si rem æternæ Veritatis Fidei mysteriis accommodare non est prohibitum) ipsius æternæ generationis Filii, qui Patris veluti imago, et ab illo ut Lumen a Lumine emanans, eidem ὁμοούσιος existit, qualiscunque adumbratio. Aut, si mavis, quia curva nostra mirabilis in ipsa mutatione semper sibi constantissime manet similis et numero eadem, poterit esse, vel fortitudinis et constantiæ in adversitatibus; vel etiam carnis nostræ post varias alterationes, et tandem ipsam quoque mortem, ejusdem numero resurrecturæ symbolum; adeo quidem, ut si Archimedem imitandi hodienum consuetudo obtineret, libenter spiram hanc tumulo meo juberem incidi cum Epigraphe: *Eadem numero mutata resurgo.*

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[*] *Anno* 1692.

(11). On the Cycloid and Cycloidal Curves.

If AE, (Fig. 81.) the radius of the generating circle $= a$, $AP = x$, $PM = y$, and the angle $AEM = \theta$, then the equations of the common cycloid are,

$$y = a\,(\sin\theta + \theta) \text{ and } x = a\,(1 - \cos\theta).$$

The equations of the trochoidal curves described by a point within or without the generating circle and moving with its plane, are

$$y = a\,(\sin\theta + m\,\theta) \text{ and } x = a\,(1 - \cos\theta).$$

These equations may be made more general, by supposing

$$y = a\,(n\sin\theta + m\,\theta) \text{ and } x = a\,(1 - \cos\theta).$$

If $n = 0$ and $m = 1$, these equations belong to a curve, called the *Companion* of the *Cycloid.*

It is unnecessary to put these equations under a more general form, since they include all those cycloidal curves, generated upon a rectilinear base, which have been the object of particular remark.

(1). We will now mention the principal properties of the common cycloid.

A tangent MT, at the point M, is parallel to AQ, the corresponding chord of the generating circle.

If Qt be a tangent to the circle at Q, meeting MT in t, then $Qt = QM =$ the arc AQ.

The locus of the point t is the *involute* of the generating circle.

If tangents be drawn to two points M and M' of the cycloid and produced to meet in t', the angle $Mt'M'$ is equal to the angle QAQ' of the circle.

If the rectangle $CLL'D$ (Fig. 82.), be completed, and if Mt' and $M't'$ intersect LL' in O and O', the line OO' is constantly equal to the arc QAQ' of the circle.

If the points Q and Q' be the extremities of a diameter, the angle $Mt'M'$ is a right angle, and the locus of the point t' is a curtate cycloid, whose equations are

$$y = \frac{\pi a}{2}\left(\sin\theta + \frac{2\theta}{\pi}\right), \text{ and } x = \frac{\pi a}{2}(1 - \cos\theta):$$

the curve passes through L and L', and its base, which is parallel to LL' at the distance $\frac{\pi a}{2}$ below it, is double of the axis.

In all other cases when the arc $QAQ' = a\phi$, and the angle $Mt'M' = \pi - \frac{\phi}{2}$, the locus of the point t' is also a curtate cycloid, whose equations are

$$y = \frac{a\phi}{2\sin\frac{\phi}{2}}\left\{\sin\theta + \frac{2\sin\frac{\phi}{2}}{\phi}\theta\right\} \text{ and } x = \frac{a\phi}{2\sin\frac{\phi}{2}}(1-\cos\theta).$$

The curve passes through two points l and l' in AL and AL' or in these lines produced, which are determined, by making $Al = Al' = a\phi$; and its base intersects AB or AB produced, in a point K determined by making $AK = \frac{a\phi}{2\tan\frac{\phi}{2}}$.

The arc AM = twice the chord AQ, and the whole arc CAD of the cycloid is equal to four times AB, the diameter of the generating circle.

If the rectangle $APMR$ be completed, the area AMR is equal to the segment AQP of the circle.

The whole area of the cycloid is three times the area of the generating circle.

The ordinate PM is proportional to the mixtilineal area ABQ, included between the arc AQ, the diameter AB and the chord BQ.

The area AMM', cut off by the double ordinate MM' which bisects AE, is equal to the equilateral triangle BQQ'. This quadrature was discovered by Huygens. (Fig. 83.)

If the ordinate EF be drawn from the centre of the generating circle, the cycloidal segment $AFMA$ is equal to the triangle EBq, or to half the square described upon the radius of the circle.

If two ordinates PM and $P'M'$ be drawn in such a manner, that $AP' = EP$, and the points M and M' be joined, the area AMM' is equal to the sum or difference of the triangles BPQ and $BP'Q'$, according as the ordinates are upon the same or different sides of the axis. (Fig. 84.)

If a double ordinate $MP'M$, pass through P' and the lines PM and PM' be drawn; then, if $AP' = EP$, the cycloidal sector $MPM'A$ is equal to the isosceles triangle BQQ'. (Fig. 85.)

If $AQ = \theta$ and $AQ' = n\theta$, and if the points P and P' be determined by the solution of the equation

$$\left\{\frac{n+1-2\cos\theta}{n}\right\}^{\frac{1}{2}} = 2\sin\frac{n\theta}{2},$$

then the area $MM'P'P$ is equal to $EAQ' + AQ'P' - EAQ - AQP$, which are known quantities*. (Fig. 84.)

The radius of curvature of the cycloid at the point $M = 4a\cos\frac{\theta}{2} = 2\,BQ$.

The evolute of each semicycloid, is an equal semicycloid in a reversed position.

If a series of equal cycloids be described, whose bases are in the same straight line, the curve which cuts them all at right angles, is an equal cycloid in a inverted position: this is one of the simplest cases of the problem of orthogonal trajectories †.

If any curve whatever AB (Fig. 85.), whose tangents a A and B are at right angles to its rectangular co-ordinate AC and BC respectively, be evolved into the curve BA', beginning from B; if again, BA' be evolved into the curve $B'A'$, beginning from A', and $B'A'$ into $B'A''$, beginning from B', and so on continually: each successive evolute will approximate more and more to a common cycloid, whose

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[*] These quadratures were discovered by John Bernoulli, in 1699, a century after the idea of this curve first occurred to Galileo: after enumerating the controversies to which this curve gave rise, and the many wonderful properties which it possessed, he proceeds thus: "At vero non prius de palæstrâ discedere velle videtur, quam per integrum exactum sæculum omnia sua exuisset mysteria, nobisque pandisset quicquid in extremis haberet recessibus suis. Prodit enim, ecce denuo, in scenam nobilis nostra curva, spectatoribus quasi cum agonizante hoc sæculo valedictura, ut tandem candide sistat infinitorum suorum segmentorum, aliorumque spatiorum quadraturam." Johannis Bernoullii. *Opera. Tom.* I. p. 324, 390.

[†] Johannis Bernoullii. *Opera.* Tom. II. p. 521.

axis is equal and parallel to AC, and will ultimately coincide with it.

This singular property, the discovery of John Bernoulli, was first demonstrated by Euler*.

Several problems concerning the solids of revolution round the axis and base, their surfaces, and the centres of gravity of those solids and surfaces, and of different portions of them, were proposed by Pascal under the assumed name of Dettonville, and resolved by him and Wallis† in 1658 and 1659. Those problems were of extreme difficulty for the time at which they were proposed, but they lead to no result worth notice.

(2). The *prolate* and *curtate* cycloids are included under the equations

$$y = a(\sin\theta + m\theta) \text{ and } x = a(1 - \cos\theta).$$

The greatest ordinate of the *curtate* cycloid, corresponds to

$$x = a(1 + m).$$

The *prolate* cycloid has a point of *inflexion* corresponding to

$$x = \frac{a(1 + m)}{m}.$$

In both cases, the base of the trochoid is a tangent to it, at the points where it meets the curve.

The whole trochoidal area $= (2m + 1)$ times the area of the generating circle.

The arc AM (Fig. 86), of the trochoid is equal to $\frac{2a(a+b)}{b}$ multiplied into an elliptic arc BM (Fig. 87.) whose semi-axes are 1 and $\frac{b-a}{b+a}$, corresponding to an abscissa CP, reckoned from the centre C, which is equal to $\sin\frac{\theta}{2}$; when b is less than a, or for the prolate cycloid, the semi-axis minor of the ellipse is $\frac{a-b}{a+b}$ *.

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* *Comm. Petrop.* 1766; Legendre *Exercices du Calc. Integ.* Tom. II. p. 541. 1817.

† Wallisii *Opera.* Tom. I. p. 491.

The *prolate* cycloid is the *trochoid* of Wallis and Newton, and was applied by them to divide the area of an ellipse in a given ratio †.

(3). The *companion of the cycloid*, whose equations are

$$y = a\,\theta \text{ and } x = a\,(1 - \cos\theta). \quad \text{Fig. (88).}$$

possesses many properties analogous to those of the common cycloid.

An ordinate to the centre meets the curve in a point of *inflexion*, and the area which it cuts off, is equal to the square described upon the radius of the circle.

The whole area of the curve is equal to twice that of its generating circle.

If the line AC be drawn, cutting the curve in H, the area AKH = the area $H\,k\,C$.

If the rectangle $APMm$ be completed, and the chord AQ be drawn, the external area AMm of the curve is equal to twice the segment ALQ of the circle.

If the points P and P' (Fig. 89.), be equidistant from the centre E of the generating circle, and the points M and M', the extremities of the corresponding ordinates, be joined, the segment MAM' of the curve is equal to the rectangle under the ordinate PQ, and the diameter of the circle ‡.

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* Wallisii *Opera*, Tom. I. p. 539.

† Id. p. 540. Newton, *Princ.* Lib. I. Sect. vi. Prop. 31.

‡ Joh. Bern. *Opera*, Tom. I. p. 334.

(4). If the equations are

$$y = a\,\theta, \text{ and } x = \frac{a}{m}\,(1 - \cos\theta),$$

the portion of the curve which they represent, which is cut off by a double ordinate, passing through the centre of the generating circle, constitutes the *harmonic* curve, or one of the figures which a vibrating chord may assume in its different positions. Fig. 90.

The radius of curvature or

$$\gamma = \frac{a\,\{m^2 + \sin^2\theta\}^{\frac{3}{2}}}{m^2\cos\theta},$$

which varies very nearly inversely as $\cos\theta$ or $p\,M$, when m is considerable.

If any two harmonic curves be constructed upon *equal* bases and upon the same straight line, but whose centres are at any given distance from each other, the curve whose ordinate is equal to the sum of the corresponding ordinates of the given curves, is likewise an harmonic curve.

(5). If one circle revolve upon another as its base and in the same plane with it, the curve described by any point in its plane, is called the *Epitrochoid*, which becomes the *Epicycloid*, when the describing point is in the circumference of the revolving circle. If a circle revolve, in a similar manner, upon the concave part of the circumference of another circle, the curve described by a point in its plane, is called the *Hypotrochoid*, which becomes the *Hypocycloid*, when that point is in the circumference.

If a be the radius of the base, b the radius of the generating circle, θ the angle ACQ, (Fig. 91.), A and Q being the points of the base, which are first and last in contact with the generating circle, $CP = x$, and $PM = y$, then the equations of the Epicycloid are

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\frac{a+b}{b}.\theta \\ y &= (a+b)\sin\theta - b\sin\frac{a+b}{b}.\theta \end{aligned}\right\}(\alpha).$$

If b be negative, these equations represent the corresponding *Hypocycloid*; in this case Fig. 92.

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b\cos\frac{a-b}{b}.\theta \\ y &= (a-b)\sin\theta - b\sin\frac{a-b}{b}.\theta \end{aligned}\right\}(\beta).$$

If b' be the distance of the describing point from the centre of the generating circle, the equations of the *Epitrochoid* are

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b'\cos\frac{a+b}{b}.\theta \\ y &= (a+b)\sin\theta - b'\sin\frac{a+b}{b}.\theta \end{aligned}\right\}(\gamma).$$

The equations of the corresponding *Hypotrochoid* are

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b'\cos\frac{a-b}{b}.\theta \\ y &= (a-b)\sin\theta - b'\sin\frac{a-b}{b}.\theta \end{aligned}\right\}(\delta).$$

If in the general equations (β), for the hypocycloid, we suppose $b = a + c$, we shall find, making $\frac{c}{a+c}\theta = \theta'$,

$$x = (a+c)\cos\theta' - c\cos\frac{a+c}{c}\theta',$$

$$y = (a+c)\sin\theta' - c\sin\frac{a+c}{c}\theta',$$

which are the equations of the *epicycloid*, whose base is a and generating circle c[*].

Again, if, in the same equations (β), we suppose $b = \dfrac{a+c}{2}$, we find

$$\left.\begin{aligned} x &= \left(\frac{a-c}{2}\right)\cos\theta + \left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{a+c}\right).\theta \\ y &= \left(\frac{a-c}{2}\right)\sin\theta - \left(\frac{a+c}{2}\right)\sin\left(\frac{a-c}{a+c}\right).\theta \end{aligned}\right\} (\epsilon).$$

If we also make $b = \dfrac{a-c}{2}$, and $\left(\dfrac{a+c}{a-c}\right)\theta = \theta'$, we find

$$x = \left(\frac{a-c}{2}\right)\cos\theta' + \left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{a+c}\right)\theta'$$

$$y = \left(\frac{a-c}{2}\right)\sin\theta - \left(\frac{a+c}{2}\right)\sin\left(\frac{a-c}{a+c}\right)\theta',$$

equations which are identical with those preceding (ϵ): the same *hypocycloids* are therefore generated, by generating circles whose radii are $\dfrac{a+c}{2}$ and $\dfrac{a-c}{2}$.

The angle MTP, between the tangent to the *epicycloid* at M and the axis is $= \left(\dfrac{a+2b}{2b}\right)\theta$: when the curve is the *hypocycloid*, the corresponding angle is $= \pi - \left(\dfrac{a-2b}{2b}\right)\theta$.

The arc AM (Fig. 91.), of the *epicycloid*

$$= \frac{4b(a+b)}{a}\left(1 - \cos\frac{a\theta}{2b}\right)$$

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[*] Euler, *Acta Petrop.* 1784; *de duplici genesi tam epicycloidum quam hypocycloidum.*

$$= \frac{4\,b\,(a + b)}{a} \text{ ver. sin } \frac{\phi}{2},$$

where $\phi = \frac{a\,\theta}{b} =$ the angle MBQ, or the angle subtended by the arc of the generating circle which has been in contact with the base; in the *hypocycloid*, this arc

$$= \frac{4\,b\,(a - b)}{a} \text{ ver. sin } \frac{\phi}{2}*.$$

The radius of curvature of the epicycloid

$$= - \frac{4\,b\,(a + b)}{a + 2\,b} \,.\, \sin \frac{\phi}{2}.$$

The *evolute* of the epicycloid is an epicycloid similar to the original, the radii of whose base and generating circle are $\frac{a^2}{a + 2\,b}$ and $\frac{a\,b}{a + 2\,b}$, and therefore in the proportion of a to b.

The centre of the base of the evolute is in the centre C of the original epicycloid (Fig. 93.); it consists of two semi-epicycloids AE and BE, which pass through A and B the extreme points of the original curve, forming a cusp at E.

The same is true, *mutatis mutandis*, of the evolute of the hypocycloid. Fig. 94.

The equation between CY or p the perpendicular upon the tangent of the epicycloid, and the radius vector CM or u, is

$$p^2 = \frac{e^2\,(u^2 - a^2)}{e^2 - a^2}, \text{ where } e = a + 2\,b;$$

in the hypocycloid this becomes

$$p^2 = \frac{e^2\,(a^2 - u^2)}{a^2 - e^2}, \text{ where } e = a - 2\,b.$$

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† Newton *Princip.* Lib. I. Sect. x. Prop. 48, 49.

The area CAM, included between the arc AM of the epicycloid, the radius CA and the radius vector CM,

$$= \frac{(a+b)(a+2b)}{2}\left\{\theta - \frac{b}{a}\sin\phi\right\};$$

and the corresponding area of the hypocycloid

$$= \frac{(a-b)(a-2b)}{2}\left\{\theta - \frac{b}{a}\sin\phi\right\}.$$

If AE be a semicycloidal arc (Fig. 95.), B the centre of the generating circle at E, and BH at right angles to ED; and if a circular arc HM be described with centre C and radius CH; the curvilinear space $EHM = \frac{(a+b)\,b^2}{a}$, and the corresponding space $e\,h\,m$ in the hypocycloid $= \frac{(a-b)\,b^2}{a}$.

If $b = \frac{a}{2}$, the hypocycloid is a straight line, and the space $e\,h\,m$ becomes equal to the triangle $h\,f\,e$; the quadrature is in this case identified with that of Hippocrates of Chios*.

We have seen before (p. 190.) that the rectification of the trochoid depends, upon that of an elliptic arc; in a similar manner, we shall find that the arc AM of the epitrochoid (Fig. 96.) is $= \frac{(a+b)(b+b')}{a}$ multiplied into an elliptic arc BM (Fig. 87.), whose semi-axes are 1 and $\frac{b'-b}{b+b}$, and whose abscissa $CP = \cos\frac{\phi}{2}$; when b' is less than b the semi-axes of the ellipse are 1 and $\frac{b-b'}{b+b'}$. The same proposition is true, *mutatis mutandis*, of the arc of the *hypotrochoid*.

All these curves are expressible by finite algebraical equations, whenever a is to b in the proportion of two finite integral numbers. For in no other case will the curve return into itself, but its course will be infinite and the same straight line will cut it in an infinite number of points.

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* Caswell. *Phil. Transact.* 1695; Halley, *Id.* 1695.

The same conclusion may be drawn from a consideration of their equations: for whenever a and b are to each other in the proportion of integral numbers, $\cos\theta$, $\sin\theta$, $\cos\frac{a+b}{2}\theta$ &c. are expressible by finite series (see p. 80.) in terms of $\sin t$ or $\cos t$, where t is some common sub-multiple of θ and $\left(\frac{a+b}{b}\right)\theta$: the elimination of $\cos t$ or $\sin t$ from the resulting expressions for x and y, will lead to an algebraical equation of finite dimensions.

Thus if $a = b$, the equations for the epicycloid become

$$x = a\ (2\cos\theta - \cos 2\theta)$$

$$y = a\ (2\sin\theta - \sin 2\theta)$$

from which, we deduce, by eliminating the trigonometrical quantities, the algebraical equation of the curve,

$$(x^2 + y^2 - ax)^2 = a^2\{(x-a)^2 + y^2\};$$

which is therefore of the fourth order.

This curve from its heart-shape form, has been called the *Cardioide**. (Fig. 97.)

Its polar equation is

$$u = a\ (1 - \cos\psi),$$

where $u = AM$ and $\psi =$ the angle MAP.

Its greatest ordinate is $= \frac{3\sqrt{3}\,a}{4}$, which corresponds to $x = \frac{a}{4}$.

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* Castiglione. *Phil. Transact.* 1741.

Its whole area is equal to six times the area of the generating circle.

This curve is one of the caustics of the circle, the radiating point being in the circumference *.

The equation of the *epitrochoid*, when $a = b$, is

$$x^4 - 2(2a - b')x^3 + (5a^2 - 6ab' + b'^2)x^2 - 2a(a^2 - 3ab' + b'^2)x$$
$$- a^2 b'(2a - b') + 2x^2y^2 - 2(2a - b')xy^2$$
$$+ a(a - 2b')y^2 + y^4 = 0:$$

which becomes, by putting $x + a$ in the place of x, or by making the point A the origin of the co-ordinates,

$$x^4 + 2b'x^3 - (a^2 - b'^2)x^2 + 2(x + b')xy^2 + y^4 - a^2y^2 = 0.$$

Again, if $b = \frac{a}{2}$, the equation of the epicycloid with two cusps, is

$$a^4y^2 = (x^2 - 2ax + y^2)^3;$$

or if A be the origin of the co-ordinates,

$$a^4y^2 = (x^2 + y^2 - a^2)^3.$$

This curve is one of the caustics of a circle, when the incident rays are parallel to a diameter †.

If $b = \frac{a}{2}$, the general equations for the *hypocycloid* (β), p. 193. become

$$\left.\begin{array}{l} x = a\cos.\theta, \\ y = 0. \end{array}\right\}$$

In this case, therefore, the *hypocycloid* becomes the diameter of the circle which passes through A.

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* Wood's *Optics*, Prop. 99.

† Id. *Ib.* Prop. 98.

The general equations for the *hypotrochoid*, in this case, become

$$\left.\begin{aligned} x &= \left(\frac{a}{2} + b'\right) \cos\theta, \\ y &= \left(\frac{a}{2} - b'\right) \sin\theta : \end{aligned}\right\}$$

these are the equations to an *ellipse*, whose semi-axes are $\frac{a}{2} + b'$ and $\frac{a}{2} - b'$.

If $b = \frac{a}{3}$, the equation of the *hypocycloid* is

$$x^4 - \frac{4}{3} a x^3 + 2 x^2 y^2 - 12 a x y^2 + y^4 + 12 a^2 y^2 = 0 ;$$

the co-ordinates being reckoned from A.

If $b = \frac{a}{4}$, the equation of the *hypocycloid* is

$$27 a^2 x^2 y^2 = (a^2 - x^2 - y^2)^3$$

the co-ordinates being also reckoned from A.

This is the curve, which has been considered by John Bernoulli, in the solution of the following problem. A line DE (Fig. 98.) is moveable between the lines AC and CB, which are at right angles to each other: to find the curve which touches it in all its positions*. In a certain sense, therefore the *hypocycloid* with four cusps, may be considered as an equitangential curve.

If a be infinite, the epicycloid and hypocycloid coincide with the common cycloid: if b be infinite, they coincide with the *involute* of the circular base. The epitrochoid described, when b is infinite, by the point C, is the Spiral of Archimedes, whose equation is

$$u = a\,\theta. \text{ (See p. 177.)}$$

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* Joh. Bern. *Opera*, Tom. III. p. 447.

On the Differentiation of Functions of two or more Variables.

ART. 120—132. Ex. (1.) Let $u = \frac{x^n}{y^m}$:

$$du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy,$$

$$= \frac{x^{n-1}(ny\,dx - mx\,dy)}{y^{m+1}}.$$

(2.) Let $u = \frac{\sqrt{(x^2 - y^2)}}{\sqrt{(x^2 + y^2)}}$,

$$du = \frac{2xy(y\,dx - x\,dy)}{(x^2 + y^2)^{\frac{3}{2}}\sqrt{(x^2 - y^2)}}.$$

(3.) Let $u = \log\left\{\frac{x + \sqrt{(x^2 - y^2)}}{x - \sqrt{(x^2 - y^2)}}\right\}$,

$$du = \frac{2(y\,dx - x\,dy)}{y\sqrt{(x^2 - y^2)}}.$$

(4.) Let $u = \sin^{-1}\frac{x}{y}$,

$$du = \frac{y\,dx - x\,dy}{y\sqrt{(y^2 - x^2)}}.$$

(5.) Let $u = \log\tan\frac{x}{y}$,

$$du = \frac{y\,dx - x\,dy}{y^2\sin\frac{x}{y}\cos\frac{x}{y}}.$$

The following theorem of Euler, is of considerable importance in the integration of *homogeneous* differential equations: if n be the sum of the exponents in each term of the homogeneous function u, of the variables x, y, z, &c., and if

$$d\,u = \frac{d\,u}{d\,x}\,d\,x + \frac{d\,u}{d\,y}\,d\,y + \frac{d\,u}{d\,z}\,d\,z + \&c.$$

then

$$n\,u = \frac{d\,u}{d\,x}\,x + \frac{d\,u}{d\,y}\,y + \frac{d\,u}{d\,z}\,z + \&c.$$

Also, if

$$d^2\,u = \frac{d^2\,u}{d\,x^2}\,d\,x^2 + \frac{2\,d^2\,u}{d\,x\,d\,y}\,d\,x\,d\,y + \frac{d^2\,u}{d\,y^2}\,d\,y^2 + \&c.$$

then

$$n\,(n-1)\,u = \frac{d^2\,u}{d\,x^2}\,.\,x^2 + \frac{2\,d^2\,u}{d\,x\,d\,y}\,.\,x\,y + \frac{d^2\,u}{d\,y^2}\,.\,y^2 + \&c.$$

(See Art. 266.)

The four last Examples afford a verification of this theorem, for the case in which $n = 0$.

(6). Let $u = \dfrac{x^3 + y^3}{x - y}$; where $n = 2$.

$$d\,u = \frac{2\,x^3\,d\,x - 3\,x^2\,y\,d\,x - y^3\,d\,x + 3\,x\,y^2\,d\,y - 2\,y^3\,d\,y + x^3\,d\,y}{(x-y)^2};$$

putting x and y in the place of $d\,x$ and $d\,y$, we get

$$\frac{2\,x^4 - 2\,x^3\,y + 2\,x\,y^3 - 2\,y^4}{(x-y)^2} = \frac{2\,x^3 + 2\,y^3}{x - y} = 2\,u.$$

(7). Let $u = \dfrac{1}{(x^2 + y^2)^2}$; where $n = -4$:

$$d\,u = \frac{-4\,(x\,d\,x + y\,d\,y)}{(x^2 + y^2)^3},$$

and $\dfrac{-4\,(x^2 + y^2)}{(x^2 + y^2)^3} = \dfrac{-4}{(x^2 + y^2)^2} = -4\,u.$

(8). Let $u = \frac{\sqrt{x} + \sqrt{y}}{x+y}$; where $n = -\frac{1}{2}$:

$$du = \frac{\frac{1}{2}\left\{\frac{y\,dx}{\sqrt{x}} + \frac{x\,dy}{\sqrt{y}} - \sqrt{x}.dx - \sqrt{y}.dy - 2\sqrt{x}.dy - 2\sqrt{y}.dx\right\}}{(x+y)^2}$$

and $\frac{-\frac{1}{2}(y\sqrt{x} + x\sqrt{y} + x\sqrt{x} + y\sqrt{y})}{(x+y)^2}$

$$= \frac{-\frac{1}{2}(\sqrt{x} + \sqrt{y})}{x+y} = -\frac{1}{2}u.$$

(9). Let $u = \frac{xy}{ax + bz}$; where $n = 1$:

$$du = \frac{ax^2\,dy + bxz\,dy + byz\,dx - bxy\,dz}{(ax + bz)^2},$$

and $\frac{ax^2y + bxyz}{(ax + bz)^2} = \frac{xy}{ax + bz} = u.$

(10). Let $u = \sin^{-1}\frac{\sqrt{(x-y)}}{\sqrt{(x+y)}}$; where $n = 0$:

$$du = \frac{y\,dx - x\,dy}{(x+y)\sqrt{\{2y(x-y)\}}},$$

and $\frac{yx - xy}{(x+y)\sqrt{\{2y(x-y)\}}} = 0 = nu.$

(11). Let $u = \log\{x + y + \sqrt{(2xy + y^2)}\}$:

$$du = \frac{dx + dy + \frac{x\,dy + y\,dx + y\,dy}{\sqrt{(2xy + y^2)}}}{x + y + \sqrt{(2xy + y^2)}};$$

and putting x and y in the place of $d\,x$ and $d\,y$, we get

$$\frac{x+y+\dfrac{2\,x\,y+y^2}{\sqrt{(2\,x\,y+y^2)}}}{x+y+\sqrt{(2\,x\,y+y^2)}}=1.$$

The reader must take care not to consider functions of this nature as homogeneous, which are only apparently so: thus, in the case before us, if we make $y=x\,t$, we find

$$u=\log\{x+x\,t+x\sqrt{(2\,t+t^2)}\}$$

$$=\log x+\log\{1+t+\sqrt{(2\,t+t^2)}\};$$

the parts of which expression are evidently not homogeneous.

In general, if $u=\phi\,v$, where v is a homogeneous function of x, y, &c., then u is not a homogeneous function of these variables, unless u be also a homogeneous function of v: this consideration would exclude the example just given, as is evident from the form of the series for $\log v$.

The example $u=\sin^{-1}\sqrt{(x^2-y^2)}=\cos^{-1}\sqrt{(1-x^2+y^2)}$ would place this remark in a still clearer light.

The functions given in Examples 2, 3, 4, 5 and 10, are strictly homogeneous, in the sense of the theorem.

(12). Let $u=x\sqrt{(2\,x\,y+y^2)}$, where $n=2$:

$$du=\left\{\sqrt{(2xy+y^2)}+\frac{x\,y}{\sqrt{(2\,xy+y^2)}}\right\}dx+\left\{\frac{x^2+x\,y}{\sqrt{(2xy+y^2)}}\right\}dy,$$

and $\dfrac{(3\,x\,y+y^2)\,x+(x^2+x\,y)\,y}{\sqrt{(2\,x\,y+y^2)}}=2\,x\sqrt{(2\,x\,y+y^2)}=2\,u.$

Again,

$$d^2\,u=\frac{d^2\,u}{d\,x^2}\,d\,x^2+\frac{2\,d^2\,u}{d\,x\,d\,y}\,d\,x\,d\,y+\frac{d^2\,u}{d\,y^2}\,d\,y^2$$

$$= \frac{(3\,x\,y^2 + 2\,y^3)}{(2\,x\,y + y^2)^{\frac{3}{2}}}\,d\,x^2 + \frac{2\,(3\,x^2\,y + 3\,x\,y^2 + y^3)}{(2\,x\,y + y^2)^{\frac{3}{2}}}\,d\,x\,d\,y$$

$$- \frac{x^3\,d\,y^2}{(2\,x\,y + y^2)^{\frac{3}{2}}};$$

and putting x and y in the place of $d\,x$ and $d\,y$, we get

$$= \frac{x\,y^2\,(8\,x^2 + 8\,x\,y + 2\,y^2)}{(2\,x\,y + y^2)^{\frac{3}{2}}} = 2\,x\,\sqrt{(2\,x\,y + y^2)} = n\,(n-1)\,u.$$

The theorem expressed by the equation

$$\frac{d^2\,u}{d\,x\,d\,y} = \frac{d^2\,u}{d\,y\,d\,x}, \text{ Art. 122,}$$

is of great importance in the integration of differential equations of the first order and first degree, involving two variables, as it enables us to ascertain whether they result from the simple differentiation of a function of x and y, or from the elimination of a constant from the equation in which they are involved. (Art. 262.)

(13). Let $u = x \sin y + y \sin x$:

$$\frac{d\,u}{d\,x} = \sin y + y \cos x$$

$$\frac{d\,u}{d\,y} = \sin x + x \cos y,$$

$$\frac{d^2\,u}{d\,x\,d\,y} = \cos y + \cos x = \frac{d^2\,u}{d\,y\,d\,x}.$$

(14). Let $u = x^y$

$$\frac{d\,u}{d\,x} = y\,x^{y-1}, \quad \frac{d\,u}{d\,y} = x^y \log x,$$

$$\frac{d^2\,u}{d\,x\,d\,y} = x^{y-1} + y\,x^{y-1} \log x = \frac{d^2\,u}{d\,y\,d\,x}.$$

(15). Let $u = \log x - \frac{y^2}{2\,x^2} - \frac{y\sqrt{(x^2+y^2)}}{2\,x^2}$

$$+ \tfrac{1}{2}\log\left\{\frac{-y^2 + y\sqrt{(x^2+y^2)}}{x}\right\}.$$

$$\frac{d\,u}{d\,x} = \frac{x^2+y^2+y\sqrt{(x^2+y^2)}}{x^3},$$

$$\frac{d\,u}{d\,y} = \frac{-y-\sqrt{(x^2+y^2)}}{x^2} + \frac{1}{2\,y};$$

$$\frac{d^2 u}{d\,x\,d\,y} = \frac{2\,y + \sqrt{(x^2+y^2)}}{x^3} + \frac{y^2}{x^3\sqrt{(x^2+y^2)}}$$

$$\frac{d^2 u}{d\,y\,d\,x} = \frac{2y+2\sqrt{(x^2+y^2)}}{x^3} - \frac{x}{x^2\sqrt{(x^2+y^2)}},$$

and by reduction we shall find

$$\frac{d^2 u}{d\,x\,d\,y} = \frac{d^2 u}{d\,y\,d\,x} = \frac{2\,y}{x^3} + \frac{x^2+2\,y^2}{x^3\sqrt{(x^2+y^2)}}.$$

The reader may make use of this or of any of the preceding Examples, in illustration of the theorems,

$$\frac{d^3 u}{d\,x\,d\,y^2} = \frac{d^3 u}{d\,y^2\,d\,x} = \frac{d^3 u}{d\,y\,d\,x\,d\,y},$$

$$\frac{d^4 u}{d\,x^2\,d\,y^2} = \frac{d^4 u}{d\,y^2\,d\,x^2} = \frac{d^4 u}{d\,y\,d\,x\,d\,y\,d\,x}$$

$$= \frac{d^4 u}{d\,x\,d\,y\,d\,x\,d\,y} = \frac{d^4 u}{d\,x\,d\,y^2\,d\,x} = \frac{d^4 u}{d\,y\,d\,x^2\,d\,y},$$

&c. &c.

(16). Let $u = \frac{x^2\,y}{a^2 - z^2}$:

(1). $\frac{d^2 u}{d\,x\,d\,y} = \frac{2\,x}{a^2 - z^2} = \frac{d^2 u}{d\,y\,d\,x}$:

$$(2).\quad \frac{d^2 u}{dx\,dz} = \frac{4xyz}{(a^2 - z^2)^2} = \frac{d^2 u}{dz\,dx}:$$

$$(3).\quad \frac{d^2 u}{dy\,dz} = \frac{2x^2 z}{(a^2 - z^2)^2} = \frac{d^2 u}{dz\,dy}:$$

$$(4).\quad \frac{d^3 u}{dx\,dy\,dz} = \frac{4xz}{(a^2 - z^2)^3} = \frac{d^3 u}{dy\,dx\,dz}$$

$$= \frac{d^3 u}{dx\,dz\,dy} = \frac{d^3 u}{dy\,dz\,dx} = \frac{d^3 u}{dz\,dx\,dy}$$

$$= \frac{d^3 u}{dz\,dy\,dx}.$$

(17). Let $u = \frac{x^2 y^2}{z^2 + v^2}$:

$$\frac{d^4 u}{dx\,dy\,dz\,dv} = \frac{32xyzv}{(z^2 + v^2)^3}:$$

the same result may be obtained by as many processes of differentiation, as are denoted by all the different permutations of the four quantities dx, dy, dz, dv, in the denominator of $d^4 u$, which are 24 in number.

We have seen (p. 72.) in what manner the Differential Calculus may be applied to eliminate one or more constants from an equation involving two variables: when the equation involves three variables, we may apply a similar process to eliminate functions, whose forms are absolutely unknown.

(18). Let the equation be

$$z = xy\,\phi y,$$

$$\frac{dz}{dx} = y\,\phi y;$$

from which two equations we deduce the partial differential equation

$$z - x\frac{dz}{dx} = 0:$$

in which ϕy does not appear.

(12). Let $z = \dfrac{y^2 \phi y + x}{1 - x\phi y}$,

$$\frac{dz}{dx} = \frac{z\phi y + 1}{1 - x\phi y};$$

and by eliminating ϕy from these two equations, we get

$$(x^2 + y^2)\frac{dz}{dx} = z^2 + y^2.$$

(20). Let $z = \phi\left(\dfrac{y^2 - x^2}{x}\right)$.

$$\frac{dz}{dx} = -\phi'\left(\frac{y^2-x^2}{x}\right) \cdot \left(\frac{y^2+x^2}{x^2}\right),$$

$$\frac{dz}{dy} = \phi'\left(\frac{y^2-x^2}{x}\right) \cdot \frac{2y}{x};$$

by eliminating $\phi'\left(\dfrac{y^2-x^2}{x}\right)$ from the two last equations, we find

$$\frac{dz}{dx} + \left(\frac{x^2+y^2}{2xy}\right)\frac{dz}{dy} = 0.$$

It is hardly necessary to state, that if $u = \dfrac{y^2 - x^2}{x}$,

$$\phi'\left(\frac{y^2-x^2}{x}\right) = \phi' u = \frac{d \,.\, \phi u}{du},$$

and is consequently the same, whether x or y be considered as variable.

(21). Let $z = a\sqrt{(x^2 + y^2)} + \phi\left(\dfrac{x}{y}\right)$:

$$\frac{dz}{dx} = \frac{ax}{\sqrt{(x^2+y^2)}} + \phi'\left(\frac{x}{y}\right) \cdot \frac{1}{y};$$

$$\frac{dz}{dy} = \frac{ay}{\sqrt{(x^2+y^2)}} - \phi'\left(\frac{x}{y}\right) \cdot \frac{x}{y^2};$$

by eliminating $\phi'\left(\frac{x}{y}\right)$, we get

$$x\frac{dz}{dx} + y\frac{dz}{dy} - a\sqrt{(x^2+y^2)} = 0.$$

(22). Let $z = (x+y)^a \phi(x^2-y^2)$

$$\frac{dz}{dx} = a(x+y)^{a-1}\phi(x^2-y^2) + 2(x+y)^a \phi'(x^2-y^2)x$$

$$\frac{dz}{dy} = a(x+y)^{a-1}\phi(x^2-y^2) - 2(x+y)^a \phi'(x^2-y^2)y;$$

and eliminating $\phi(x^2-y^2)$ and $\phi'(x^2-y^2)$ from these three equations, we get

$$y\frac{dz}{dx} + x\frac{dz}{dy} - az = 0.$$

(23). Let $z = \frac{y^2}{2} + \phi\left(\frac{1}{x} + \log y\right)$

$$\frac{dz}{dx} = -\phi'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{x^2}$$

$$\frac{dz}{dy} = y + \phi'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y},$$

and eliminating $\phi'\left(\frac{1}{x} + \log y\right)$, we get

$$x^2\frac{dz}{dx} + y\frac{dz}{dy} - y^2 = 0.$$

(24). Let $z = x\,\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$.

Since this equation involves two arbitrary functions of x and y, we must proceed to the second differential coefficients of z, in order to eliminate them: this process, however, will not terminate generally, without proceeding to the third differential coefficients of z:

$$\frac{dz}{dx} = \phi\left(\frac{y}{x}\right) - \frac{y}{x}\,\phi'\left(\frac{y}{x}\right) - \frac{y}{x^2}\,\psi'\left(\frac{y}{x}\right)$$

$$\frac{dz}{dy} = \phi'\left(\frac{y}{x}\right) + \frac{1}{x}\,\psi'\left(\frac{y}{x}\right):$$

If we eliminate $\psi'\left(\frac{y}{x}\right)$, from these equations, we find

$$\frac{dz}{dx} + \frac{y}{x}\,\frac{dz}{dy} = \phi\left(\frac{y}{x}\right): \qquad (\beta).$$

in this case $\phi\left(\frac{y}{x}\right)$ also disappears: but in general we shall find $\phi(x, y)$ and $\phi'(x, y)$ in the equation resulting from this operation, and their elimination from it would present the same difficulties, as that of $\phi(x, y)$ and $\psi(x, y)$ from the primitive equation.

Again, in order to eliminate $\phi\left(\frac{y}{x}\right)$ from equation (β), we must proceed thus:

$$\frac{d^2z}{dx^2} + \frac{y}{x}\cdot\frac{d^2z}{dy\,dx} - \frac{y}{x^2}\cdot\frac{dz}{dy} = -\,\phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x^2},$$

$$\frac{d^2z}{dx\,dy} + \frac{y}{x}\cdot\frac{d^2z}{dy^2} + \frac{1}{x}\cdot\frac{dz}{dy} = \phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x};$$

and eliminating $\phi'\left(\frac{y}{y}\right)$ from these equations, we find

$$x^2\,\frac{d^2z}{dx^2} + 2\,xy\,\frac{d^2z}{dx\,dy} + y^2\,\frac{d^2z}{dy^2} = 0,$$

a partial differential equation of the second order.

(25.) Let $z = x^n \phi\left(\frac{y}{x}\right) + y^n \psi\left(\frac{y}{x}\right)$,

$$\frac{dz}{dx} = n x^{n-1} \phi\left(\frac{y}{x}\right) - x^{n-2} \phi'\left(\frac{y}{x}\right) - \frac{y^{n+1}}{x^2} \psi'\left(\frac{y}{x}\right),$$

$$\frac{dz}{dy} = x^{n-1} \phi'\left(\frac{y}{x}\right) + n y^{n-1} \psi\left(\frac{y}{x}\right) + \frac{y^n}{x} \psi'\left(\frac{y}{x}\right):$$

the elimination of $\psi\left(\frac{y}{x}\right)$ and $\psi'\left(\frac{y}{x}\right)$ from these three equations, will likewise cause $\phi\left(\frac{y}{x}\right)$ and $\phi'\left(\frac{y}{x}\right)$ to disappear, and we shall have $x \frac{dz}{dx} + y \frac{dz}{dy} - nz = 0$.

In this case, however, z cannot be considered as involving two different arbitrary functions of x and y, as they may both be comprehended in one:

$$\text{for } z = x^n \left\{ \phi\left(\frac{y}{x}\right) + \frac{y^n}{x^n} \psi\left(\frac{y}{x}\right) \right\} = x^n f\left(\frac{y}{x}\right).$$

(26.) Let $z = \phi(y + ax) + \psi(y - ax)$.

In this case, we find

$$\frac{dz}{dx} + \frac{a\,dz}{dy} = 2a\phi'(y + ax)$$

and $\frac{d^2 z}{dx^2} - \frac{a\,d^2 z}{dy^2} = 0$.

(27.) Let $z = \phi(y + ax)\psi(y - ax)$.

If we denote the functions $\phi(y + ax)$ and $\psi(y - ax)$, by ϕ and ψ, we find

$$\frac{dz}{dx} + \frac{a\,dz}{dy} = 2a \cdot \psi \cdot \phi' = 2az\frac{\phi'}{\phi}.$$

Again,

$$\frac{d^2 z}{d x^2} + \frac{a\, d^2 z}{d y\, d x} = 2\, a\, \frac{d z}{d x} \cdot \frac{\phi'}{\phi} + 2\, a^2 z \cdot \frac{\phi''}{\phi}$$

$$- 2\, a^2 z \cdot \frac{\phi'^2}{\phi^2}$$

$$\frac{d^2 z}{d x\, d y} + \frac{a\, d^2 z}{d y^2} = 2\, a\, \frac{d z}{d y} \cdot \frac{\phi'}{\phi} + 2\, a\, z \cdot \frac{\phi''}{\phi}$$

$$- 2\, a\, z \frac{\phi'^2}{\phi^2}.$$

Multiplying the last equation by a, and subtracting it from the former, we get

$$\frac{d^2 z}{d x^2} - \frac{a^2 d^2 z}{d y^2} = 2\, a \left(\frac{d z}{d x} - \frac{a\, d z}{d y}\right) z\, \frac{\phi'}{\phi}$$

$$= \left(\frac{d z}{d x} - \frac{a\, d z}{d y}\right) \left(\frac{d z}{d x} + \frac{a\, d z}{d y}\right)$$

$$= \frac{d z^2}{d x^2} - \frac{a^2\, d z^2}{d y^2}.$$

(28.) Let $z = \phi\,(y + a\,x) + x\,y\,\psi\,(y - a\,x)$ (1).

$$\frac{d z}{d x} = a\,\phi' + y\,\psi - a\,x\,y\,\psi' \qquad (2).$$

$$\frac{d z}{d y} = \phi' + x\,\psi + x\,y\,\psi'. \qquad (3).$$

We have now four functions ϕ, ϕ', ψ, ψ', to eliminate, and only three equations: we must therefore proceed to the second differential coefficients of z.

$$\frac{d^2 z}{d x^2} = a^2 \phi'' - 2\, a\, y\, \psi' + a^2 x\, y\, \psi'' \qquad (4).$$

$$\frac{d^2 z}{dx\,dy} = a\phi'' + \psi + (y - ax)\psi' - axy\psi'', \quad (5).$$

$$\frac{d^2 z}{dy^2} = \phi'' + 2x\psi' + xy\psi'': \quad (6).$$

we have thus six equations and six functions ϕ, ϕ', ϕ'', ψ, ψ', and ψ'' to eliminate from them, which is impossible, unless, the operation for the elimination of one of them should occasion the disappearance of one or more of the others, as in the four last Examples: if we proceed with the differentiation of z, we find

$$\frac{d^3 z}{dz^3} = a^2 \{ a\phi''' - axy\psi''' + 3y\psi'' \} \quad (7).$$

$$\frac{d^3 z}{dx^2\,dy} = a \{ a\phi''' + axy\psi''' - 2\psi' - (2y - ax)\psi'' \} \quad (8).$$

$$\frac{d^3 z}{dx\,dy^2} = a\phi''' - axy\psi''' + 2\psi' - (2ax - y)\psi'' \quad (9).$$

$$\frac{d^3 z}{dy^3} = \phi''' + xy\psi''' + 3x\psi''. \quad (10).$$

We have now ten equations and only eight functions ϕ, ϕ', ϕ'', ϕ''', ψ, ψ', ψ'' and ψ''', to eliminate from them: we ought therefore generally to have *two* partial differential equations of the third order.

One of these is

$$2a\left(\frac{dz}{dx^2} - \frac{a^2 d^2 z}{dy^2}\right) - (y + ax)\left\{\frac{d^3 z}{dx^3} + \frac{a\,d^3 z}{dx^2\,dy} \quad \frac{a^2 d^3 z}{dx\,dy^2} - \frac{a^3 d^3 z}{dy^3}\right\} = 0: \quad (\alpha).$$

This results from eliminating from equations (4), (5), (6), (7), (8), (9), (10).

By eliminating from equations (2), (3), (4), (5), (6), (8), (10), we get

$$\begin{aligned}&\frac{dz}{dx}-\frac{a\,dz}{dy}-\frac{(y^3+axy^2-a^2x^2y+3a^3x^3)}{2a(y+ax)^2}\cdot\frac{d^2z}{dx^2}\\&-(y-ax)\frac{d^2z}{dx\,dy}\\&+\frac{(3y^3+3axy^2-3a^2x^2y+a^3x^3)}{2(y+ax)^2}\,\frac{a\,d^2z}{dy^2}\\&+\frac{xy(y-ax)}{(y+ax)}\left\{\frac{d^3z}{dx^2\,dy}-\frac{a^2\,d^3z}{dy^3}\right\}=0. \qquad (\beta).\end{aligned}$$

A third might also be found in this case, by eliminating from equations (2), (3), (4), (5), (6), (7), (9): which is one more than we should have got, unless the operation had exterminated a number of unknown quantities equal to or greater than the number of equations employed: this however connot be considered as independent of the other two; for if it be subtracted from the second equation (3), the first (α) will be the result.

On the Maxima and Minima of Functions of two Variables.

Ex. (1). Let $u=y^4-8y^3+18y^2-8y+x^3-3x^2-3x$.

If $\dfrac{du}{dy}=0$, $y=2,\ 2+\sqrt{3},\ 2-\sqrt{3}$.

If $\dfrac{du}{dx}=0$, $x=1+\sqrt{2},\ 1-\sqrt{2}$.

If $y=2$ and $x=1-\sqrt{2}$, we have

$$\left.\begin{aligned} D &= \frac{d^2 u}{dy^2} = -12 \\ E &= \frac{d^2 u}{dy\,dx} = 0 \\ F &= \frac{d^2 u}{dx^2} = -6\sqrt{2} \end{aligned}\right\}.$$

In this case D and F are *both* negative, and $DF > E^2$: therefore

$$u = 3 + 4\sqrt{2}, \text{ a } \textit{maximum}.$$

If $y=2-\sqrt{3}$, or $2+\sqrt{3}$, and $x=1+\sqrt{2}$, we find

$$D = 24,\ E = 0,\ F = 6\sqrt{2}:$$

we have therefore D and F both positive, and $DF > E^2$: consequently

$$u = -6 - 4\sqrt{2}, \text{ a } \textit{minimum}.$$

If $y=2+\sqrt{3}$, or $2-\sqrt{3}$, and $x=1-\sqrt{2}$, we find $D=24$, $E=0$, and $F=-6\sqrt{2}$: in this case D and F have different signs, and the function u is neither a *maximum* nor a *minimum*. Euler. *Calc. Diff. Pars* II. *Cap*. xi. *Art*. 287.

(2). Let $u=y^4-8y^3+18y^2-8y-x^3+3x^2+3x$.

If $y=2$, and $x=1-\sqrt{2}$, then $D=-12$, and $F=6\sqrt{2}$; therefore u is neither a *maximum* nor a *minimum*.

If $y=2+\sqrt{3}$ or $2-\sqrt{3}$, and $x=1-\sqrt{2}$; then $D=24$, $E=0$ and $F=6\sqrt{2}$: therefore

$$u = 4 - 4\sqrt{2}, \text{ a } \textit{minimum}.$$

If $y = 2$, and $x = 1 + \sqrt{2}$: then

$$D = -12,\ E = 0, \text{ and } F = -6\sqrt{2}:$$

therefore $u = 13 + 4\sqrt{2}$, a *maximum*. Euler. *Ib.*

(3). Let $u = x^2 + xy + y^2 - ax - by$:

If $x = \frac{2a - b}{3}$, and $y = \frac{2b - a}{3}$, we have

$$D = 2,\ E = 1,\ F = 2, \text{ and } DF > E^2:$$

and $u = \frac{ab - a^2 - b^2}{3}$, a *minimum*. **Euler.** *Ib.*

(4). Let $u = x^3 + y^3 - 3axy$:

If $x = 0$, and $y = 0$, u is neither a *maximum* nor a *minimum*.

If $x = a$, and $y = a$, we have

$$D = 6a,\ E = -3a, \text{ and } F = 6a, \text{ and } DF > E^2:$$

consequently $u = -a^3$ is a *minimum* when a is a positive, and a *maximum* when a is a negative quantity. Euler. *Ib.*

(5) Let $u = x^4 + y^4 - ax^2y - axy^2 + c^2x^2 + c^2y^2$:

$$\left.\begin{aligned}\frac{du}{dx} &= 4x^3 - 2ayx - ay^2 + 2c^2x = 0\\ \frac{du}{dy} &= 4y^3 - 2ayx - ax^2 + 2c^2y = 0\end{aligned}\right\}.$$

If we subtract these equations from each other, we find

$$4(x^3 - y^3) - a(x^2 - y^2) + 2c^2(x - y) = 0,$$

of which equation $x - y$ is a factor: consequently $y = x$, and $4x^3 - 3ax^2 + 2c^2x = 0$: which gives

$$x = 0, \text{ and } x = \frac{3a \pm \sqrt{(9a^2 - 32c^2)}}{8}.$$

If $x = 0$, and $y = 0$, we have

$$D = 2c^2,\ E = 0, \text{ and } F = 2c^2:$$

therefore $u = 0$, is a *minimum.*

If $x = \frac{3a + \sqrt{(9a^2 - 32c^2)}}{8} = y$, we have

$$D = \frac{21a^2 - 32c^2 + 7a\sqrt{(9a^2 - 32c^2)}}{8} = F,$$

$$\text{and } E = -\frac{12a^2 + 4a\sqrt{(9a^2 - 32c^2)}}{8}:$$

and $DF > E^2$, for $9a^2 > 32c^2$; consequently $u = -\frac{27}{256}a^4 + \frac{9}{16}a^2c^2 - \frac{1}{2}c^4 - \frac{a}{256}(9a^2 - 32c^2)^{\frac{3}{2}}$, which is a *minimum.*

If $x = \frac{3a - \sqrt{(9a^2 - 32c^2)}}{8} = y$, we have

$$D = \frac{21a^2 - 32c^2 - 7a\sqrt{(9a^2 - 32c^2)}}{8} = F,$$

$$\text{and } E = -\frac{12a^2 - 4a\sqrt{(9a^2 - 32c^2)}}{8}:$$

$$\text{and } DF < E^2;$$

in this case, therefore,

$$u = -\frac{27}{256}a^4 + \frac{9}{16}a^2c^2 - \frac{1}{2}c^4 + \frac{a}{256}(9a^2 - 32c^2)^{\frac{3}{2}},$$

is neither a *maximum* nor a *minimum.*

If we consider the other factor of $\frac{du}{dx} - \frac{du}{dy} = 0$ or

$$4x^2 + 4xy + 4y^2 + ax + ay + 2c^2 = 0,$$

and eliminate y from this equation and the equation $\frac{du}{dx} = 0$, we shall get a final equation of six dimensions, the solution of which would be required in order to determine the other values of x and y. Euler. *Ib.*

(6). Let $u = x^4 + y^4 - x^2 + xy - y^2$:

$$\left.\begin{aligned} \frac{du}{dx} &= 4x^3 - 2x + y = 0 \\ \frac{du}{dy} &= 4y^3 - 2y + x = 0 \end{aligned}\right\};$$

eliminating y from these equations, we get

$$256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = 0$$
$$= x(4x^2 - 1)^3(4x^2 - 3).$$

The values of x are 0, $\pm\frac{1}{2}$, $\pm\frac{\sqrt{3}}{2}$.

If $x = y = 0$, we have $D = -2$, $E = 1$ and $F = -2$; therefore $u = 0$, is a *maximum*.

If $x = y = \pm\frac{1}{2}$, $D = E = F = 1$; in this case $DF = E^2$, and $u = -\frac{1}{8}$, a *minimum*.

If $x = \pm\frac{\sqrt{3}}{2}$, and $y = \mp\frac{\sqrt{3}}{2}$, we have $D = F = 7$, and $E = 1$: consequently $DF > E^2$ and $u = -\frac{9}{8}$, is *a minimum*.

(7). Let $z^2 = a x^2 + 2 b x y + c y^2 - e x - f y$: this is the general equation of curve surfaces of the second order, and it is proposed to find in what cases the ordinate z is a *maximum* or a *minimum*.

The equations $\frac{d u}{d x} = 0$, and $\frac{d u}{d y} = 0$, give us

$$x = \frac{e c - f b}{2 (a c - b^2)}, \; y = \frac{e b - f a}{2 (a c - b^2)}.$$

Also $D = 2 a$, $E = 2 b$, $F = 2 c$: u and therefore z is a *minimum* when a and c are positive, and $a c > b^2$: a *maximum* when a and c are *negative* and $a c > b^2$; but neither a *maximum* nor a *minimum*, when a and c have different signs, or when $a c < b^2$, in which last cases the surface is infinite with a conical asymptote. *Lagrange, Misc. Taurinensia.* Tom. I p. 31.

(8). Let $u = a x^2 - b x y + x z + y z$.

If we make $\frac{d u}{d x} = 0$, $\frac{d u}{d y} = 0$, and $\frac{d u}{d z} = 0$, we find $x = y = z = 0$: again, to find whether $u = 0$, is a *maximum* or a *minimum*, we must proceed to the second differential coefficients of u:

$$D = \frac{d^2 u}{d x^2} = 2 a, \; E = \frac{d^2 u}{d x \, d y} = -b, \; F = \frac{d^2 u}{d y^2} = 0,$$

$$G = \frac{d^2 u}{d y \, d z} = 1, \; H = \frac{d^2 u}{d z^2} = 0, \; K = \frac{d^2 u}{d z \, d x} = 1:$$

and $2 a h^2 - 2 b h k + 2 k i + 2 h i$

$$= 2 a \left\{ h - \frac{b k}{2 a} \right\}^2 + 2 a \left\{ \frac{(k + h) i}{a} - \frac{b^2}{4 a^2} \cdot k^2 \right\},$$

the second part of which expression may evidently become negative and greater than the first; consequently $u=0$, is neither a *maximum* nor a *minimum*.

(9). Let $u = \dfrac{2^4\, a\, c\, x\, y\, z}{(a+x)(x+y)(y+z)(z+e)}$.

Considering $\log u = u'$, as the function, and making $\dfrac{d\,u'}{d\,x} = 0$, $\dfrac{d\,u'}{d\,y} = 0$, $\dfrac{d\,u'}{d\,z} = 0$, we get

$$a\,y = x^2,\ x\,z = y^2, \text{ and } e\,y = z^2, \text{ or}$$

$$a : x :: x : y :: y : z :: z : e,$$

and $x = a\sqrt[4]{\dfrac{e}{a}} = a\,m$, $y = a\,m^2$, and $z = a\,m^3$, if $m = \sqrt[4]{\dfrac{e}{a}}$.

Again, if we put $a^2\,m\,(1+m)^2 = A$, we find

$$D = \frac{d^2u'}{d\,x^2} = -\frac{2}{A},\ E = \frac{d^2\,u'}{d\,x\,d\,y} = \frac{1}{m\,A},\ F = \frac{d^2u'}{d\,y^2} = -\frac{2}{m^2\,A},$$

$$G = \frac{d^2u'}{d\,y\,d\,z} = \frac{1}{m^3\,A},\ H = \frac{d^2u'}{d\,z^2} = -\frac{2}{m^4\,A},\ K = \frac{d^2\,u'}{d\,x\,d\,z} = 0:$$

$$\text{consequently } (DF - E^2)(DH - K^2) = \frac{12}{m^6\,A^4},$$

$$\text{and } DG - EK = -\frac{2\,a}{m^3\,A^2}: \text{ therefore}$$

$$(DF - E^2)(DH - K^2) > (DG - EK)^2,$$

and the value of u', and therefore of u, determined by this process, is a *maximum*.

This is the solution of a well-known mechanical problem: "to find the magnitudes of the perfectly elastic balls x, y, z, interposed between the given balls a and e, so that the velocity communicated by a, moving with the velocity c, through x, y and z to e, may be the greatest possible." Lagrange. *Misc. Taur.* Tom. I. p. 29.

(10). Let $u = 2xy + 2xz + 2yz$: which is subject to the equation of condition

$$xyz = a^3 : \quad (\alpha).$$

Eliminating z by means of equation (α), we find

$$u = 2xy + \frac{2a^3}{y} + \frac{2a^3}{x}:$$

from which we readily find, making $\frac{du}{dx} = 0$, and $\frac{du}{dz} = 0$,

$x = y = a$ and therefore also $z = a$; and $u = 6a^2$ a *minimum*.

This is the solution of the following problem: "Amongst all parallelopipedons of given volume, to find that which has the least surface."

(11). Let $u = (x+1)(y+1)(z+1)$: which is subject to the equation of condition

$$a^x b^y c^z = A.$$

Eliminating z, we find

$$u = (x+1)(y+1)\left\{1 + \frac{\log A}{\log c} - \frac{y \log b}{\log c} - \frac{x \log a}{\log c}\right\};$$

and making $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, we get

$a^{x+1} = b^{y+1}$ and therefore also $= c^{z+1}$.

Therefore $x = \frac{\log bcA - 2\log a}{3\log a}$, $y = \frac{\log acA - 2\log b}{3\log b}$,

and $z = \frac{\log abA - 2\log c}{3\log c}$; consequently

$$u = \frac{\left\{\log \sqrt[3]{(abcA)}\right\}^3}{\log a \log b \log c},$$

which is a *maximum*.

This is the solution of the following problem: "If a, b, c, be the prime factors of a number A, to find the number of times each factor must enter into that number, so that it may admit of the greatest possible number of divisors."

If the converse problem had been proposed: "Given the number of divisors and the prime factors a, b, c of a number, to find the exponents x, y, z of these factors, so that the number may be the greatest possible;" we should have found the same relation subsisting between these exponents as in the solution of the preceding problem. Waring. *Med. Alg.* p. 344.

(11). Let $u = \left(\frac{a^{x+1}-1}{a-1}\right)\left(\frac{b^{y+1}-1}{b-1}\right)\left(\frac{c^{z+1}-1}{c-1}\right)$,

the equation of condition being

$$u^x b^y c^z = A.$$

Eliminating z, we find

$$u = \frac{c A}{(a-1)(b-1)(c-1)} \left\{ (a^{x+1}-1)(b^{y+1}-1)\left(\frac{1}{a^x b^y} - \frac{1}{c A}\right) \right\}$$

Making $\frac{d u}{d x} = 0$, and $\frac{d u}{d y} = 0$, we find

$$\frac{a^{x+1} - b^{y+1}}{a^x b^y} = \frac{a^{x+1} - b^{y+1}}{c A},$$

an impossible equation, unless $a^{x+1} = b^{y+1}$ and therefore also $= c^{z+1}$.

The values of x, y, z are determined as in the preceding example, and

$$u = \frac{\{ \sqrt[3]{(a b c A)} - 1 \}^3}{(a-1)(b-1)(c-1)}, \text{ a maximum.}$$

This is the solution of the following problem: "If a, b, c be the prime factors of a number A, to find the relation subsisting among their exponents, so that the sum of the divisors of A may be the greatest possible." Waring. *Ib.*

(12). Let $u = A\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 - \frac{1}{z}\right)$, the equation of condition being

$$x^a\, y^b\, z^c = A.$$

Eliminating z, we find

$$u = A\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 - \frac{x^{\frac{a}{c}}\, y^{\frac{b}{c}}}{A^{\frac{1}{c}}}\right):$$

making $\dfrac{du}{dx} = 0$, and $\dfrac{du}{dy} = 0$, we get

$$a\,(x-1) = b\,(y-1) \text{ and therefore} = c\,(z-1).$$

The nature of the equation

$$x^a\, y^b\, z^c = A,$$

prevents our proceeding further in the direct determination of x or of u.

In this as well as in the two preceding examples, when the parts of which the functions are composed, are symmetrical, the conclusion which has been deduced for two or three variables, may be extended to any number: the same remark may be extended to the application of the *test* for determining whether the result is a *maximum* or a *minimum*.

In the last example, if A be a number, and x, y, z its prime factors, u is the expression for the number of integers less than A, which are also prime to it. Legendre. *Théorie des Nombres*. p. 8.

In considering any of these examples as the solutions of problems in the theory of numbers, it is of course understood that the integers which approach most nearly to the true values of x, y, z &c. are to be assumed.

(13). Let $u = a + \sqrt[3]{(x^2 + y^2)}$.

This expression is evidently a *minimum*, when $x = 0$, and $y = 0$; but in this case, we have

$$\frac{du}{dx} = \frac{2x}{3(x^2+y^2)^{\frac{2}{3}}}, \quad \frac{du}{dy} = \frac{2y}{3(x^2+y^2)^{\frac{2}{3}}},$$

which are both infinite, when x and y vanish.

A function of two or more variables, may therefore be a *maximum* or *a minimum*, when its first differential coefficients are infinite. See Example 15, p. 113. For an explanation of the meaning of this circumstance, in the theory of curve surfaces, see Lacroix, *Traitè du Calc. Diff.* p. 588.

(14). Given the three sides AB, BC, and CD, of a quadrilateral figure: to find the angles at B and C, when its area is a *maximum*. Fig. 100.

If $AB=a$, $BC=b$, $CD=c$, $\pi-\theta=$ the angle ABC and $\pi-\theta'=$ the angle BCD, we have the area or

$$u = ab \sin\theta + bc \sin\theta' + ac \sin(\theta+\theta')$$

$$\frac{du}{d\theta} = a\ \{ b\cos\theta + c\cos(\theta+\theta') \} = 0 \quad (\alpha).$$

$$\frac{du}{d\theta'} = c\ \{ b\cos\theta' + a\cos(\theta+\theta') \} = 0. \quad (\beta).$$

By eliminating θ', we find that the determination of $\cos\theta$, depends upon the solution of the cubic equation

$$(\cos\theta)^3 + \left(\frac{a^2+b^2+c^2}{2ab}\right)(\cos\theta)^2 - \frac{c^2}{2ab} = 0;$$

only one root of which is possible.

The conditions expressed by the equations (α) and (β) are satisfied, by supposing the quadrilateral figure $ABCD$ inscribed in a semicircle, of which the unknown side AD, is the diameter* : for in this case, if we describe a circle upon BC, cutting AC and DB in n and n', and AB and DC produced in N and N', we have $BN = Cn$ or $b\cos\theta = -c\cos(\theta+\theta')$, and $CN' = Bn'$ or $b\cos\theta' = -a\cos(\theta+\theta')$. It is very easily shewn that these conditions cannot be fulfilled upon any other hypothesis.

The diameter $AD = \sqrt{\{a^2+b^2+c^2+2ab\,.\cos\theta\}}$

$$= \sqrt{\{a^2+b^2+c^2+2bc\,.\cos\theta'\}};$$

which expressions are identical, since $a\cos\theta = c\cos\theta'$.

If we suppose the figure to be pentagonal, four of whose sides a, b, c, d are given; and if $\pi-\theta$, $\pi-\theta'$, $\pi-\theta''$ be the angles included by a and b, b and c, c and d, then the area or

$$u = ab\sin\theta + bc\sin\theta' + cd\sin\theta'' + ac\sin(\theta+\theta')$$

$$+ bd\sin(\theta'+\theta'') + ad\sin(\theta+\theta'+\theta'')$$

$$\frac{du}{d\theta} = a\{b\cos\theta + c\cos(\theta+\theta') + d\cos(\theta+\theta'+\theta'')\} = 0 \quad (\alpha).$$

$$\frac{du}{d\theta''} = d\{c\cos\theta'' + b\cos(\theta'+\theta'') + a\cos(\theta+\theta'+\theta'')\} = 0 \quad (\beta).$$

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* Cresswell. *Maxima and Minima.* p. 71.

$$\frac{du}{d\theta'} = b \{ c \cos \theta' + d \cos (\theta' + \theta'') \}$$

$$+ a \{ c \cos (\theta + \theta') + d \cos (\theta + \theta' + \theta'' \} = 0.$$

This last equation being reduced, by means of the first, becomes

$$c \cos \theta' - a \cos \theta + d \cos (\theta' + \theta') = 0. \quad (\gamma).$$

These equations are also satisfied, by supposing the pentagonal figure inscribed in a semicircle, of which the unknown side is the diameter. Thus suppose *ABCDE* (Fig. 101.) the pentagon and *AE* the diameter of the circumscribed circle: produce *AB* and *ED* to meet in *F*, *AB* and *DC* to meet in *H*, and *BC* and *ED* to meet in *K*: draw *Dd*, *Cc* perpendiculars upon *AF*: then $Bc = b \cos \theta$, $Bd = -d \cos (\theta + \theta' + \theta'')$ and $dc = c \cos (\theta + \theta')$: therefore since $Bc + dc - Bd = 0$, the first equation is satisfied by this hypothesis: and the same may be readily shewn of the second. Again, if we join *A* and *C*, *C* and *E*, we find

$$AE^2 = a^2 + b^2 + c^2 + d^2 + 2ab \cos \theta + 2cd \cos \theta''$$

$$= a^2 + b^2 + c^2 + d^2 + 2cd \cos \theta'' + 2b \,.\, CE \,.\, \cos ECK:$$

consequently $CE \cos . ECK = a \cos \theta$: draw *D m* and *EM* perpendiculars upon *CK* or *CK* produced:

then $Cm = c \cos \theta'$, and

$$Mm = ED \sin DEM = d \sin \left\{ \frac{\pi}{2} - \theta' - \theta'' \right\} = d \cos (\theta' + \theta''),$$

consequently,

$$CM = c \cos \theta' + d \cos (\theta' + \theta'') = a \cos \theta,$$

which is the condition expressed by the third equation (γ).

If the figure be hexagonal, five of its sides being a, b, c, d, e and $\pi - \theta$, $\pi - \theta'$, $\pi - \theta''$, $\pi - \theta'''$, the angles included by a and b, b and c, c and d, d and e respectively, then the area or

$$u = ab\sin\theta + bc\sin\theta' + cd\sin\theta'' + de\sin\theta'''$$
$$+ ac\sin(\theta+\theta') + bd\sin(\theta'+\theta'') + ce\sin(\theta''+\theta''')$$
$$+ ad\sin(\theta+\theta'+\theta'') + be\sin(\theta'+\theta''+\theta''')$$
$$+ ae\sin(\theta+\theta'+\theta''+\theta''').$$

Making $\frac{du}{d\theta}=0$, $\frac{du}{d\theta'}=0$, $\frac{du}{d\theta''}=0$, $\frac{du}{d\theta'''}=0$, we should get four equations of condition, which would be satisfied by supposing the hexagon inscribed in a semicircle, of which the sixth side x is determined by the resolution of the cubic equation

$$x^3-(a^2+C^2+D^2)x-2CDa=0, \quad \text{where}$$

$$C=\sqrt{\{b^2+c^2+2bc\cos\theta'\}} \text{ and } D=\sqrt{\{d^2+e^2+2de\cos\theta'''\}}.$$

(14). To find a point within a triangle, from which, if lines be drawn to the angular points, their sum is the least possible.

Let ABC be the triangle (Fig. 102), P the point required: let $BC=a$, $AC=b$, $AB=c$, PM a perpendicular upon $BC=y$, and $CM=x$: then

$$u=\sqrt{\{x^2+y^2\}}+\sqrt{\{(a-x)^2+y^2\}}$$
$$+\sqrt{\{x^2+y^2+b^2-2b(x\cos C+y\sin C)\}},$$

$$\frac{du}{dx}=\frac{x}{\sqrt{\{x^2+y^2\}}}-\frac{(a-x)}{\sqrt{\{(a-x)^2+y^2\}}}$$
$$+\frac{x-b\cos C}{\sqrt{\{x^2+y^2+b^2-2b(x\cos C+y\sin C)\}}}=0.$$

$$\frac{du}{dy}=\frac{y}{\sqrt{\{x^2+y^2\}}}+\frac{y}{\sqrt{\{(a-x)^2+y^2\}}}$$
$$+\frac{y-b\sin C}{\sqrt{\{x^2+y^2+b^2-2b(x\cos C+y\sin C)\}}}=0.$$

The elimination of x or of y from these equations, would lead to equations of great complexity: by a simple artifice, however, they may be made to exhibit the geometrical conditions, upon which the *minimum* depends: let $\theta =$ the angle CPM, $\theta' =$ the angle BPM: then

$$\sin\theta - \sin\theta' = \frac{b\cos C - x}{\sqrt{\{x^2+y^2+b^2-2b\,(x\cos C+y\sin C)\}}}$$

$$\cos\theta + \cos\theta' = \frac{b\sin C - y}{\sqrt{\{x^2+y^2+b^2-2b\,(x\cos C+y\sin C)\}}},$$

if we square each of these equations and add them together, we find

$\cos(\theta+\theta') = -\frac{1}{2}$, and therefore $\theta+\theta' =$ the angle $BPC = 120°$:

in the same manner we may shew that the angle APB, and therefore also the angle APC is $= 120°$*.

If we represent the distances AP, BP, CP, by t, v, z, we find

$$\left.\begin{array}{l} t^2 + tv + v^2 = a^2 \\ t^2 + tz + z^2 = b^2 \\ v^2 + vz + z^2 = c^2 \end{array}\right\}.$$

The elimination of v and z from these equations would lead to an equation of four dimensions, from which the value of t may be determined: the same method may be applied to the determination of v and z.

(15). To find the point within a quadrilateral figure, from which, if lines be drawn to the angular points, their aggregate is the least possible.

Let $ABCD$ (Fig. 103.) be the four-sided figure, and P the point required; and let $AB=a$, $BC=b$, $CD=d$, and

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* Cresswell. *Maxima* and *Minima*, p. 120.

$DA=c$, PM, which is perpendicular to $AB=y$, and $AM=x$: then

$$u=\sqrt{\{x^2+y^2\}}+\sqrt{\{(a-x)^2+y^2\}}$$

$$+\sqrt{\{c^2+x^2+y^2-2c\,(x\cos A+y\sin A)\}}$$

$$+\{b^2+(a-x)^2+y^2-2b\,[(a-x)\cos B+y\sin B]\}$$

$$=t+t'+t''+t''':$$

$$\frac{du}{dx}=\frac{x}{t}-\frac{(a-x)}{t'}+\frac{x-c\cos A}{t''}-\left(\frac{(a-x)-b\cos B}{t'''}\right)=0.\quad(\alpha).$$

$$\frac{du}{dy}=\frac{y}{t}+\frac{y}{t'}+\frac{y-c\sin A}{t''}+\frac{y-b\sin B}{t'''}=0.\quad(\beta).$$

From C and D, draw Cc and Dd perpendiculars upon AB or AB produced: call the angle $APM=\theta$, $BPM=\theta'$, $cCP=\phi$, and $dDP=\phi'$: then the equations (α) and (β) give us

$$\sin\theta-\sin\theta'=\sin\phi-\sin\phi',$$

$$\cos\theta+\cos\theta'=\cos\phi+\cos\phi',$$

equations which can only be satisfied, by making $\theta=\phi$, and $\theta'=\phi'$: the point P, therefore, is situated in the concourse of the diagonals AC and BD*.

(16). To find a point within a triangle, from which if lines be drawn to the angular points, the sum of their squares is the least possible.

Making use of the same construction and notation as in Ex. 14., we have

$$u=x^2+y^2+(a-x)^2+y^2+x^2+y^2+b^2-2b\,(x\cos C+y\sin C).$$

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* Cresswell. *Maxima and Minima*, p. 117.

Making $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$, we find

$$x = \frac{a + b\cos C}{3}, \text{ and } y = \frac{b\sin C}{3}:$$

$$\text{consequently } AP = \frac{\sqrt{\{2c^2 + 2b^2 - a^2\}}}{3},$$

$$BP = \frac{\sqrt{\{2a^2 + 2c^2 - b^2\}}}{3}, \text{ and } CP = \frac{\sqrt{\{2a^2 + 2b^2 - c^2\}}}{3}.$$

This point is the centre of gravity of the triangle.

(17). To find a point within a quadrilateral figure from which, if lines be drawn to the angular points, the sum of their squares is the least possible.

Making use of the same notation and construction as in Ex. 15., we shall find

$$u = (x^2 + y^2) + \{(a - x)^2 + y^2\}$$

$$+ \{c^2 + x^2 + y^2 - 2c(x\cos A + y\sin B\}$$

$$+ \{b^2 + (a - x)^2 + y^2 - 2b(a - x)\cos B + y\sin B\}.$$

Making $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$, we find

$$x = \frac{2a + c\cos A - b\cos B}{4},$$

$$y = \frac{c\sin A + b\sin B}{4}.$$

If we apply the *test* of a *maximum* or *minimum* in this case, we shall find

$$\frac{d^2u}{dx^2} = 8, \quad \frac{d^2u}{dx\,dy} = 0, \quad \frac{d^2u}{dy^2} = 8:$$

consequently $DF > E^2$, and D is positive, and the values of x and y which we have found, make u a *minimum*.

If it was required to determine a similar point within the pentagonal figure $ABCDE$ (Fig. 104.), we should find, making $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EA = e$, $AM = x$, and $PM = y$, that

$$u = (x^2 + y^2) + \{ (a - x)^2 + y^2 \}$$

$$+ \{ e^2 + x^2 + y^2 - 2e(x \cos A + y \sin A) \}$$

$$+ \{ b^2 + (a - x)^2 + y^2 - 2b[(a - x) \cos B + y \sin B] \}$$

$$+ \{ d^2 + e^2 + x^2 + y^2 - 2e(x \cos A + y \sin A)$$

$$+ 2d[x \cos (A + E) + y \sin (A + E)] \} ;$$

and making $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$, we get

$$x = \frac{2a + 2e \cos A - b \cos B - d \cos (A + E)}{5},$$

$$y = \frac{2e \sin A + b \sin B - d \sin (A + E)}{5}.$$

If the figure was hexagonal and its sides a, b, c, d, e, f, we should find

$$u = (x^2 + y^2) + \{ (a - x)^2 + y^2 \}$$

$$+ \{ b^2 + x^2 + y^2 - 2f(x \cos A + y \sin A) \}$$

$$+ \{ b^2 + (a - x)^2 + y^2 - 2b[(a - x) \cos B + y \sin B] \}$$

$$+ \{ e^2 - 2ef \cos F + f^2 + x^2 + y^2 - 2f(x \cos A + y \sin A)$$

$$+ 2e[x \cos (A + F) + y \sin (A + F)] \}$$

$$+ \{ c^2 - 2bc \cos C + b^2 + (a - x)^2$$

$$+ y^2 - 2b[(a-x)\cos B + y\sin B]$$
$$+ 2c[(a-x)\cos(B+C) + y\sin(B+C)].$$

The co-ordinates of the point P are

$$x = \frac{3a + 2f\cos A - 2b\cos B - e\cos(A+F) + c\cos(B+C)}{6},$$

$$y = \frac{2f\sin A + 2b\sin B - e\sin(A+F) - c\sin(B+C)}{6}.$$

If the figure was heptagonal, and its sides a, b, c, d, e, f, g, we should find

$$\begin{gathered}
u = (x^2 + y^2) + \{(a-x)^2 + y^2\} \\
+ \{g^2 + x^2 + y^2 - 2g(x\cos A + y\sin A)\} \\
+ \{b^2 + (a-x)^2 + y^2 - 2b[(a-x)\cos B + y\sin B]\} \\
+ \{f^2 - 2fg\cos G + g^2 + x^2 + y^2 - 2g(x\cos A + y\sin A) \\
+ 2f[x\cos(A+G) + y\sin(A+G)]\} \\
+ \{c^2 - 2bc\cos C + b^2 + (a-x)^2 + y^2 - 2b[(a-x)\cos B \\
+ y\sin B] + 2c[(a-x)\cos(B+C) + y\sin(B+C)]\} \\
+ \{e^2 - 2ef\cos F + f^2 - 2fg\cos G + g^2 \\
- 2eg\cos(G+F) + x^2 + y^2 - 2g(x\cos A + y\sin A) \\
+ 2f[x\cos(A+G) + y\sin(A+G)] \\
- 2e[x\cos(A+G+F) + y\sin(A+G+F)]\}
\end{gathered}$$

and the co-ordinates of the point P are $x =$

$$\frac{g\cos A - 2b\cos B - 2f\cos(A+G) + c\cos(B+C) + e\cos(A+G+F)}{7},$$

$y =$

$$\frac{\text{in}\,A + 2b\sin B - 2f\sin(A+G) - c\sin(B+C) + e\sin(A+G+F)}{7};$$

These formulæ sufficiently indicate the law of formation of corresponding formulæ for polygons of any number of sides whatever.

(18). To find the point within a triangular pyramid, from which if lines be drawn to the angular points, the sum of their squares is the least possible.

Let $ABCD$ (Fig. 105.) be the triangular pyramid, P the point required; let PM be perpendicular to the base DBC, and MQ perpendicular to the edge DC: let d, d', d'' be the edges opposite to the solid angle at D, and D, D', D'' the plane angles by which it is formed: let $AD=a$, $DC=b$, $DB=c$, $DQ=x$, $MQ=y$, $PM=z$, and let ϕ represent the angle of inclination of the face ADC to DBC: then

$$u=(x^2+y^2+z^2)+\{(b-x)^2+y^2+z^2\}$$

$$+\{x^2+y^2+z^2+c^2-2c(x\cos D+y\sin D\}$$

$$+\{x^2+y^2+z^2+a^2-2ax\cos D'$$

$$-2a\sin D'(y\cos\phi+z\sin\phi)\};$$

and making $\dfrac{du}{dx}=0$, $\dfrac{du}{dy}=0$, and $\dfrac{du}{dz}=0$, we find

$$x=\frac{b+a\cos D'+c\cos D}{4},$$

$$y=\frac{c\sin D+a\sin D'\cos\phi}{4},$$

$$z=\frac{a\sin D'\sin\phi}{4}.$$

Also, if we put $\cos\phi=\dfrac{\cos D''-\cos D\cos D'}{\sin D\sin D'}$, we shall find

$$DP=\frac{1}{4}\sqrt{\{a^2+2cb\cos D'+b^2+2bc\cos D+c^2+2ac\cos D''\}}$$

$$=\frac{1}{4}\sqrt{\{3\,a^2+3\,b^2+3\,c^2-(d^2+d'^2+d''^2)\}},$$

$$CP=\frac{1}{4}\sqrt{\{3\,b^2+3\,d^2+3\,d'^2-(a^2+c^2+d''^2)\}},$$

$$BP=\frac{1}{4}\sqrt{\{3\,d^2+3\,d''^2+3\,c^2-(a^2+b^2+d'^2)\}},$$

$$AP=\frac{1}{4}\{3\,a^2+3\,d'^2+3\,d''^2-(b^2+c^2+d^2)\}.$$

The point P is the centre of gravity of the triangular pyramid.

If the point P was required within a pyramid, whose base is a quadrilateral figure, from which, if lines be drawn to the angular points, the sum of their squares is the least possible; if we make $AD=a$ (Fig. 106), $DC=d$, $DE=d'$, $CB=a$, the angle $CDE=D$, the angle $ADC=D'$, the angle $DCB=C$, the angle of inclination of the face ADC to the base $DEBC=\phi$, x, y, z, the co-ordinates of the point P, referred to the plane $DEBC$ and the axis DC, their origin being at D: then we shall find

$$u=\{x^2+y^2+z^2\}+\{(d-x)^2+y^2+z^2\}$$

$$+\{x^2+y^2+z^2+d'^2-2\,d'(x\cos D+y\sin D)\}$$

$$+\{(d-x)^2+y^2+z^2+b^2-2\,b\,[(d-x)\cos C+y\sin C]\}$$

$$+\{x^2+y^2+z^2+a^2-2\,a\,x\cos D'$$

$$-2\,a\sin D'(y\cos\phi+z\sin\phi)\};$$

from whence we get, as before

$$x=\frac{2\,d+d'\cos D+a\cos D'-b\cos C}{5},$$

$$y = \frac{d' \sin D + b \sin C + a \sin D' \cos \phi}{5},$$

$$z = \frac{a \sin D' \sin \phi}{5}.$$

If the solid be an irregular triangular prism, within which a similar point P is required to be determined; if we make $CB = a$ (Fig. 107.), $CA = b$, $AB = c$, $BE = l$, $FC = l'$, $DA = l''$, the angle $EBC = B'$, $BCF = C'$, $DAB = A'$, ϕ and ϕ' the angles of inclination of the quadrilateral planes $EBCF$, $DABE$ to the base ABC, $x = BQ$, $y = QM$ and $z = PM$; then we shall find

$$\begin{aligned} u = (x^2 + y^2 + z^2) + \{ (a-x)^2 + y^2 + z^2 \} \\ + \{ x^2 + y^2 + z^2 + c^2 - 2c(x \cos B + y \sin B) \} \\ + \{ x^2 + y^2 + z^2 + l^2 - 2lx \cos B' \\ - 2l \sin B' (y \cos \phi + z \sin \phi) \} \\ + \{ (a-x)^2 + y^2 + z^2 + l'^2 - 2l'(a-x) \cos C' \\ - 2l' \sin C' (y \cos \phi + z \sin \phi) \} \\ + \{ x^2 + y^2 + z^2 + c^2 + l''^2 - 2cl'' \cos A' \\ - 2(c - l'' \cos A')(x \cos B + y \sin B) \\ - 2l'' \sin A' [(x \sin B - y \cos B) \cos \phi' + z \sin \phi'] \} : \end{aligned}$$

and making $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, $\frac{du}{dz} = 0$, we find

$$x = \frac{2a + 2c \cos B + l \cos B' - l' \cos C' - l'' \cos A' \cos B + l'' \sin A' \sin B}{6}$$

$$y = \frac{2c \sin B + (l \sin B' + l' \sin C') \cos \phi - l'' \cos A' \sin B - l'' \sin A' \cos B}{6}$$

$$z = \frac{(l \sin B' + l' \sin C') \sin \phi + l'' \sin A' \sin \phi'}{6}.$$

If $\phi = \phi' = 90°$, and $l = l' = l''$, or if the quadrilateral planes are perpendicular to the bases which are equal and parallel, then

$$x = \frac{a + c\cos B}{3}, \; y = \frac{c\sin B}{3}, \; z = \frac{l}{2};$$

the point P is the centre of gravity of the prism.

(19). To find a point within a triangle, from which, if perpendiculars be let fall upon the sides, the sum of their squares is the least possible:

Let ABC (Fig. 108.) be the triangle, P the point required, PM, PQ, PR the perpendiculars upon the sides, $BM = x$, $PM = y$; $PQ = y'$, $PR = y''$: then we have

$$u = y^2 + y'^2 + y''^2$$

$$u = y^2 + (x\sin B - y\cos B)^2 + \{(a - x)\sin C - y\cos C\}^2;$$

and making $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$, we find

$$y = \frac{abc\sin A}{a^2 + b^2 + c^2}, \; y' = \frac{abc\sin B}{a^2 + b^2 + c^2}, \; y'' = \frac{abc\sin C}{a^2 + b^2 + c^2}$$

$$\text{and} \;\; x = \frac{a(3c^2 + a^2 - b^2)}{2(a^2 + b^2 + c^2)}.$$

$$\text{Also} \;\; BP = \frac{ac}{a^2 + b^2 + c^2}\sqrt{\{2a^2 + 2c^2 - b^2\}}$$

$$CP = \frac{ab}{a^2 + b^2 + c^2}\sqrt{\{2a^2 + 2b^2 - c^2\}}$$

$$AP = \frac{bc}{a^2 + b^2 + c^2}\sqrt{\{2b^2 + 2c^2 - a^2\}}.$$

(20). To find a point within a triangular pyramid, from which if lines be drawn perpendicular to the several faces, the sum of their squares is the least possible.

Making use of the same notation and construction as in

Ex. 18., the angles of inclination of the faces to the base, being represented by ϕ, ϕ', ϕ'', and the perpendiculars by z, z', z'', z''', we shall find

$$u = z^2 + z'^2 + z''^2 + z'''^2$$

$$= z^2 + \{ y \sin \phi - z \cos \phi \}^2$$

$$+ \{ (x \sin D - y \cos D) \sin \phi' - z \cos \phi' \}^2$$

$$+ \{ [(b - x) \sin C - y \cos C] \sin \phi'' - z \cos \phi'' \}^2.$$

Making $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, $\frac{du}{dz} = 0$, we shall get three equations for the determination of x, y, z, of the form

$$\left.\begin{array}{l} A x + B y + C z = D \\ A' x + B' y + C' z = D' \\ A'' x + B'' y + C'' z = D'' \end{array}\right\}$$

but the coefficients A, B, C, &c. are very complicated, and a complete solution of these equations would be a work of considerable labour.

A more simple method of solving this problem, is the following: let E, E', E'', E''', represent the areas of the faces DBC, ADC, ADB, ABC, and let V be the solid content or volume of the pyramid: then we have $x = z^2 + z'^2 + z''^2 + z'''^2$, subject to the equation of condition

$$3\, V = E\, z + E'\, z' + E''\, z'' + E'''\, z'''.$$

Eliminating z''', we shall find by the ordinary process, making

$$\frac{du}{dz} = 0, \quad \frac{du}{dz'} = 0, \quad \frac{du}{dz''} = 0,$$

$$z = \frac{3\, V E}{E^2 + E'^2 + E''^2 + E'''^2}$$

$$y = \frac{3\, V}{\sin \phi} \frac{(E' + E \cos \phi)}{E^2 + E'^2 + E''^2 + E'''^2}$$

$$x = \frac{3V}{\sin D \sin \phi \sin \phi'}$$

$$\left\{\frac{E'' \sin\phi + E' \cos D \sin\phi' + E(\cos\phi' \sin\phi + \cos D \cos\phi \sin\phi')}{E^2 + E'^2 + E''^2 + E'''^2}\right\}.$$

(21). Amongst all triangular pyramids, of given base and altitude, to find that which has the least surface.

Making use of the same notation as in the preceding Example, if h be the altitude of the pyramid, and b, c, d the sides of the base, we find

$$u = \frac{h}{2}(b \operatorname{cosec} \phi + c \operatorname{cosec} \phi' + d \operatorname{cosec} \phi'')$$

which is subject to the equation of condition

$\frac{h}{2}(b \cot \phi + c \cot \phi' + d \cot \phi'') = E$, where E is the area of the base.

Eliminating ϕ'' and making $\frac{du}{d\phi} = 0$, $\frac{du}{d\phi'} = 0$,

we get $\phi = \phi'$ and therefore also $= \phi''$: or the faces of the pyramid are equally inclined to the base.

Amongst all triangular pyramids of given volume, the regular tetraedron has the least surface. This is an immediate corollary to the last problem.

The same proposition is true of pyramids whose bases are any polygonal figures whatever.

(22). From a given point, to draw the shortest line to the surface of a given sphere.

Let a, b, c, be the co-ordinates of the given point; α, β, γ, the co-ordinates of the centre of the sphere, whose radius is r: the equation of the sphere is

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2,$$

and

$$u = \sqrt{\{(x - a)^2 + (y - b)^2 + (z - c)^2\}},$$

is the length of the line drawn from the given point to the point in the sphere, whose co-ordinates are x, y, z.

Eliminating $(z - c)^2$ from u, making $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$ and putting $d = \sqrt{\{(a - \alpha)^2 + (b - \beta)^2 + (c - \gamma)^2\}}$, the distance of the given point from the centre of the sphere, we find

$$x = \alpha \pm \frac{r}{d}(a - \alpha)$$

$$y = \beta \pm \frac{r}{d}(b - \beta)$$

$$z = \gamma \pm \frac{r}{d}(c - \gamma)$$

and $u = d \mp r$, according as the upper or lower sign prevails, in the expressions for x, y, and z; the first being the expression for the *minimum* and the second for the *maximum* value of u.

(23). To find the shortest distance between two given lines, not in the same plane.

Let $x = az + \alpha$, $y = bz + \beta$,

and $x' = a'z' + \alpha'$, $y' = b'z' + \beta'$,

be the equations of the given lines: the distance of the points whose co-ordinates are x, y, z, and x', y', z', or

$$u = \sqrt{\{(z - z')^2 + (x - x')^2 + (y - y')^2\}}$$

$$= \sqrt{\{(z - z')^2 + (\alpha - \alpha' + az - a'z')^2 + (\beta - \beta' + bz - b'z')^2\}}$$

$$\frac{du}{dz} = \frac{1}{u}\{z - z' + a(\alpha - \alpha' + az - a'z')$$

$$+ b(\beta - \beta' + bz - b'z')\} = 0,$$

$$\frac{du}{dz'} = -\frac{1}{u}\left\{ z - z' + a'(\alpha - \alpha' + a z - a' z') \right.$$

$$\left. + b'(\beta - \beta' + b z - b' z' \right\} = 0.$$

From which equations we find

$$z - z' = \frac{(a'b - ab')\left\{(b - b')(\alpha - \alpha') - (a - a')(\beta - \beta')\right\}}{(a - a')^2 + (b - b')^2 + (a'b - ab')^2}$$

$$x - x' = \frac{(b' - b)(z - z')}{a'b - ab'}$$

$$y - y' = \frac{(a' - a)(z - z')}{a'b - ab'}$$

$$\text{and } u = \frac{(b - b')(\alpha - \alpha') - (a - a')(\beta - \beta')}{\sqrt{\left\{(a - a')^2 + (b - b')^2 + (a'b - ab')^2\right\}}}.$$

If the angles which this shortest line makes with the given lines be required, we find, since x, y, z and x', y', z' are co-ordinates of two points in it, that

$$x - x' = a''(z - z') \text{ and } y - y' = b''(z - z'')$$

where $a'' = \frac{b' - b}{a'b - ab'}$ and $b'' = \frac{a - a'}{a'b - ab'}$, the equations of this line being $x'' = a''z'' + \alpha''$ and $y'' = b''z'' + \beta''$: consequently

$$\cos\theta = \frac{aa'' + bb'' + 1}{\sqrt{\{1 + a^2 + b^2\}}\sqrt{(1 + a''^2 + b''^2)\}}} = 0,$$

$$\text{since } aa'' + bb'' + 1 = 0;$$

and therefore $\theta = 90$: in the same manner, it may be shewn that $\theta' = 90$; the shortest line is therefore perpendicular to both the given lines. Lacroix. *Traitè du Calc. Diff.* p. 524.

(24). To inscribe the greatest rectangular parallelopipedon in a given ellipsoid.

Let the equation of the ellipsoid be

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{a}\right)^2 = 1,$$

which is therefore an equation of condition to

$$u = 8\,x\,y\,z,$$

which is the expression for the inscribed parallelopipedon.

If we eliminate z, and make $\frac{d\,u}{d\,x} = 0$, and $\frac{d\,u}{d\,y} = 0$, we find

$$x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{3}},\ \text{and } z = \frac{c}{\sqrt{3}},$$

$$\text{and } u = \frac{8\,a\,b\,c}{3\sqrt{3}},\ \text{a } maximum.$$

(25). From a given point in the surface of an ellipsoid, to draw the longest line to the surface again.

Let α, β, γ be the co-ordinates of the given point; then we have

$$u = \sqrt{\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}},$$

which is subject to the equation of condition

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Eliminating z and making $\frac{d\,u}{d\,x} = 0$, and $\frac{d\,u}{d\,y} = 0$, we find the following equation for the determination of x,

$$(a\,\alpha)^4 + 2\,(a\,\alpha)^3\,(e + \epsilon')\,a\,x + \{\,a^4\,(e^2 + 4\,e\,e' + e'^2)$$

$$-\,(a\,\alpha)^2 - (b\,\beta)^2 - (c\,\gamma)^2\,\}\,\alpha^2\,x^2 -$$

$$2\left\{(a\alpha)^2(e+e')+(b\beta)^2e'+(c\gamma)^2e-a^4ee'(e+e')\right\}\alpha x^3$$

$$-\left\{(a\alpha)^2(e^2+4ee'+e'^2)+(b\beta e')^2+(c\gamma e)^2-a^4e^2e'^2\right\}x^4$$

$$-2a^2\alpha ee'(e+e')x^5-a^2e^2e'^2x^6=0,$$

where $e=\dfrac{a^2-b^2}{a^2}$, and $e'=\dfrac{a^2-c^2}{a^2}$. See Ex. 67, p. 132.

In most of the preceding examples, we have not put down the actual verification of the *maxima* and *minima* by Lagrange's test, as the reader will, in general, find little difficulty in supplying it.

On Curve Surfaces and Curves of Double Curvature.

Art. 137—144. Ex. (1). Let the surface be that of a cone, the sections perpendicular to whose axis are circles.

Its equation is

$$z=e\left\{x^2+y^2\right\}^{\frac{1}{2}},$$

the line upon which z is measured or the intersection of the planes of $x\,z$ and $y\,z$, being the axis of the cone, the origin of x, y and z being at its vertex, and e being $=\cot\theta$, where θ is the semi-angle of the cone.

Since $\dfrac{dz}{dx}=\dfrac{e^2x}{z}$, and $\dfrac{dz}{dy}=\dfrac{e^2y}{z}$, the equation of its tangent plane is

$$z'-z=\frac{dz}{dx}(x'-x)+\frac{dz}{dy}(y'-y)$$

$$= \frac{e^2 x}{z}(x' - x) + \frac{e^2 y}{z}(y' - y)$$

$$\text{or } z\,z' = e^2\,(x\,x' + y\,y').$$

In order to find where this plane cuts the axis of the cone, we must make $x' = 0$, $y' = 0$: this gives also $z' = 0$: or the tangent plane passes through the vertex of the cone, as is evident from the nature of its genesis.

The equations of the *normal* are

$$\left.\begin{aligned} x' - x + \frac{e^2 x}{z}(z' - z) = 0 \\ y' - y + \frac{e^2 y}{z}(z' - z) = 0 \end{aligned}\right\}.$$

If the point of its intersection with the plane of x, z, be required, we must make $y' = 0$: this gives us $\frac{e^2\,(z' - z)}{z} = 1$, and therefore $x' - x + x = x' = 0$; or the normal intersects the axis of the cone.

The general equation of conical surfaces is

$$z = x\,\phi\left(\frac{y}{x}\right)$$

the origin of the co-ordinates being at the vertex: they are all characterized by the same property, as the cone of geometry, that their tangent plane constantly passes through the vertex: thus since $\frac{dz}{dx} = \frac{z}{x} - \frac{y}{x}\cdot\frac{dz}{dy}$, the equation of this plane is

$$z' - z = \frac{z}{x}(x' - x) + \frac{y'x - x'y}{x}\cdot\phi'\left(\frac{y}{x}\right):$$

and if we make $x' = 0$, $y' = 0$, we have also $z' = 0$, as in the common cone.

(2). Let the curve surface be the ellipsoid, whose equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1. \qquad (\alpha).$$

where a, b, c are the semi-axes and x, y, z are reckoned from the centre.

If we make $x=0$, $y=0$, $z=0$, successively, we shall have the intersections of the surface with the planes of $x\,y$, $x\,z$, $y\,z$, which are expressed by the equations

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \text{ or } y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, \text{ or } z^2 = \frac{c^2}{a^2}(a^2 - x^2),$$

$$\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, \text{ or } z^2 = \frac{c^2}{b^2}(b^2 - y^2):$$

these sections are ellipses, and are sometimes called the *principal ellipses* of the ellipsoid. It is also evident from the form of this equation, that all sections parallel to each principal section, are ellipses similar to the corresponding *principal* ellipse.

Since $\frac{dz}{dx} = -\frac{c^2 x}{a^2 z}$ and $\frac{dz}{dy} = -\frac{c^2 y}{b^2 z}$, the equation of the tangent plane is

$$z' - z = -\frac{c^2 x}{a^2 z}(x' - x) - \frac{c^2 y}{b^2 z}(y' - y) \text{ or}$$

$$\frac{x\,x'}{a^2} + \frac{y\,y'}{b^2} + \frac{z\,z'}{c^2} = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{a}\right)^2 = 1. \qquad (\beta).$$

If we make $z' = 0$, we find

$$\frac{x\,x'}{a^2} + \frac{y\,y'}{b^2} = 1, \qquad (\gamma).$$

which is the equation of the line of intersection of the tangent plane with the plane of x, y: this line makes with the axis of x an angle whose tangent is $-\frac{b^2 x}{a^2 y}$.

The distances from the centre or origin of the intersections of the tangent plane with the axes of x, y and z respectively, or its rectangular subtangents are $\frac{a^2}{x}$, $\frac{b^2}{y}$, $\frac{c^2}{z}$: these three points, or any two of them and the given point upon the surface, determine the tangent plane.

The equation to a line *perpendicular* to that expressed by equation (γ), is

$$y'' = \frac{a^2 y}{b^2 x} x'' + \frac{b^2}{y}, \qquad (\delta).$$

which is the projection or *trace* upon the plane of x, y, of that line in the tangent plane which makes the greatest angle with this plane, which is obviously perpendicular to their common intersection: this is called the *line of greatest inclination**: the equation of its *trace* upon the plane of $x\,z$, is

$$z'' = \frac{a^2 z}{c^2 x} x'' + \frac{c^2}{z}. \qquad (\epsilon).$$

These two equations (δ) and (ϵ) determine the line in space.

The equations of the normal of the ellipsoid are

$$\left.\begin{aligned} x' - x - \frac{c^2 x}{a^2 z}(z' - z) = 0, \\ y' - y - \frac{c^2 y}{b^2 z}(z' - z) = 0. \end{aligned}\right\} \qquad (\eta).$$

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* *Ligne de plus grande pente.* Monge. *Application de l'Analyse à la Gèomètrie.* p. 45.

If $c=b$, or if the ellipsoid become a spheroid, we find, when $z'=0$, that also $y'=0$, or the normal intersects the axis of x, which is that of revolution, at a distance $x' = \frac{a^2-b^2}{a^2}x$: if $a=b=c$, the ellipsoid becomes a sphere, and when $z''=0$, we have also $y'=0$, and $x'=0$: or the normal passes through the centre of the sphere.

The length of the normal between the surface and a point whose co-ordinates are x', y' z', is equal to

$$\sqrt{\{(z'-z)^2+(x'-x)^2+(y'-y)^2\}},$$

$$=(z'-z)\sqrt{\left\{1+\frac{c^4x^2}{a^4z^2}+\frac{c^4y^2}{b^4z^2}\right\}},$$

which becomes, when $z'=0$,

$$\lambda=-c^2\sqrt{\left\{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right\}}.$$

If θ, θ', θ'' be the angles of the triangle formed by the intersections of the tangent plane with the planes of xy, yz, xz, we find

$$\tan\theta=-\frac{a^2b^2z\lambda}{c^4xy},\quad \tan\theta'=-\frac{b^2x\lambda}{a^2yz},\text{ and }\tan\theta''=-\frac{a^2y\lambda}{b^2xz};$$

and the area of this triangle

$$=\frac{a^2b^2c^2}{2xyz}\sqrt{\left\{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right\}}=-\frac{a^2b^2\lambda}{2xyz};$$

and the areas of its projections upon the planes of xy, xz, yz are

$$\frac{a^2b^2}{2xy},\ \frac{a^2c^2}{2xz},\ \frac{b^2c^2}{2yz},\text{ respectively.}$$

If c be infinite, the equation (α) becomes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

or the equation of a cylinder, whose sections perpendicular to the axis of z are ellipses, equal to the principal ellipse of the plane of $x\,y$: the equation of its tangent is

$$\frac{x\,x'}{a^2} + \frac{y\,y'}{b^2} = 1,$$

which is that of its *trace* upon the plane of $x\,y$: it is therefore constantly parallel to the axis of the cylinder, which is in this case the axis of z: a property which is characteristic of all cylindric surfaces *.

If the equation be

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1,$$

it will represent a surface of the second order, called the *hyperboloid of one surface* † or the *hyperboloid of four summits*: its principal sections in the planes of $x\,y$, $x\,z$, $y\,z$ are expressed by the equations

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \text{ or } y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{z}{c}\right)^2 = 1, \text{ or } z^2 = \frac{c^2}{a^2}(x^2 - a^2),$$

$$\left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1, \text{ or } z^2 = \frac{c^2}{b^2}(y^2 - b^2),$$

the first of these is an ellipse and the others are hyperbolas;

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* Monge, Ib. p. 4.

† *Hyperboloide à une nappe.*

and all the sections parallel to each of these, are figures respectively similar to the originals.

The *four summits* of this *hyperboloid* are the extremities of the axes of the *principal* ellipse.

The conical surface whose equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 0,$$

whose vertex is in the origin of the co-ordinates, is an asymptotic surface to this hyperboloid.

If $a = b$, the principal ellipse becomes a circle and all sections made by planes passing the axis of z are hyperbolas equal to each other: in this case the surface is generated by the revolution of the conical hyperbola, whose semi-axes are a and c, round its minor axis.

The surface of the second order whose equation is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1,$$

is called the *hyperboloid of two surfaces** or the hyperboloid of *two summits*: its principal sections are represented by the equations

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1, \text{ or } y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{z}{c}\right)^2 = 1, \text{ or } z^2 = \frac{c^2}{a^2}(x^2 - a^2)$$

$$-\left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1,$$

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* *Hyperboloide à deux nappes.*

the first two are hyperbolas : the third equation is impossible, and shews that no part of the surface is situated in the plane of $y\,z$: its *two summits* are *the vertices of the principal hyperbolas.*

This hyperboloid consists of two distinct surfaces separated from each other, like the two portions of the conical hyperbola: the former is one continuous surface.

It also admits of a conical asymptote, whose equation is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 0.$$

If $c = b$, all sections made by planes passing through the axis of x are hyperbolas equal to each other : the surface is, in this case, generated by the revolution of the hyperbola, which constitutes one of its sections, round its major axis. The reader will have no difficulty in the solution of the same problems concerning the tangent planes and normals of these surfaces, which have been previously solved in the case of the ellipsoid.

(3). Let the curve surface be the *paraboloid,* whose equation is

$$z^2 = 4\,a\,x - \frac{a\,y^2}{a'},$$

which is likewise of the second order ; its principal sections are represented by the equations

$$y^2 = 4\,a'\,x, \quad z^2 = 4\,a\,x, \quad z = y\sqrt{\left(-\frac{a'}{a}\right)}:$$

the first two are parabolas, whose *latera recta* are $4\,a'$ and $4\,a$: the third is imaginary, and shews that no part of the surface is situated in the plane of $y\,z$: all sections parallel to this plane, when x has any positive value α, are ellipses whose equations are

$$z^2 + \frac{a}{a'} y^2 = 4 a \alpha:$$

this surface has from hence been called the *elliptic paraboloid:* it may be conceived to be generated by the second section moving parallel to itself, in such a manner that its vertex may be constantly found in the first. If $a = a'$, the elliptic sections become circular, and the surface is the *paraboloid* of *revolution.*

The equation of its tangent plane is

$$z z' = 2 a (x' + x) - \frac{a y y'}{a'}.$$

Its sub-tangents upon the axes of x, y, z, are

$$-x, \quad \frac{2 a' x}{y}, \text{ and } \frac{2 a x}{z} \text{ respectively.}$$

The equations of the normal are

$$x' = -\frac{2 a}{z} z' + 2 a + x.$$

$$y' = \frac{a y}{a' z} z' + \frac{(a' - a)}{a'} y.$$

The co-ordinates of the point where it meets the plane of $x y$ are,

$x' = 2 a + x$ and $y' = \dfrac{a' - a}{a'} y$, the second of which vanishes when $a' = a$.

Its length from the point of contact to this point, is

$$\lambda = -\sqrt{\left\{4 a x + 4 a^2 + \frac{a}{a'} \left(\frac{a}{a'} - 1\right) y^2\right\}};$$

and if θ, θ', θ'' be the angles formed by the intersections of the tangent plane with the planes $x y$, $x z$, $y z$, we find

$$\tan \theta = -\frac{a' z \lambda}{2 a^2 y}, \quad \tan \theta' = -\frac{y \lambda}{2 a' z} \text{ and } \tan \theta'' = -\frac{2 a' \lambda}{z y}.$$

If the equation of the surface be

$$z^2 = \frac{a}{a'} y^2 - 4\, a\, x,$$

the principal sections are

$$y^2 = 4\, a'\, x, \quad z^2 = -4\, a\, x, \text{ and } z = \pm\, y \sqrt{\frac{a}{a'}};$$

the first two of which are parabolas, whose branches are on different sides of the plane of $y\,z$: the third represents two straight lines passing through the origin, on different sides of the plane of $x\,y$ and making an angle with the axis of y whose tangent is $= \sqrt{\frac{a}{a'}}$.

This surface would be generated by the second of the principal sections moving parallel to itself, whilst its vertex moves in the curve of the first section.

All sections parallel to the third are hyperbolas, whether x be positive or negative: it has hence been called the *hyperbolic paraboloid*.

The centres of the hyperbolic sections are in the axis of x: their major axes will be parallel to the axis of z, when x is positive and to that of y, when x is negative; and if these sections be projected upon the plane of yz, their asymptotes will be the straight lines which together constitute the principal section in that plane: two planes therefore which pass through the axis of x and through these lines respectively, will include the whole surface of this *paraboloid* between them and may be considered as asymptotes to it. If $a = a'$, these planes make each an angle of 45° with the plane of $x\,y$ and are therefore at right angles to each other.

(4). Let the equation of the curve surface be

$$x\,y\,z = a^3,$$

or such that the parallelopipedons inscribed between it and the planes of $x\,y$, $x\,z$ and $y\,z$ may be constantly equal to a given cube.

The planes of $x y$, $x z$ and $y z$ are asymptotes to the surface, and all sections made by planes respectively parallel to each of them, are rectangular hyperbolas.

The equation of the tangent plane is

$$x\,y\,z' + y\,z\,x' + x\,z\,y' = 3\,a^3.$$

Its subtangents are $3\,x$, $3\,y$, $3\,z$, respectively.

The area of the tangent plane included between the co-ordinate planes is

$$= -\frac{9\,a^3\,\lambda}{2\,z^2}$$

where $\lambda = -z^2\sqrt{\left\{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right\}}$, is the length of the *normal* between the point of contact and the plane of $x\,y$.

The triangular pyramid included by the co-ordinate planes and the tangent plane is a constant quantity and equal to $\frac{9\,a^3}{2}$.

(5). Let the surface be generated by the revolution of a circle round a given line as an axis, which is in the same plane with it.

Let $C\,D$, Fig. 109. be the given line, B the centre of the revolving circle, BA a perpendicular upon $C\,D$ cutting the circle in E: let A be the origin, $A\,D$ the axis of x: let $AQ = x$, $QM = y$, $MP = z$, $A\,B = a$, $BE = r$: then we have

$$Q\,P = \sqrt{\{y^2 + z^2\}},$$

$$\text{and } A\,Q^2 = x^2 = r^2 - \{a - \sqrt{(y^2 - z^2)}\}^2$$

$$\text{or } \{x^2 + y^2 + z^2 + a^2 - r^2\}^2 = 4\,a^2\,(y^2 + z^2)$$

for the equation of the surface.

This is the equation of the surface of an *annulus* or ring.

The equations of the normal are

$$x' - x = -\frac{x\sqrt{(y^2 + z^2)}}{z\sqrt{(r^2 - x^2)}}(z' - z),$$

$$y' - y = \frac{y}{z}(z' - z).$$

If $z' = 0$, we have also $y' = 0$, or the normal passes through the axis of revolution.

The length of the normal between the point of contact and the axis of x, is

$$= - r\sqrt{\left\{\frac{y^2 + z^2}{r^2 - x^2}\right\}},$$

which is infinite and therefore parallel to that axis, when $x = \pm r$.

The equation of its tangent plane is

$$z z' + y y' - x x'\sqrt{\left\{\frac{y^2 + z^2}{r^2 - x^2}\right\}}$$

$$= z^2 + y^2 - x^2\sqrt{\left\{\frac{y^2 + z^2}{r^2 - x^2}\right\}}.$$

If the generating circle be inclined at an angle θ, to the plane passing through the axis of revolution and its centre, and if ϕ be the angle which its intersection with that plane makes with the line AB, then the equation of the surface generated is

$$r^2 = x^2 + u^2 + (u \sin\phi - x \cos\phi)^2 \tan^2\theta,$$

where u is determined in terms of x, y, and z, by means of the equation

$$2au = a^2 + r^2 - x^2 - y^2 - z^2.$$

(6). Let the surface be the *cono-cuneus* of Wallis, which is generated as follows.

Let $ABCD$ (Fig. 110.) be a rectangular parallelogram in the plane of xy: upon CD describe a semicircle, whose plane is parallel to the plane of $x\,z$: the surface is described by a line PR parallel to the plane of $y\,z$, one extremity of which moves in the line AB and the other in the circumference of the semicircle CRD.

The solid partakes of the form and generation of a *cone* and a *wedge*, from whence its name is derived.

If $AP = x$, $PQ = y$, $QM = z$, $AC = c$, and $CD = 2\,r$, its equation is

$$\frac{c\,z}{y} = \sqrt{(r^2 - x^2)}.$$

The equation of its tangent plane is

$$c^2\,z\,y\,z' - c^2\,z^2\,y' + x\,y^3\,x' = x^2 y^3.$$

This solid was the subject of a particular dissertation of Wallis, who determined its volume, centre of gravity, &c.*

If we suppose the point P to move in a semicircle, described upon AB and in the plane of $x\,y$, we shall have a surface whose equation is

$$r^2 - x^2 - c\,z = (z - y)\sqrt{(r^2 - x^2)}.$$

Instead of supposing the curves, which guide the generating line to be straight lines or circles, we may suppose them to be any curves whatever, and we shall thus get a series of surfaces of a similar character†, whose equations may be determined in the same manner as in the cases we have considered.

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* Wallisii *Opera*, Tom. II. p. 661.

† Denominated by the French Analysts "*Surfaces gauches:*" see *Mémoires présentés à l' Académie des Sciences par les Savans Etrangers.* Tom. IX. p. 623. 1780.

(7). Let the curve of *double curvature* be the *helix*, or the curve formed by a thread, wrapped round the surface of a cylinder, so as always to make the same angle with the axis. Or a cylinder being supposed to be generated by the uniform revolution of a parallelogram round one of its sides, the *helix* is traced out by a point moving uniformly upon the other. (Fig. 111.)

Let the axis of z be the axis of the cylinder: the plane of $x\,y$ which is perpendicular to this axis, and all planes parallel to it, will form circular sections with the surface of the cylinder, whose equations are

$$x^2 + y^2 = a^2.$$

Also z bearing a constant ratio to the arc described, whose cosine is x, sine y and radius a, we find

$$\left.\begin{aligned} z &= e \cos^{-1} \frac{x}{a} \\ z &= e \sin^{-1} \frac{y}{a} \end{aligned}\right\}$$

which are the equations of the *helix*.

The projection of this curve upon the plane of $y\,z$ is a curve called the *Line of Sines*, which becomes the *Quadratrix* of Tschirnhausen

$$\text{when } e = \frac{2a}{\pi}. \quad \text{See p. 172.}$$

Since $\dfrac{dz}{dx} = -\dfrac{e}{y}$, and $\dfrac{dz}{dy} = \dfrac{e}{x}$, the equations of its tangent are

$$z' - z = -\frac{e}{y}(x' - x),$$

$$z' - z = \frac{e}{x}(y' - y).$$

The co-ordinates of the point where it meets the plane of $x\,y$, when $z' = 0$, are

$$x' = x + \frac{z\,y}{e} = x + y \cos^{-1}\frac{x}{a},$$

$$y' = y - \frac{z\,x}{e} = y - x \cos^{-1}\frac{x}{a};$$

and its distance from the origin of the co-ordinates

$$= \sqrt{\{x'^2 + y'^2\}} = \frac{a}{e}\sqrt{\{e^2 + x^2\}}$$

$= a\sqrt{\left\{1 + \left(\cos^{-1}\frac{x}{a}\right)^2\right\}}$, which is therefore independent of e.

If α be the constant angle which the curve makes with the plane of $x\,y$, we have

$$\tan\,\alpha = \frac{d\,z}{\sqrt{\{d\,x^2 + d\,y^2\}}} = \frac{e}{\sqrt{(x^2 + y^2)}} = \frac{e}{a}.$$

The position of the osculating plane is determined by the equation

$$(x' - x)(d\,y\,d^2 z - d\,z\,d^2 y) + (y' - y)(d\,z\,d^2 x - d\,x\,d^2 z)$$
$$+ (z' - z)(d\,x\,d^2 y - d\,y\,d^2 x) = 0:$$

but if we consider z and y as functions of the third variable x, we shall have $d^2 x = 0$, by which the formula will be simplified: thus, we have, in this case

$\frac{dz}{dy} = \frac{e}{x}$; therefore $\frac{dy\,d^2z - dz\,d^2y}{dy^2} = -\frac{e\,dx}{x^2}$

and $\frac{dy\,d^2z - dz\,d^2y}{dx^2} = -\frac{e}{x^2} \cdot \frac{dy^2}{dx^2} \cdot dx = -\frac{e}{x^2} \cdot \frac{x^2}{y^2} \cdot dx$

$= -\frac{e}{y^2}\,dx$:

$\frac{dz}{dx} = -\frac{e}{y}$ and $\frac{d^2z}{dx^2} = \frac{e}{y^2} \cdot \frac{dy}{dx} = -\frac{ex}{y^3}$,

$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}}$ and $\frac{d^2y}{dx^2} = -\frac{\{a^2 - x^2 + x^2\}}{(a^2 - x^2)^{\frac{3}{2}}} = -\frac{a^2}{y^3}$:

and the equation of the osculating plane is

$$-(x' - x)\frac{e}{y^2} + (y' - y)\frac{ex}{y^3} - (z' - z)\frac{a^2}{y^3} = 0, \text{ or}$$

$$a^2(z' - z) - e(xy' - yx') = 0.$$

If we make $x' = 0$, and $y' = 0$, we find $z' = z$, or this plane cuts the axis of the cylinder, at the distance z from the origin, as is likewise evident from a consideration of the genesis of the curve.

The equation of the *normal plane* is

$$(x' - x)\frac{dx}{dz} + (y' - y)\frac{dy}{dz} + (z' - z) = 0,$$

which becomes in this case

$$e(z' - z) + (xy' - yx') = 0:$$

the equation of its intersection with the plane of xy, when $z' = 0$, is

$$y' = \frac{y}{x}x' + \frac{ez}{x}.$$

(8). Let us take the curve surface, which is generated by a horizontal line passing constantly through the vertical axis of z and whose extremity is always found in the *helix* of the preceding example.

If a plane pass through the axis of z, it will intersect the generating line in one of its positions : and if any point whatever be taken in this line, whose co-ordinates are z, x, y, then z will bear the same ratio to the angle described by the plane from its first position, as in the case of the *helix*, which *guides* the generating line : and this angle is evidently

$$= \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{\sqrt{(x^2+y^2)}}$$

$$= \cos^{-1}\frac{x}{\sqrt{(x^2+y^2)}} : \text{ consequently}$$

$$z = e\tan^{-1}\frac{y}{x} = e\sin^{-1}\frac{y}{\sqrt{(x^2+y^2)}} = e\cos^{-1}\frac{x}{\sqrt{(x^2+y^2)}}$$

is the equation of the curve surface in question.

A more general method of determining the equations of surfaces, subject to this mode of generation, is given by Monge*.

The equation of its tangent plane is

$$z' - z = \frac{e(xy' - yx')}{x^2+y^2} :$$

if we make $x' = 0$, $y' = 0$, we have $z' = z$, as was the case in the osculating plane of the *helix*, with which it coincides when $x^2 + y^2 = a^2$.

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* *Application de l' Analyse à la Géométrie*. p. 26.

The equations of the *normal* are

$$\left.\begin{aligned} x' - x - \frac{e y}{x^2 + y^2}(z' - z) = 0, \\ y' - y + \frac{e x}{x^2 + y^2}(z' - z) = 0; \end{aligned}\right\}$$

and its length from the point of contact to the plane of $x y$ or

$$\lambda = - z \sqrt{\left\{1 + \frac{e^2}{x^2 + y^2}\right\}}.$$

If x or y be very large, we have $\lambda = - z$, or the normal is perpendicular to the plane of $x y$ and the tangent plane parallel to it: if $x = 0$, and $y = 0$, we have $\lambda = \infty$, or the normal is parallel to the plane of $x y$ and the tangent plane perpendicular to it.

This surface possesses some remarkable properties: it is *one* of those, whose area included by any given curve through which it passes, is a *minimum**.

If the *helix*, through which the generating line passes, was described upon the surface of a cylinder whose sections parallel to the base are elliptical, the surface would be the same as before: for these sections in the *helix* merely determine a relation between x and y, which have no such dependence in the equation of the surface.

This is the surface presented by the superior and inferior surface of a staircase, attached to a vertical column round which it winds, or to the concave wall of a circular or elliptical tower; its thickness being supposed to be uniform and no regard being paid to irregularities caused by the form of the steps.

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* *Meusnier. Mémoires présentés à l'Académie*...Tom. X. 1785.

(9). Let the curve of *double curvature* be the spiral of Pappus.

This spiral is generated by a point moving uniformly upon a quadrantal section of a hemisphere, from the pole to the base, whilst the section moves uniformly through the circumference of the base. Fig. 112.

Let C be the centre of the base, D the pole of the hemisphere, AD the generating quadrant in its first position: let DB be any other position of the quadrant: PM a perpendicular upon the base, MQ a perpendicular upon AC: let $CA = r$, $CQ = x$, $QM = y$, $PM = z$: the angle $ACB = \varphi$: then since the arc DP is one fourth part of the arc AB, we have

$$\frac{z}{r} = \cos\frac{\phi}{4},\ \frac{y}{\sqrt{(r^2 - z^2)}} = \sin\varphi, \text{ and } \frac{x}{\sqrt{(r^2 - z^2)}} = \cos\varphi:$$

from whence we get

$$\left.\begin{array}{r} 2z^5 - 3r^2z^3 + r^4z + \dfrac{yr^4}{4} = 0 \\ \{8z^4 - 8r^2z^2 + r^4\}\sqrt{(r^2 - z^2)} - r^4x = 0 \end{array}\right\};$$

the equations of the spiral.

The spherical area which is included between the quadrant AD, the spiral and the base of the hemisphere, is perfectly quadrable and equal to the square described upon the diameter of the hemisphere. This singular property is demonstrated geometrically by Pappus*.

The equation of the surface generated by a line parallel to the plane of $x\,y$, which passes through the axis of z, and is guided by this spiral, is

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* *Math. Collect.* Lib. iv. Prop. 30.

$$4 \cos^{-1} \frac{z}{r} = \cos^{-1} \frac{x}{\sqrt{(x^2+y^2)}}.$$

The equation of its tangent plane is

$$z' - z = \frac{\sqrt{(r^2-z^2)}}{4\,(x^2+y^2)}\,(y\,x' - x\,y').$$

If $x'=0$, $y'=0$, we have $z'=z$, as in all surfaces generated in this manner: the other subtangents are

$$\frac{4\,z\,(x^2+y^2)}{x\,\sqrt{(r^2-z^2)}}, \text{ and } -\frac{4\,z\,(x^2+y^2)}{y\,\sqrt{(r^2-z^2)}}.$$

If the spiral be generated by a point describing the quadrant uniformly, whilst it moves uniformly through a quadrant of the base; we shall find, assuming the same axes and origin as before, that its equations are

$$\left.\begin{array}{r} x^2 + y^2 - r\,y = 0 \\ z^2 - r^2 + r\,y = 0 \end{array}\right\};$$

and the equation of the surface, generated in the same manner as in the former case, is

$$z = \frac{r\,x}{\sqrt{(x^2+y^2)}}.$$

(7). Let the curve of double curvature be formed by the intersections of a sphere and cylinder, the axis of the cylinder not passing through the centre of the sphere.

Let the origin of the co-ordinates be the centre of the sphere, and let the axis of the cylinder be in the plane of $x\,z$, parallel to the axis of z, and distant from it by a line equal to

c: let a and b be the radii of the sphere and cylinder: then we have

$$x^2+y^2+z^2=a^2, \text{ the equation of the sphere;}$$

$$(x-c)^2+y^2=b^2, \text{ the equation of the cylinder:}$$

from which we get

$$z^2=e^2-2cx, \text{ if } e^2=a^2+c^2-b^2;$$

$$z^2=e'^2-2c\sqrt{\{b^2-y^2\}}, \text{ if } e'^2=a^2-c^2-b^2,$$

which are the equations of the projections of the curve upon the planes of $x\,z$ and $y\,z$.

If $a>b+c$, the cylinder penetrates entirely within the sphere and the second projection consists of two separate ovals: the limits of the values of z are $\sqrt{\{a^2-(b-c)^2\}}$, and $\sqrt{\{a^2-(b+c)^2\}}$: for all other values, y is impossible, or the curve has no existence, (Fig. 113. No. 1.): if $a=b+c$, the cylinder falls just within the sphere and the two ovals meet, (Fig. 113. No. 2.): if $a>c<b+c$, a part only of the diameter of the cylinder penetrates the sphere, and the projection is a single oval with points of flexure, (Fig. 113. No. 3.): as $b+c-a$ still diminishes, these points disappear, and when the cylinder cuts off a very small portion of the surface, it assumes the form of an ellipse. (Fig. 113. No. 4.)

The equations of the tangent are

$$\left.\begin{aligned} z'-z &= -\frac{c}{z}(x'-x) \\ z'-z &= \frac{cy}{z\sqrt{(b^2-y^2)}}(y'-y) \end{aligned}\right\}.$$

The length of the tangent between the point of contact, and the plane of $x\,y$ is

$$\frac{z}{cy}\sqrt{(c^2y^2+b^2z^2)}.$$

The equation of the osculating plane, is

$$(z'-z)\,b^2z^3-(y'-y)\,c^2y^3+(x'-x)\,c\{b^2z^2-c\,y^2\sqrt{(b^2-y^2)}\}=0.$$

Thus, if we apply this to the points P or P' of the curve, which are most distant from each other, and in the plane of $x\,z$: we have $y=0$, $x=c-b$, $z^2=a^2-(c-b)^2$ and the equation becomes

$$(z'-z)\,z+c\,(x'-x)=0, \quad \text{or}$$

$$z\,z'+c\,x'=z^2+c\,x, \quad \text{or}$$

$$z'\sqrt{\{a^2-(c-b)^2\}}+c\,x'=a^2+b\,c-b^2;$$

the plane is parallel to the axis of y and the equation is that of its trace upon the plane of $z\,x$, which makes an angle with the axis of x, whose tangent is $\pm\dfrac{c}{\sqrt{\{a^2-(c-b)^2\}}}$.

If $b=c=\dfrac{a}{2}$, the equations of the curve become

$$z^2=a\,(a-x)$$

$$z^2=a\left\{\frac{a}{2}-\sqrt{\left(\frac{a^2}{4}-y^2\right)}\right\}:$$

if similar sections be made on the other side of the centre, we shall have four similar and equal sections upon the surface of the sphere, which possess these remarkable properties, that if they be subtracted from the surface of the sphere, the portion that remains is absolutely quadrable and equal to the square of the diameter, which is likewise equal to the area of the cylindrical surfaces, which are enveloped by the sphere. The first of these properties constitutes the celebrated problem of Viviani.

(8). Let the curve of double curvature arise from the intersection of two cylinders, whose axes intersect at right angles.

Let the origin be the intersection of the axes of the cylinders, and let these axes be assumed as the axes of x and y: the equations of the intersections are

$$\left.\begin{array}{l} z^2 + x^2 = a^2 \\ z^2 + y^2 = b^2 \end{array}\right\}.$$

The equations of the tangent are

$$\left.\begin{array}{l} z\,z' + x\,x' = a^2 \\ z\,z' + y\,y' = b^2 \end{array}\right\}.$$

The co-ordinates of the point where it meets the plane of $x\,y$ are $x' = \frac{a^2}{x}$, and $y' = \frac{b^2}{y}$; and its length from the point of contact to that plane

$$= \frac{z}{x\,y} \sqrt{\{x^2 y^2 + z^2 x^2 + z^2 y^2\}}.$$

The equation of the osculating plane is

$$(x' - x)\frac{b^2}{(b^2 - z^2)^{\frac{3}{2}}} - (y' - y)\frac{a^2}{(a^2 - z^2)^{\frac{3}{2}}}$$

$$+ (z' - z)\frac{(a^2 - b^2)\,z^3}{(a^2 - z^2)^{\frac{3}{2}}\,(b^2 - z^2)^{\frac{3}{2}}} = 0, \quad \text{or}$$

$$b^2 (a^2 - z^2)^{\frac{3}{2}}\,x' - a^2 (b^2 - z^2)^{\frac{3}{2}}\,y' + (a^2 - b^2)\,z^3\,z'$$

$$+ (a^2 - b^2)\,a^2\,b^2 = 0.$$

Thus, if we take a point where $y = 0$, and therefore $z = b$ and $x = \sqrt{(a^2 - b^2)}$, the equation becomes

$$x' \sqrt{(a^2 - b^2)} + z'\,b + a^2 = 0.$$

If $z = 0$, and therefore $x = a$, and $y = b$, it becomes

$$a\,x' - b\,y' + (a^2 - b^2) = 0.$$

In the first case the tangent plane is parallel to the axis of y; in the second, it is parallel to that of z.

The equation of the normal plane is

$$x y z' - z y x' - x z y' + x y z = 0.$$

(10). Let the curve of double curvature, be formed by the intersections of two cones, whose axes intersect at right angles. Fig. 114.

Let V and V' be the vertices of the cones, $A V$ and $A V'$ their axes which intersect at right angles at A; assume A as the origin, AV the axis of x, and AV' that of y; then if $AV = a$, and $AV' = a'$, the equations of the cones are

$$a + x = e \sqrt{\{ z^2 + y^2 \}}, \text{ and } a' + y = e' \sqrt{\{ z^2 + x^2 \}};$$

from which we get

$$\frac{(a+x)^2}{e^2} + x^2 = \frac{(a'+y)^2}{e'^2} + y^2,$$

$$\text{and} \quad a' + \sqrt{\left\{ \frac{(a+x)^2}{e^2} - z^2 \right\}} = e' \sqrt{\{ z^2 + x^2 \}},$$

for the equations of the projections of the curve of double curvature upon the planes of $x y$ and $x z$.

If $\frac{a}{\sqrt{\{1 + e^2\}}}$ be greater than $\frac{a'}{\sqrt{\{1 + e'^2\}}}$, the second cone penetrates entirely within the first, and the projections upon the plane of $x y$ are two figures with their convexities opposed to each other, such as are represented in Fig. 115. No. 1.; if $\frac{a}{\sqrt{(1+e^2)}} = \frac{a'}{\sqrt{(1+e'^2)}}$, these figures touch each other as in No. 2.: if $\frac{a}{\sqrt{(1+e^2)}} < \frac{a'}{\sqrt{(1+e'^2)}}$, the figures possess points of flexure, and assume the form represented in No. 3. which approximate more and more to straight lines converging to the vertex of the second cone, as the excess of the second of these quantities above the first, continues to increase. The corresponding projections upon the plane of $x z$ are ovals included within each other in the first instance as in No. 4; which in the second touch in two extreme points, as in No. 5; and in the third they become pairs of curves, with their concavities opposite to each other, the lines joining the extreme points of each pair of curves being parallel, as in No. 6.

The equations of the tangent are

$$(x'-x)=\frac{e^2\{a'+(1+e'^2)y\}}{e'^2\{a+(1+e^2)x\}}(y'-y),$$

$$(x'-x)=\frac{e^2 z\{(1-e'^2)\sqrt{(z^2+x^2)}-a'e'^2\}}{\{a+x(1+e^2e'^2)\sqrt{(z^2+x^2)}-a'e'e^2x\}}(z'-z).$$

The only difficulty attending the determination of the osculating plane arises from the complication of the expressions which it is necessary to differentiate.

(11). To find the equation of the curve of double curvature, in which, if any point be taken, the sum of its distances from two given points is equal to a given line and also the sum of its distances from two other points is equal to another given line. Fig 116.

Let B and C be two of the given points and B' and C' the two others: let a plane pass through B, C and C', which may be assumed as that of $x y$: bisect BC in A, and assume A as the origin and AC as the axis of x: let $AB=AC=a$: and let c, c' be the co-ordinates of C', and b, b', b'' the co-ordinates of B': then the distance of a point whose co-ordinates are x, y, z,

from B $=\sqrt{\{(x+a)^2+y^2+z^2\}}$,

from C $=\sqrt{\{(x-a)^2+y^2+z^2\}}$,

from $C' = \surd \{ (x-c)^2 + (y-c')^2 + z^2 \}$,

and from $B' = \surd \{ (x-b)^2 + (y-b')^2 + (z-b'')^2 \}$:

consequently if e and e' be the given lines, we have

$$\surd \{ (x+a)^2 + y^2 + z^2 \} + \surd \{ (x-a)^2 + y^2 + z^2 \} = e. \quad (\alpha).$$

$$\surd \{ (x-c)^2 + (y-c')^2 + z^2 \}$$
$$+ \surd \{ (x-b)^2 + (y-b')^2 + (z-b'')^2 \} = e' ; \quad (\beta).$$

from which the equations of the projections of the curve of double curvature upon the co-ordinate planes may be obtained.

The equations (α) and (β) are those of two spheroids, whose foci are B and C, B' and C' and whose axes of revolution are e and e': the curve itself results from the intersections of these spheroids.

These examples may be sufficient to give the student some notions of the elements of the theory of curve surfaces, and curves of double curvature, and to excite his curiosity in the pursuit of a more accurate and extended knowledge of them: but the nature of this work and the want of any treatise upon the subject in our own language, to which reference could be made, has prevented our discussing many questions of great and essential importance in this subject: such as the curvature of curve surfaces, their evolutes, inflections, remarkable points and lines: the nature and characteristics of surfaces which admit of developement upon a plane, &c.: the whole of which are treated with singular elegance and generality in the celebrated work of Monge, to which we have referred before.

We have chosen examples, which are, for the most, only simple cases of more general problems, as the best calculated for a student to whom this subject is new: the contemplation of figures possessing three dimensions, is at all times attended with considerable difficulty, which is not much diminished by any representation upon a plane surface: and it is only by a careful study of particular examples, that the mind is enabled to grasp the full force and meaning of general theories, where all reasoning must be conducted by symbolical language.

PART II.

INTEGRAL CALCULUS.

On the Integration of Rational Functions.

(1). $$\int \frac{x\,dx}{a+b\,x} = \frac{x}{b} - \frac{a}{b^2}\log(a+b\,x).$$

(2). $$\int \frac{x^3\,dx}{a+b\,x} = \frac{x^3}{3\,b} - \frac{a\,x^2}{2\,b^2} + \frac{a^2\,x}{b^3} - \frac{a^3}{b^4}\log(a+b\,x).$$

(3). $$\int \frac{x^2\,dx}{(a+b\,x)^2} = \left(\frac{x^2}{b} - \frac{2\,a^2}{b^3}\right)\frac{1}{(a+b\,x)} - \frac{2\,a}{b^3}\log(a+b\,x).$$

(4). $$\int \frac{x^4\,dx}{(a+b\,x)^2} = \left(\frac{x^4}{3\,b} - \frac{2\,a\,x^3}{3\,b^2} + \frac{2\,a^2\,x^2}{b^3} - \frac{4\,a^4}{b^5}\right) \times$$
$$\frac{1}{(a+b\,x)} - \frac{4\,a^3}{b^5}\log(a+b\,x).$$

(5). $$\int \frac{x\,dx}{(a+b\,x)^3} = -\left(\frac{x}{b} + \frac{a}{2\,b^2}\right)\frac{1}{(a+b\,x)^2}.$$

(6). $$\int \frac{x^3\,dx}{(a+b\,x)^3} = \left(\frac{x^3}{b} - \frac{6\,a^2\,x}{b^3} - \frac{9\,a^3}{2\,b^4}\right)\frac{1}{(a+b\,x)^2}$$
$$- \frac{3\,a}{b^4}\log(a+b\,x).$$

(7). $$\int \frac{x^4\,dx}{(a+b\,x)^3} = \left(\frac{x^4}{2\,b} - \frac{2\,a\,x^3}{b^2} + \frac{12\,a^3\,x}{b^4} + \frac{9\,a^4}{b^5}\right) \times$$

$$\frac{1}{(a+b\,x)^2} + \frac{6\,a^2}{b^5} \log\,(a+b\,x).$$

(8). $$\int \frac{x^3\,dx}{(a+b\,x)^4} = \left(\frac{3\,a\,x^2}{b^2} + \frac{9\,a^2\,x}{2\,b^3} + \frac{11\,a^3}{6\,b^4}\right) \times$$

$$\frac{1}{(a+b\,x)^3} + \frac{1}{b^4} \log\,(a+b\,x).$$

(9). $$\int \frac{x^3\,dx}{(a+b\,x)^5} = -\left(\frac{x^3}{b} + \frac{3\,a\,x^2}{2\,b^2} + \frac{a^2\,x}{b^3} + \frac{a^3}{4\,b^4}\right) . \frac{1}{(a+b\,x)^4}.$$

(10). $$\int \frac{x^{-1}\,d\,x}{a+b\,x} = \frac{1}{a} \log \frac{x}{a+b\,x}.$$

(11). $$\int \frac{x^{-3}\,d\,x}{a+b\,x} = -\frac{1}{2\,a\,x^2} + \frac{b}{a^2\,x} - \frac{b^2}{a^3} \log \frac{a+b\,x}{x}.$$

(12). $$\int \frac{x^{-1}\,d\,x}{(a+b\,x)^2} = \frac{1}{a\,(a+b\,x)} - \frac{1}{a} \log \frac{a+b\,x}{x}.$$

(13). $$\int \frac{x^{-2}\,d\,x}{(a+b\,x)^2} = -\left(\frac{1}{a\,x} + \frac{2\,b}{a^2}\right) \frac{1}{(a+bx)} + \frac{2\,b}{a^3} \log \frac{a+b\,x}{x}.$$

(14). $$\int \frac{x^{-1}\,d\,x}{(a+b\,x)^3} = \left(\frac{3}{2\,a} + \frac{b\,x}{a^2}\right) \frac{1}{(a+b\,x)^2} - \frac{1}{a^3} \log \frac{a+b\,x}{x}.$$

(15). $$\int \frac{x^{-2}\,d\,x}{(a+b\,x)^3} = -\left(\frac{1}{a\,x} + \frac{9\,b}{2\,a^2} + \frac{3\,b^2\,x}{a^3}\right)$$

$$\frac{1}{(a+b\,x)^2} + \frac{3\,b}{a^4} \log \frac{a+b\,x}{x}.$$

(16). $$\int \frac{x^{-1}\,dx}{(a+bx)^4} = \left(\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3}\right) \times$$

$$\frac{1}{(a+bx)^3} - \frac{1}{a^4} \log \frac{a+bx}{x}.$$

(17). $$\int \frac{x^{-2}\,dx}{(a+bx)^4} = -\left(\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4}\right) \times$$

$$\frac{1}{(a+bx)^3} + \frac{4b}{a^5} \log \frac{a+bx}{x}.$$

(18). $$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{(ab)}} \tan^{-1}\left(x\sqrt{\frac{b}{a}}\right)$$

$$= \frac{1}{\sqrt{(ab)}} \sin^{-1} \sqrt{\left(\frac{bx^2}{a+bx^2}\right)} = \frac{1}{\sqrt{(ab)}} \cos^{-1}$$

$$\sqrt{\left(\frac{a}{a+bx^2}\right)} = \frac{1}{\sqrt{(ab)}} \sec^{-1} \sqrt{\left(\frac{a+bx^2}{a}\right)}.$$

(19). $$\int \frac{dx}{a-bx^2} = \frac{1}{2\sqrt{(ab)}} \log \frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}$$

$$= \frac{1}{\sqrt{(ab)}} \log \frac{\sqrt{a}+x\sqrt{b}}{\sqrt{(a-bx^2)}}.$$

(20). $$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}.$$

(21). $$\int \frac{dx}{(a+bx^2)^3} = \left(\frac{1}{4a(a+bx^2)^2} + \frac{3}{8a^2(a+bx^2)}\right)x$$

$$+ \frac{3}{8a^2}\int \frac{dx}{a+bx^2}. \quad \text{(Ex. 18.)}$$

(22). $$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}. \quad \text{(Ex. 18.)}$$

(23). $$\int \frac{x^3\,dx}{a+b\,x^2} = \frac{x^2}{2\,b} - \frac{a}{2\,b^2}\log\,(a+b\,x^2).$$

(24). $$\int \frac{x^4\,dx}{a+b\,x^2} = \frac{x^3}{3\,b} - \frac{a\,x}{b^2} + \frac{a^2}{b^2}\int \frac{d\,x}{a+b\,x^2}.$$ (Ex. 18.)

(25). $$\int \frac{x^3\,d\,x}{(a+b\,x^2)^2} = \frac{a}{2\,b^2\,(a+b\,x^2)} + \frac{1}{2\,b^2}\log\,(a+b\,x^2).$$

(26). $$\int \frac{x^2\,d\,x}{(a+b\,x^2)^3} = \left(\frac{x^3}{8\,a} - \frac{x}{8\,b}\right)\frac{1}{(a+b\,x^2)^2}$$

$$+ \frac{1}{8\,a\,b}\int \frac{d\,x}{a+b\,x^2}.$$ (Ex. 18.)

(27). $$\int \frac{x^4\,d\,x}{(a+b\,x^2)^3} = -\left(\frac{5\,x^3}{8\,b} + \frac{3\,a\,x}{8\,b^2}\right)\frac{1}{(a+b\,x^2)^2}$$

$$+ \frac{3}{8\,b^2}\int \frac{d\,x}{a+b\,x^2}.$$ (Ex. 18.)

(28). $$\int \frac{x^3\,d\,x}{(a+b\,x^2)^4} = -\left(\frac{x^2}{4\,b} + \frac{a}{12\,b^2}\right)\frac{1}{(a+b\,x^2)^3}.$$

(29). $$\int \frac{x^4\,d\,x}{(a+b\,x^2)^5} = \left(\frac{3\,b\,x^7}{128\,a^2} + \frac{11\,x^5}{128\,a} - \frac{11\,x^3}{128\,b} - \frac{3\,a\,x}{128\,b^2}\right)\times$$

$$\frac{1}{(a+b\,x^2)^4} + \frac{3}{128\,a^2\,b^2}\int \frac{d\,x}{a+b\,x^2}.$$ (Ex. 18.)

(30). $$\int \frac{d\,x}{x^2\,(a+b\,x^2)} = -\frac{1}{a\,x} - \frac{b}{a}\int \frac{d\,x}{a+b\,x^2}.$$ (Ex. 18.)

(31). $$\int \frac{d\,x}{x^3\,(a+b\,x^2)} = -\frac{1}{2\,a\,x^2} - \frac{b}{2\,a^2}\log\frac{x^2}{a+b\,x^2}.$$

(32). $$\int \frac{d\,x}{x\,(a+b\,x^2)^2} = \frac{1}{2\,a\,(a+b\,x^2)} + \frac{1}{2\,a^2}\log\frac{x^2}{a+b\,x^2}.$$

(33). $$\int \frac{dx}{x^2(a+bx^2)^2} = -\left(\frac{1}{ax} + \frac{3bx}{2a^2}\right)\frac{1}{a+bx^2} - \frac{3b}{2a^2}\int\frac{dx}{a+bx^2}. \quad \text{(Ex. 18.)}$$

(34). $$\int \frac{dx}{x(a+bx^2)^3} = \left(\frac{3}{4a} + \frac{bx^2}{2a^2}\right)\frac{1}{(a+bx^2)^2} + \frac{1}{2a^3}\log\frac{x^2}{a+bx^2}.$$

(35). $$\int \frac{dx}{x^2(a+bx^2)^4} = -\left(\frac{1}{ax} + \frac{77bx}{16a^2} + \frac{35b^2x^3}{6a^3} + \frac{35b^3x^5}{16a^4}\right)\frac{1}{(a+bx^2)^3} - \frac{35b}{16a^4}\int\frac{dx}{a+bx^2}. \quad \text{(Ex. 18.)}$$

(36). $$\int \frac{dx}{a+bx+cx^2} = \frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}$$
$$= \frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}.$$

(37). $$\int \frac{dx}{(a+bx+cx^2)^2} = \frac{2cx+b}{(4ac-b^2)(a+bx+cx^2)} + \frac{2c}{(4ac-b^2)}\int\frac{dx}{a+bx+cx^2}. \quad \text{(Ex. 36.)}$$

(38). $$\int \frac{dx}{(a+bx+cx^2)^3} = \left\{\frac{1}{2(4ac-b^2)(a+bx+cx^2)^2} + \frac{3c}{(4ac-b^2)^2(a+bx+cx^2)}\right\}(2cx+b)$$
$$+ \frac{6c^2}{(4ac-b^2)^3}\int\frac{dx}{a+bx+cx^2}. \quad \text{(Ex. 36.)}$$

(39). $$\int \frac{x\,dx}{a+bx+cx^2} = \frac{1}{2c}\log(a+bx+cx^2)$$

$$-\frac{b}{2c}\int\frac{dx}{a+bx+cx^2}. \qquad \text{(Ex. 36.)}$$

(40). $$\int \frac{x\,dx}{(a+bx+cx^2)^2} = -\frac{1}{2c\,(a+bx+cx^2)} - \frac{b}{2c}\times$$

$$\frac{2cx+b}{(4ac-b^2)(a+bx+cx^2)} - \frac{b}{(4ac-b^2)}\int\frac{dx}{a+bx+cx^2}.$$

(41). $$\int \frac{dx}{x\,(a+bx+cx^2)} = \frac{1}{2a}\log\frac{x^2}{a+bx+cx^2}$$

$$-\frac{b}{2a}\int\frac{dx}{a+bx+cx^2}. \qquad \text{(Ex. 36.)}$$

(42). $$\int \frac{dx}{x^2\,(a+bx+cx^2)} = -\frac{1}{ax} - \frac{b}{2a^2}\log\frac{x^2}{a+bx+cx^2}$$

$$+\left(\frac{b^2}{2a^2}-\frac{c}{a}\right)\int\frac{dx}{a+bx+cx^2}. \qquad \text{(Ex. 36.)}$$

Let $\frac{a}{b} = p^3$ in the (11) following Examples.

(43). $$\int \frac{dx}{a+bx^3} = \frac{1}{3bp^2}\left\{\frac{1}{2}\log\frac{(x+p)^2}{x^2-px+p^2}\right.$$

$$\left.+\sqrt{3}\cdot\tan^{-1}\frac{x\sqrt{3}}{2p-x}\right\}.$$

(44). $$\int \frac{x\,dx}{a+bx^3} = -\frac{1}{3bp}\left\{\frac{1}{2}\log\frac{(x+p)^2}{x^2-px+p^2}\right.$$

$$\left.-\sqrt{3}\,.\tan^{-1}\frac{x\sqrt{3}}{2p-x}\right\}.$$

(45). $$\int \frac{x^3\,dx}{a+b\,x^3} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+b\,x^3}.$$ (Ex. 43.)

(46). $$\int \frac{dx}{(a+b\,x^3)^2} = \frac{x}{3\,a\,(a+b\,x^3)} + \frac{2}{9\,a\,b\,p^2} \times$$
$$\left\{\frac{1}{2}\log\frac{(x+p)^2}{x^2-p\,x+p^2} + \sqrt{3}\cdot\tan^{-1}\frac{x\sqrt{3}}{2p-x}\right\}.$$

(47). $$\int \frac{x\,dx}{(a+b\,x^3)^2} = \frac{x^2}{3\,a\,(a+b\,x^3)} - \frac{1}{9\,a\,b\,p}$$
$$\left\{\frac{1}{2}\log\frac{(x+p)^2}{x^2-p\,x+p^2} - \sqrt{3}\tan^{-1}\frac{x\sqrt{3}}{2p-x}\right\}.$$

(48). $$\int \frac{dx}{(a+b\,x^3)^3} = \left(\frac{5\,b\,x^4}{18\,a^2} + \frac{4\,x}{9\,a}\right)\frac{1}{(a+b\,x^3)^2}$$
$$+ \frac{5}{9\,a^2}\int \frac{dx}{a+b\,x^3}.$$ (Ex. 43.)

(49). $$\int \frac{dx}{x\,(a+b\,x^3)} = \frac{1}{3\,a}\log\frac{x^3}{a+b\,x^3}.$$

(50). $$\int \frac{dx}{x^3\,(a+b\,x^3)} = -\frac{1}{2\,a\,x^2} - \frac{b}{a}\int \frac{dx}{a+b\,x^3}.$$ (Ex. 43.)

(51). $$\int \frac{dx}{x^4\,(a+b\,x^3)} = -\frac{1}{3\,a\,x^3} + \frac{b}{3\,a^2}\log\frac{a+b\,x^3}{x^3}.$$

(52). $$\int \frac{dx}{x\,(a+b\,x^3)^2} = \frac{1}{3\,a\,(a+b\,x^3)} - \frac{1}{3\,a^2}\log\frac{a+bx^3}{x^3}.$$

(53). $$\int \frac{dx}{x^3\,(a+b\,x^3)^2} = -\left(\frac{1}{2\,a\,x^2} + \frac{5\,b\,x}{6\,a^2}\right)\frac{1}{a+b\,x^3}$$
$$-\frac{5\,b}{3\,a^2}\int \frac{dx}{a+b\,x^3}.$$ (Ex. 43.)

(54). $$\int \frac{dx}{1+x} = \log(1+x).$$

(55). $$\int \frac{dx}{1+x^2} = \tan^{-1} x.$$

(56). $$\int \frac{dx}{1+x^3} = -\frac{2}{3}\cos\frac{\pi}{3} \,.\, \log \sqrt{\left(1 - 2x\cos\frac{\pi}{3} + x^2\right)}$$

$$+ \frac{2}{3}\sin\frac{\pi}{3} \,.\, \tan^{-1}\frac{x \sin\frac{\pi}{3}}{1 - x\cos\frac{\pi}{3}} + \frac{1}{3}\log(1+x)$$

$$= \frac{1}{3}\log\frac{1+x}{1-x+x^2} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}.$$

(57). $$\int \frac{dx}{1+x^4} = \left\{\begin{array}{l} -\frac{1}{4}\cos\frac{\pi}{4} \,.\, \log\left(1 - 2x\cos\frac{\pi}{4} + x^2\right) \\ \\ +\frac{1}{2}\sin\frac{\pi}{4} \,.\, \tan^{-1}\dfrac{x\sin\frac{\pi}{4}}{1 - x\cos\frac{\pi}{4}} \\ \\ -\frac{1}{4}\cos\frac{3\pi}{4} \,.\, \log\left(1 - 2x\cos\frac{3\pi}{4} + x^2\right) \\ \\ +\frac{1}{2}\sin\frac{3\pi}{4} \,.\, \tan^{-1}\dfrac{x\sin\frac{3\pi}{4}}{1 - x\cos\frac{3\pi}{4}} \end{array}\right.$$

$$= \frac{1}{4\sqrt{2}}\log\frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}.$$

$$(58).\quad \int \frac{dx}{1+x^5} = \left\{ \begin{aligned} & -\frac{1}{5}\cos\frac{\pi}{5}\,.\log\left(1-2x\cos\frac{\pi}{5}+x^2\right) \\ & +\frac{2}{5}\sin\frac{\pi}{5}\,.\tan^{-1}\frac{x\sin\frac{\pi}{5}}{1-x\cos\frac{\pi}{5}} \\ & -\frac{1}{5}\cos\frac{3\pi}{5}\log\left(1-2x\cos\frac{3\pi}{5}+x^2\right) \\ & +\frac{2}{5}\sin\frac{3\pi}{5}\,.\tan^{-1}\frac{x\sin\frac{3\pi}{5}}{1-x\cos\frac{3\pi}{5}} \\ & +\frac{1}{5}\log(1+x). \end{aligned} \right.$$

$$(59).\quad \int \frac{dx}{1+x^6} = \left\{ \begin{aligned} & -\frac{1}{6}\cos\frac{\pi}{6}\,.\log\left(1-2x\cos\frac{\pi}{6}+x^2\right) \\ & +\frac{1}{6}\sin\frac{\pi}{6}\tan^{-1}\frac{x\sin\frac{\pi}{6}}{1-x\cos\frac{\pi}{6}} \\ & +\frac{1}{3}\tan^{-1}x \\ & -\frac{1}{6}\cos\frac{5\pi}{6}\,.\log\left(1-2x\cos\frac{5\pi}{6}+x^2\right) \\ & +\frac{1}{3}\sin\frac{5\pi}{6}\,.\tan^{-1}\frac{x\sin\frac{5\pi}{6}}{1-x\cos\frac{5\pi}{6}}. \end{aligned} \right.$$

$$= \frac{1}{4\sqrt{3}}\log\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2} + \frac{1}{6}\tan^{-1}\frac{3x(1-x^2)}{1-4x^2+x^4}.$$

$$(60).\ \int \frac{dx}{1-x} = -\log(1-x).$$

$$(61).\ \int \frac{dx}{1-x^2} = -\frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x)$$

$$= \frac{1}{2}\log\frac{1+x}{1-x}.$$

$$(62).\ \int \frac{dx}{1-x^3} = \left\{\begin{array}{l} -\frac{1}{3}\log(1-x) \\ -\frac{1}{3}\cos\frac{2\pi}{3}.\log\left(1-2x\cos\frac{2\pi}{3}+x^2\right) \\ +\frac{2}{3}\sin\frac{2\pi}{3}.\tan^{-1}\dfrac{x\sin\frac{2\pi}{3}}{1-x\cos\frac{2\pi}{3}}. \end{array}\right.$$

$$= -\frac{1}{3}\log\frac{1-x}{\sqrt{(1+x+x^2)}} + \frac{1}{3\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2+x}.$$

$$(63).\ \int \frac{dx}{1-x^4} = \left\{\begin{array}{l} -\frac{1}{4}\log(1-x) + \frac{1}{2}\tan^{-1}x. \\ +\frac{1}{4}\log(1+x) \end{array}\right.$$

$$= \frac{1}{4}\log\frac{1+x}{1-x} + \frac{1}{2}\tan^{-1}x.$$

$$(64).\quad \int \frac{dx}{1-x^5} = \left\{\begin{array}{l} -\frac{1}{5}\log(1-x) \\[2ex] -\frac{1}{5}\cos\frac{2\pi}{5}.\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right) \\[2ex] +\frac{2}{5}\sin\frac{2\pi}{5}\tan^{-1}\dfrac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}} \\[2ex] -\frac{1}{3}\cos\frac{4\pi}{5}.\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right) \\[2ex] +\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\dfrac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}. \end{array}\right.$$

$$(65).\quad \int \frac{dx}{1-x^6} = \left\{\begin{array}{l} -\frac{1}{6}\log(1-x) - \frac{1}{12}\log(1-x+x^2) \\[2ex] \qquad + \frac{1}{2\sqrt{3}}\tan^{-1}\dfrac{x\sqrt{3}}{2-x} \\[2ex] +\frac{1}{6}\log(1+x) + \frac{1}{12}\log(1+x+x^2) \\[2ex] \qquad + \frac{1}{2\sqrt{3}}\tan^{-1}\dfrac{x\sqrt{3}}{2+x}; \end{array}\right.$$

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$$(66).\quad \int \frac{dx}{(x+a)(x+b)} = \frac{1}{(b-a)}\log\frac{x+a}{x+b}.$$

$$(67).\quad \int \frac{dx}{(x+a)(x+b)^2} = \frac{1}{(b-a)(x+b)}$$

$$+ \frac{1}{(b-a)^2}\log\frac{x+a}{x+b}.$$

$$(68).\quad \int \frac{x\,d\,x}{(x+a)\,(x+b)} = \frac{b}{(b-a)}\log(x+b)$$

$$-\frac{a}{(b-a)}\log(x+a).$$

$$(69).\quad \int \frac{x\,d\,x}{(x+a)\,(x+b)^2} = -\frac{b}{(b-a)\,(x+b)}$$

$$-\frac{a}{(b-a)^2}\log\frac{x+a}{x+b}.$$

$$(70).\quad \int \frac{x^2\,d\,x}{(x+a)\,(x+b)^2} = \frac{b^2}{(b-a)\,(x+b)} + \frac{a^2}{(b-a)^2}$$

$$\log(x+a) + \frac{b^2-2\,a\,b}{(b-a)^2}\log(x+b).$$

$$(71).\quad \int \frac{d\,x}{(x+a)^2\,(x+b)^2} = -\frac{1}{(b-a)^2}\left(\frac{1}{x+a}+\frac{1}{x+b}\right)$$

$$-\frac{2}{(b-a)^3}\log\frac{x+a}{x+b}.$$

$$(72).\quad \int \frac{x\,d\,x}{(x+a)^2\,(x+b)^2} = \frac{1}{(b-a)^2}\left(\frac{a}{x+a}+\frac{b}{x+b}\right)$$

$$+\frac{a+b}{(b-a)^3}\log\frac{x+a}{x+b}.$$

$$(73).\quad \int \frac{x^2\,d\,x}{(x+a)^2\,(x+b)^2} = -\frac{1}{(b-a)^2}\left(\frac{a^2}{x+a}+\frac{b^2}{x+b}\right)$$

$$-\frac{2\,a\,b}{(b-a)^3}\log\frac{x+a}{x+b}.$$

$$(74).\ \int \frac{x^3\,dx}{(x+a)^2\,(x+b)^2} = \frac{1}{(b-a)^2}\left(\frac{a^3}{x+a} + \frac{b^3}{x+b}\right)$$

$$-\frac{a^2\,(3\,b-a)}{(b-a)^3}\log\,(x+a) + \frac{b^2\,(b-3\,a)}{(b-a)^3}\log\,(x+b).$$

$$(75).\ \int \frac{dx}{(x+a)\,(x+b)\,(x+c)} = \frac{1}{(b-a)\,(c-a)}\log\,(x+a)$$

$$+\frac{1}{(a-b)\,(c-b)}\log\,(x+b) + \frac{1}{(a-c)\,(b-c)}\log\,(x+c).$$

$$(76).\ \int \frac{x\,dx}{(x+a)\,(x+b)\,(x+c)} = -\frac{a}{(b-a)\,(c-a)}\times$$

$$\log\,(x+a) - \frac{b}{(a-b)\,(c-b)}\log\,(x+b)$$

$$-\frac{c}{(a-c)\,(b-c)}\log\,(x+c).$$

$$(77).\ \int \frac{x^2\,dx}{(x+a)\,(x+b)\,(x+c)} = \frac{a^2}{(b-a)\,(c-a)}\times$$

$$\log\,(x+a) + \frac{b^2}{(a-b)\,(c-b)}\log\,(x+b)$$

$$+\frac{c^2}{(a-c)\,(b-c)}\log\,(x+c).$$

$$(78).\ \int \frac{dx}{(x^2+a)\,(x+b)} = \frac{1}{(b^2+a)}\left\{\log\frac{x+b}{\sqrt{(x^2+a)}}\right.$$

$$\left.+\frac{b}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\right\}.$$

(79). $$\int \frac{x\,dx}{(x^2+a)(x+b)} = \frac{1}{(b^2+a)}\left\{ b \log \frac{\sqrt{(x^2+a)}}{x+b} + \sqrt{a}\,.\tan^{-1}\frac{x}{\sqrt{a}}\right\}.$$

(80). $$\int \frac{x^2\,dx}{(x^2+a)(x+b)} = \frac{1}{(b^2+a)}\left\{ b^2 \log (x+b) + \frac{a}{2}\log (x^2+a) - b\sqrt{a}\,.\tan^{-1}\frac{x}{\sqrt{a}}\right\}.$$

(81). $$\int \frac{dx}{(x^2+a)(x^2+b)} = \frac{1}{(b-a)}\left\{ \frac{1}{\sqrt{a}} \tan^{-1}\frac{x}{\sqrt{a}} - \frac{1}{\sqrt{b}}\tan^{-1}\frac{x}{\sqrt{b}}\right\}.$$

(82). $$\int \frac{x\,dx}{(x^2+a)(x^2+b)} = \frac{1}{2\,(b-a)}\log\frac{x^2+a}{x^2+b}.$$

(83). $$\int \frac{x^2\,dx}{(x^2+a)(x^2+b)} = \frac{1}{(a-b)}\left\{ \sqrt{a}\tan^{-1}\frac{x}{\sqrt{a}} - \sqrt{b}\tan^{-1}\frac{x}{\sqrt{b}}\right\}.$$

(84). $$\int \frac{dx}{(x^2+a)(x+b)^2} = \frac{1}{(b^2+a)^2}\left\{ b\log\frac{(x+b)^2}{x^2+a} + \frac{(b^2-a)}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} - \frac{(b^2+a)}{x+b}\right\}.$$

(85). $$\int \frac{x\,dx}{(x^2+a)(x+b)^2} = \frac{1}{(b^2+a)^2}\left\{ \frac{a-b^2}{2}\log\frac{(x+b)^2}{x^2+a} + \frac{2ab}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} + \frac{b\,(b^2+a)}{x+b}\right\}.$$

(86). $$\int \frac{x^2\,dx}{(x^2+a)\,(x+b)^2} = -\frac{1}{(b^2+a)^2}\left\{a\,b\log\frac{(x+b)^2}{x^2+a}\right.$$

$$\left. + (b^2-a)\sqrt{a}\tan^{-1}\frac{x}{\sqrt{a}} + \frac{b^2\,(b^2+a)}{x+b}\right\}.$$

(87). $$\int \frac{x^3\,dx}{(x^2+a)\,(x+b)^2} = \frac{1}{(b^2+a)^2}\left\{b^2\,(b^2+3\,a)\log(x+b)\right.$$

$$\left. - \frac{a\,(b^2-a)}{2}\log(x^2+a) - 2\,a^{\frac{3}{2}}\,b\tan^{-1}\frac{x}{\sqrt{a}}\right\}$$

$$+ \frac{1}{b^2+a}\cdot\frac{b^3}{x+b}.$$

(88). $$\int \frac{dx}{x^2+6x+8} = \frac{1}{2}\log\frac{x+2}{x+4}.$$

(89). $$\int \frac{x\,dx}{x^2+6\,x+8} = \log\frac{(x+4)^2}{x+2}.$$

(90). $$\int \frac{dx}{x^3+3\,x^2-4} = \frac{1}{3\,(x+2)} + \frac{1}{9}\log\frac{x-1}{x+2}.$$

(91). $$\int \frac{dx}{x^3+5\,x^2+8\,x+4} = \frac{1}{x+2} + \log\frac{x+1}{x+2}.$$

(92). $$\int \frac{x\,dx}{x^3+5\,x^2+8\,x+4} = -\frac{2}{x+2} - \log\frac{x+1}{x+2}.$$

(93). $$\int \frac{x^2\,dx}{x^3+5\,x^2+8\,x+4} = \frac{4}{x+2} + \log(x+1).$$

(94). $$\int \frac{dx}{x^3+6\,x^2+11\,x+6} = \log\frac{1}{x+2}\sqrt{\frac{x+1}{x+3}}.$$

(95). $$\int \frac{x\,dx}{x^3+6x^2+11x+6} = \log \frac{(x+2)^2}{\sqrt{\{(x+1)(x+3)^3\}}}.$$

(96). $$\int \frac{x^2\,dx}{x^3+6x^3+11x+6} = \log \frac{\sqrt{\{(x+1)(x+3)^9\}}}{(x+2)^4}.$$

(97). $$\int \frac{dx}{x^3+x^2+x+1} = \frac{1}{2}\log \frac{x+1}{\sqrt{(x^2+1)}} + \frac{1}{2}\tan^{-1} x.$$

(98). $$\int \frac{x\,dx}{x^3+x^2+x+1} = \frac{1}{2}\log \frac{\sqrt{(x^2+1)}}{x+1} + \frac{1}{2}\tan^{-1} x.$$

(99). $$\int \frac{x^2\,dx}{x^3+x^2+x+1} = \frac{1}{2}\log (x+1)$$

$$+\frac{1}{2}\log \sqrt{(x^2+1)} - \frac{1}{2}\tan^{-1} x.$$

(100). $$\int \frac{dx}{x^3-x^2+x-1} = -\frac{1}{2}\log \frac{\sqrt{(x^2+1)}}{x-1} - \frac{1}{2}\tan^{-1} x.$$

(101). $$\int \frac{dx}{x^4+x^2-2} = -\frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{6}\log \frac{1+x}{1-x}.$$

(102). $$\int \frac{x^2\,dx}{x^4+x^2-2} = \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{6}\log \frac{1+x}{1-x}.$$

(103). $$\int \frac{dx}{x^4+4x^3+5x^2+4x+4} = \frac{2}{25}\log \frac{(x+2)^2}{x^2+1}$$

$$+\frac{3}{25}\tan^{-1} x.$$

On the Integration of Irrational Functions.

(1). $$\int \frac{dx}{\sqrt{(a+bx)}} = \frac{2}{b}\sqrt{(a+bx)}.$$

(2). $$\int \frac{dx}{x\sqrt{(a+bx)}} = \frac{1}{\sqrt{a}}\log\frac{\sqrt{(a+bx)}-\sqrt{a}}{\sqrt{(a+bx)}+\sqrt{a}}$$

$$= \frac{2}{\sqrt{-a}}\tan^{-1}\frac{\sqrt{(a+bx)}}{\sqrt{-a}}.$$

(3). $$\int \frac{dx}{\sqrt{(a+bx^2)}} = \frac{1}{\sqrt{b}}\log\left\{x\sqrt{b}+\sqrt{(a+bx^2)}\right\}$$

$$= \frac{1}{\sqrt{-b}}\sin^{-1}\left\{x\sqrt{\frac{-b}{a}}\right\}.$$

(4). $$\int \frac{dx}{\sqrt{(1+x^2)}} = \log\left\{x+\sqrt{(1+x^2)}\right\}.$$

(5). $$\int \frac{dx}{\sqrt{(x^2-1)}} = \log\left\{x+\sqrt{(x^2-1)}\right\}.$$

(6). $$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x = \cos^{-1}\sqrt{(1-x^2)}$$

$$= \frac{1}{2}\cos^{-1}(1-2x^2) = \tan^{-1}\frac{x}{\sqrt{(1-x^2)}}.$$

(7). $$\int \frac{dx}{x\sqrt{(a+bx^2)}} = \frac{1}{2\sqrt{a}}\log\frac{\sqrt{(a+bx^2)}-\sqrt{a}}{\sqrt{(a+bx^2)}+\sqrt{a}}$$

$$= \frac{1}{\sqrt{-a}}\sec^{-1}x\sqrt{\frac{-b}{a}}.$$

(8). $$\int \frac{dx}{x\sqrt{(1+x^2)}} = \log \frac{\sqrt{(1+x^2)} - 1}{x}.$$

(9). $$\int \frac{dx}{x\sqrt{(1-x^2)}} = \log \frac{\sqrt{(1-x^2)} - 1}{x}.$$

(10). $$\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1} x = \tan^{-1} \sqrt{(x^2-1)} =$$

$$\cos^{-1} \frac{1}{x} = \frac{1}{2} \cos^{-1} \frac{2-x^2}{x^2} = \sin^{-1} \frac{\sqrt{(x^2-1)}}{x}.$$

(11). $$\int \frac{dx}{\sqrt{(ax+bx^2)}} = \frac{1}{\sqrt{b}} \log \frac{\sqrt{(ax+bx^2)} + x\sqrt{b}}{\sqrt{(ax+bx^2)} - x\sqrt{b}}$$

$$= \frac{2}{\sqrt{-b}} \tan^{-1} \frac{x\sqrt{-b}}{\sqrt{(ax+bx^2)}}.$$

(12). $$\int \frac{dx}{\sqrt{(x-x^2)}} = 2\tan^{-1} \frac{x}{\sqrt{(x-x^2)}} = 2\tan^{-1} \sqrt{\frac{x}{1-x}}$$

$$= 2\sin^{-1} \sqrt{x} = 2\cos^{-1} \sqrt{\frac{1-x}{x}}.$$

(13). $$\int \frac{dx}{\sqrt{(x^2+x)}} = \log \{ 2x + 1 + 2\sqrt{(x^2+x)} \}$$

$$= -\log \{ 2x + 1 - 2\sqrt{(x^2+x)} \}.$$

(14). $$\int \frac{dx}{\sqrt{(x^2-x)}} = \log \{ 1 - 2x - \sqrt{(x^2-x)} \}$$

$$= -\log \{ 1 - 2x + 2\sqrt{(x^2-x)} \}.$$

(15). $$\int \frac{dx}{\surd a + bx + cx^2)}$$

$$=\frac{1}{\sqrt{c}}\log\{2cx+b+2\sqrt{c}\,.\,\sqrt{(a+bx+cx^2)}\}$$

$$=\frac{-1}{\sqrt{(-c)}}\sin^{-1}\frac{2cx+b}{\sqrt{(b^2-4ac)}}.$$

(16). $$\int\frac{dx}{\sqrt{(1+x+x^2)}}=\log\{2x+1+2\sqrt{(1+x+x^2)}\}.$$

(17). $$\int\frac{dx}{\sqrt{(1+x-x^2)}}=\sin^{-1}\frac{2x-1}{\sqrt{5}}$$

$$=\cos^{-1}\frac{2\sqrt{(1+x-x^2)}}{\sqrt{5}}=\tan^{-1}\frac{2x-1}{2\sqrt{(1+x-x^2)}}.$$

(18). $$\int\frac{dx}{\sqrt{(1-x+x^2)}}=\log\{2x-1+2\sqrt{(1-x+x^2)}\}.$$

(19). $$\int\frac{dx}{\sqrt{(x^2-x-1)}}=\log\{2x-1+2\sqrt{(x^2-x-1)}\}.$$

(20). $$\int\frac{dx}{\sqrt{(1-x-x^2)}}=\sin^{-1}\frac{2x+1}{\sqrt{5}}$$

$$=\cos^{-1}\frac{2\sqrt{(1-x-x^2)}}{\sqrt{5}}=\tan^{-1}\frac{2x+1}{2\sqrt{(1-x-x^2)}}.$$

(21). $$\int\frac{dx}{x\sqrt{(a+bx+cx^2)}}$$

$$=\frac{1}{\sqrt{a}}\log\frac{2a+bx-2\sqrt{a}\,.\,\sqrt{(a+bx+cx^2)}}{x}$$

$$=\frac{1}{\sqrt{-a}}\tan^{-1}\frac{2a+bx}{2\sqrt{-a}\,.\,\sqrt{(a+bx+cx^2)}}.$$

(22). $$\int\frac{dx}{x\sqrt{(1+x+x^2)}}=\log\frac{2+x-2\sqrt{(1+x+x^2)}}{x}.$$

(23). $$\int \frac{d\,x}{x\sqrt{(-1+x+x^2)}} = \tan^{-1}\frac{x-2}{2\sqrt{(-1+x+x^2)}}$$

$$= \sin^{-1}\frac{x-2}{x\sqrt{5}} = \cos^{-1}\frac{2\sqrt{(-1+x+x^2)}}{x\sqrt{5}}.$$

(24). $$\int \frac{d\,x}{x\sqrt{(1-x+x^2)}} = \log\left\{\frac{2-x-2\sqrt{(1-x+x^2)}}{x}\right\}.$$

(25). $$\int \frac{d\,x}{x\sqrt{(-1-x+x^2)}} = \tan^{-1}\frac{-x-2}{2\sqrt{(-1-x+x^2)}}$$

$$= \sin^{-1}\frac{-x-2}{x\sqrt{5}} = \cos^{-1}\frac{2\sqrt{(-1-x+x^2)}}{x\sqrt{5}}.$$

(26). $$\int \frac{d\,x}{(c+e\,x)\sqrt{(a+b\,x)}}$$

$$= \frac{2}{\sqrt{[(b\,c-a\,e)\,e]}}\tan^{-1}\sqrt{\frac{e\,a+e\,b\,x}{b\,c-a\,e}} = \frac{1}{\sqrt{[-e\,(b\,c-a\,e)]}}\times$$

$$\log\left\{\frac{b\,c-2\,a\,e-b\,e\,x+2\sqrt{[-e\,(b\,c-a\,e)]}\,.\,\sqrt{(a+b\,x)}}{c+e\,x}\right\}.$$

(27). $$\int \frac{d\,x}{(1+x)\sqrt{(1-x)}}$$

$$= \frac{1}{\sqrt{2}}\log\frac{-3+x+2\sqrt{2}\,.\,\sqrt{(1-x)}}{1+x}.$$

(28). $$\int \frac{d\,x}{(1-x)\sqrt{(1+x)}}$$

$$= \frac{1}{\sqrt{2}}\log\frac{3+x+2\sqrt{2}\,.\,\sqrt{(1+x)}}{1-x}$$

(29). $$\int \frac{dx}{(c+ex)\sqrt{(a+bx^2)}}$$

$$= \frac{1}{\sqrt{(ae^2+bc^2)}} \log \frac{ae-bcx-\sqrt{(ae^2+bc^2)}\,.\,\sqrt{(a+bx^2)}}{c+ex}$$

$$= \frac{1}{\sqrt{\{-(ae^2+bc^2)\}}} \tan^{-1} \frac{ae-bcx}{\sqrt{(-ae^2-bc^2)}\,.\,\sqrt{(a+bx^2)}}.$$

(30). $$\int \frac{dx}{(1+x)\sqrt{(1+x^2)}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{1-x-\sqrt{2}\,.\,\sqrt{(1+x^2)}}{1+x}.$$

(31). $$\int \frac{dx}{(1-x)\sqrt{(1+x^2)}}$$

$$= -\frac{1}{\sqrt{2}} \log \frac{-1-x+\sqrt{2}\,.\,\sqrt{(1+x^2)}}{1-x}.$$

(32). $$\int \frac{dx}{(c+ex^2)\sqrt{(a+bx^2)}}$$

$$= \frac{1}{\sqrt{(bc^2-ace)}} \log \frac{c\sqrt{(a+bx^2)}+x\sqrt{(bc^2-ace)}}{\sqrt{(c+ex^2)}}$$

$$= \frac{1}{\sqrt{(ace-bc^2)}} \tan^{-1} \frac{x\sqrt{(ace-bc^2)}}{c\sqrt{(a+bx^2)}} = \cos^{-1} \sqrt{\frac{ac+bcx^2}{ac+aex^2}}$$

$$= \sin^{-1} x \sqrt{\frac{ae-bc}{ac+aex^2}}.$$

(33). $$\int \frac{dx}{(1-x^2)\sqrt{(1+x^2)}} = \frac{1}{\sqrt{2}} \log \frac{\sqrt{(1+x^2)}+x\sqrt{2}}{\sqrt{(1-x^2)}}.$$

$$(34).\quad \int \frac{dx}{(1+x^2)\surd(1-x^2)} = \frac{1}{\surd 2}\tan^{-1}\frac{x\surd 2}{\surd(1-x^2)}$$

$$= \cos^{-1}\sqrt{\frac{1-x^2}{1+x^2}} = \sin^{-1}\frac{x\surd 2}{\surd(1+x^2)}.$$

$$(35).\quad \int \frac{dx}{(1+x^2)\surd(x^2-1)} = \frac{1}{\surd 2}\log\frac{\surd(x^2-1)+x\surd 2}{\surd(1+x^2)}.$$

$$(36).\quad \int \frac{dx}{(e+fx)\surd(a+bx+cx^2)} = \frac{1}{\surd k}\times$$

$$\log\frac{2af-be+(bf-2ce)x-2\surd k\,.\,\surd(a+bx+cx^2)}{e+fx}$$

$$= \frac{1}{\surd(-k)}\tan^{-1}\frac{2af-be+(bf-2ce)x}{2\surd(-k)\,.\,\surd(a+bx+cx^2)},$$

where $k = af^2 - bef + ce^2$.

$$(37).\quad \int \frac{dx}{(1+x)\surd(1+x+x^2)} = \log\frac{1-x-2\surd(1+x+x^2)}{1+x}.$$

$$(38).\quad \int \frac{dx}{(1+x)\surd(1-x+x^2)} = \frac{1}{\surd 3}\times$$

$$\log\frac{3-3x-2\surd 3\,.\,\surd(1-x+x^2)}{1+x}.$$

$$(39).\quad \int \frac{dx}{(1-x)\surd(1+x+x^2)} = -\frac{1}{\surd 3}\times$$

$$-\log\frac{3-3x+2\surd 3\,.\,\surd(1+x+x^2)}{1-x}.$$

$$(40).\quad \int \frac{dx}{(1+x)\surd(1+x-x^2)} = \tan^{-1}\frac{1+3x}{2\surd(1+x-x^2)}.$$

(41). $$\int \frac{d\,x}{(1+x)\sqrt{(1-x-x^2)}}$$

$$= \log \frac{3+x-2\sqrt{(1-x-x^2)}}{1+x}.$$

(42). $$\int \frac{d\,x}{(1-x)\sqrt{(1-x-x^2)}}$$

$$= \log \frac{-1+3\,x-2\sqrt{(1-x-x^2)}}{1-x}.$$

(43). $$\int \frac{x\,d\,x}{\sqrt{(a+b\,x)}} = \left\{\frac{1}{3}(a+b\,x)-a\right\}\frac{2\sqrt{(a+b\,x)}}{b^2}.$$

(44). $$\int \frac{x^2\,d\,x}{\sqrt{(a+b\,x)}} = \left\{\frac{1}{5}(a+b\,x)^2 - \frac{2\,a}{3} \times\right.$$

$$\left.(a+b\,x)+a^2\right\}\frac{2\sqrt{(a+b\,x)}}{b^3}.$$

(45). $$\int \frac{x^3\,d\,x}{\sqrt{(x-1)}} = \left\{\frac{1}{7}(x-1)^3+\frac{3}{5}(x-1)^2\right.$$

$$\left.+(x-1)+1\right\}2\sqrt{(x-1)}.$$

(46). $$\int \frac{d\,x}{x^2\sqrt{(1+x)}} = -\frac{\sqrt{(1+x)}}{x}$$

$$-\frac{1}{2}\log\left\{\frac{\sqrt{(1-x)}-1}{\sqrt{(1+x)}+1}\right\}.$$

(47). $$\int \frac{d\,x}{x^3\sqrt{(x-1)}} = \left\{\frac{1}{2\,x^2}+\frac{3}{4\,x}\right\}\sqrt{(x-1)}$$

$$+\frac{3}{4}\tan^{-1}\sqrt{(x-1)}.$$

$$(48).\ \int \frac{x^2\,dx}{(a+b\,x)^{\frac{3}{2}}} = \left\{\frac{1}{3}\,(a+b\,x)^2 - 2\,a\,(a+b\,x) - a^2\right\} \times \frac{2}{b^3\,\sqrt{(a+b\,x)}}.$$

$$(49).\ \int \frac{d\,x}{x^2\,(1+x)^{\frac{3}{2}}} = -\left\{\frac{1}{x} + 3\right\}\frac{1}{\sqrt{(1+x)}} - \frac{3}{2}\log\left\{\frac{\sqrt{(1+x)}-1}{\sqrt{(1+x)}+1}\right\}.$$

$$(50).\ \int \frac{x\,d\,x}{(a+b\,x)^{\frac{5}{2}}} = -\left\{\sqrt{(a+b\,x)} - \frac{1}{3}\,a\right\} \times \frac{2}{b^2\,(a+b\,x)^{\frac{3}{2}}}.$$

$$(51).\ \int \frac{x\,d\,x}{(a+b\,x)^{\frac{7}{2}}} = -\left\{\frac{1}{3}\,\sqrt{(a+b\,x)} - \frac{1}{5}\,a\right\} \times \frac{2}{b^2\,(a+b\,x)^{\frac{5}{2}}}.$$

$$(52).\ \int \frac{x\,d\,x}{(a+b\,x)^{\frac{9}{2}}} = -\left\{\frac{1}{5}\,\sqrt{(a+b\,x)} - \frac{1}{7}\,a\right\} \times \frac{2}{b^2\,(a+b\,x)^{\frac{7}{2}}}.$$

$$(53).\ \int \frac{d\,x}{x\,(a+b\,x)^{\frac{5}{2}}} = \left(\frac{8}{3\,a} + \frac{2\,b\,x}{a^2}\right)\frac{1}{(a+b\,x)^{\frac{3}{2}}} + \frac{1}{a^2}\int \frac{d\,x}{x\,\sqrt{(a+b\,x)}}.\ \text{(Ex. 2.)}$$

(54). $$\int \frac{dx}{x(a+bx)^{\frac{7}{2}}} = \left(\frac{46}{15a} + \frac{14bx}{3a^2} + \frac{2b^2x^2}{a^3}\right)$$

$$\frac{1}{(a+bx)^{\frac{5}{2}}} + \frac{1}{a^3}\int \frac{dx}{x\sqrt{(a+bx)}}.$$ (Ex. 2.)

(55). $$\int \frac{dx}{x^2(a+bx)^{\frac{7}{2}}} = -\frac{1}{ax(a+bx)^{\frac{5}{2}}}$$

$$-\frac{7b}{2a}\int \frac{dx}{x(a+bx)^{\frac{7}{2}}}.$$ (Ex. 54.)

(56). $$\int \frac{dx\sqrt{(a+bx)}}{x^2} = -\frac{\sqrt{(a+bx)}}{x}$$

$$+\frac{b}{2}\int \frac{dx}{x\sqrt{(a+bx)}}.$$ (Ex. 2.)

(57). $$\int \frac{dx(a+bx)^{\frac{3}{2}}}{x} = \left\{\frac{1}{3}(a+bx)+a\right\}2\sqrt{(a+bx)}$$

$$+a^2\int \frac{dx}{x\sqrt{(a+bx)}}.$$ (Ex. 2.)

(58). $$\int \frac{dx(a+bx)^{\frac{3}{2}}}{x^3} = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right)\surd(a+bx)^{\frac{5}{2}}$$

$$+\frac{3b^2}{8a^2}\int \frac{dx(a+bx)^{\frac{3}{2}}}{x}.$$ (Ex. 57.)

(59). $$\int \frac{dx}{x(a+bx)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{a}}\left\{\frac{3}{2}\log\frac{\sqrt[3]{(a+bx)}-\sqrt[3]{a}}{\sqrt[3]{x}}\right.$$

$$\left.+\sqrt{3}\tan^{-1}\frac{\sqrt{3}\sqrt[3]{(a+bx)}}{\sqrt[3]{(a+bx)}+2\sqrt[3]{a}}\right.$$.

Assume $a+bx = \dfrac{a}{u^3}$.

(60). $$\int \frac{dx}{x^2 (a+bx)^{\frac{1}{3}}} = -\frac{(a+bx)^{\frac{2}{3}}}{ax}$$

$$-\frac{b}{3a}\int \frac{dx}{x(a+bx)^{\frac{1}{3}}}.$$ (Ex. 59.)

(61). $$\int \frac{dx}{x(a+bx)^{\frac{2}{3}}} = \frac{1}{a^{\frac{2}{3}}}\left\{\frac{3}{2}\log\frac{\sqrt[3]{(a+bx)}-\sqrt[3]{a}}{\sqrt[3]{x}}\right.$$

$$\left. -\sqrt{3}\tan^{-1}\frac{\sqrt{3}(a+bx)^{\frac{1}{3}}}{(a+bx)^{\frac{1}{3}}+2\sqrt[3]{a}}\right\}.$$

(62). $$\int \frac{dx\,(a+bx)^{\frac{1}{3}}}{x} = 3\sqrt[3]{(a+bx)}$$

$$+a\,.\int \frac{dx}{x\sqrt[3]{(a+bx)^2}}.$$ (Ex. 61.)

(63). $$\int \frac{dx}{(a+bx^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{(a+bx^2)}}.$$

(64). $$\int \frac{dx}{(a+bx^2)^{\frac{5}{2}}} = \left(\frac{1}{3a(a+bx^2)}+\frac{2}{3a^2}\right)\frac{x}{\sqrt{(a+bx^2)}}.$$

(65). $$\int \frac{x\,dx}{\sqrt{(a+bx^2)}} = \frac{\sqrt{(a+bx^2)}}{b}.$$

(66). $$\int \frac{x^2\,dx}{\sqrt{(a+bx^2)}} = \frac{x\sqrt{(a+bx^2)}}{2b}$$

$$-\frac{a}{2b}\int \frac{dx}{\sqrt{(a+bx^2)}}.$$ (Ex. 3.)

(67). $$\int \frac{x^3\,dx}{\sqrt{(a+bx^2)}} = \left(\frac{x^2}{3b}-\frac{2a}{3b^2}\right)\sqrt{(a+bx^2)}.$$

(68). $$\int \frac{x^4\,dx}{\sqrt{(1-x^2)}} = -\left\{\frac{1}{4}x^3 + \frac{3}{4\,.\,2}x\right\}\sqrt{(1-x^2)}$$

$$+\frac{3}{4\,.\,2}\sin^{-1}x.$$

(69). $$\int \frac{x^5\,dx}{\sqrt{(x^2-1)}} = \left\{\frac{1}{5}x^4 + \frac{4}{5\,.\,3}x^2\right.$$

$$\left.+\frac{4\,.\,2}{5\,.\,3\,.\,1}\right\}\sqrt{\{x^2-1\}}.$$

(70). $$\int \frac{x^6\,dx}{\sqrt{(x^2+1)}} = \left\{\frac{1}{6}x^5 - \frac{5}{6\,.\,4}x^3\right.$$

$$\left.+\frac{5\,.\,3}{6\,.\,4\,.\,2}x\right\}\sqrt{(x^2+1)}.$$

(71). $$\int \frac{dx}{x^2\sqrt{(a+bx^2)}} = -\frac{\sqrt{(a+bx^2)}}{ax}.$$

(72). $$\int \frac{dx}{x^3\sqrt{(a+bx^2)}} = -\frac{\sqrt{(a+bx^2)}}{2ax^2}$$

$$-\frac{b}{2a}\int\frac{dx}{x\sqrt{(a+bx^2)}}. \qquad \text{(Ex. 7.)}$$

(73). $$\int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\left(\frac{1}{3x^3}+\frac{2}{3\,.\,1\,.\,x}\right)\sqrt{(1-x^2)}.$$

(74). $$\int \frac{dx}{x^5\sqrt{(1-x^2)}} = -\left(\frac{1}{4x^4}+\frac{3}{4\,.\,2\,.\,x^2}\right)\sqrt{(1-x^2)}$$

$$+\frac{3}{4\,.\,2}\log\left\{\frac{1-\sqrt{(1-x^2)}}{x}\right\}.$$

(75). $$\int \frac{d\,x}{x^6 \sqrt{(1+x^2)}} = -\left(\frac{1}{5\,x^5} - \frac{4}{5\,.\,3\,.\,x^3}\right.$$

$$\left. + \frac{4\,.\,2}{5\,.\,3\,.\,1\,.\,x}\right) \sqrt{(1+x^2)}.$$

(76). $$\int \frac{d\,x}{x^7 \sqrt{(x^2-1)}} = \left\{\frac{1}{6\,x^6} + \frac{5}{6\,.\,4\,.\,x^4}\right.$$

$$\left. + \frac{5\,.\,3}{6\,.\,4\,.\,2\,.\,x^2}\right\} \sqrt{(x^2-1)}$$

$$+ \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2} \sec^{-1} x.$$

(77). $$\int \frac{x\,d\,x}{(a+b\,x^2)^{\frac{3}{2}}} = -\frac{1}{b\sqrt{(a+b\,x^2)}}.$$

(78). $$\int \frac{x^2\,d\,x}{(a+b\,x^2)^{\frac{3}{2}}} = -\frac{x}{b\sqrt{(a+b\,x^2)}}$$

$$+ \frac{1}{b}\int \frac{d\,x}{\sqrt{(a+b\,x^2)}}. \qquad \text{(Ex. 3.)}$$

(79). $$\int \frac{x^3\,d\,x}{(a+b\,x^2)^{\frac{3}{2}}} = \left(\frac{x^2}{b} + \frac{2\,a}{b^2}\right)\frac{1}{\sqrt{(a+b\,x^2)}}.$$

(80). $$\int \frac{x^4\,d\,x}{(1-x^2)^{\frac{3}{2}}} = -\left(\frac{x^3}{2} - \frac{3\,x}{2}\right)\frac{1}{\sqrt{(1-x^2)}} - \frac{3}{2}\sin^{-1}x.$$

(81). $$\int \frac{d\,x}{x\,(a+b\,x^2)^{\frac{3}{2}}} = \frac{1}{a\sqrt{(a+b\,x^2)}}$$

$$+ \frac{1}{a}\int \frac{d\,x}{x\sqrt{(a+b\,x^2)}}. \qquad \text{(Ex. 7.)}$$

(82). $$\int \frac{d\,x}{x^2(a+b\,x^2)^{\frac{3}{2}}} = -\left(\frac{1}{a\,x} + \frac{2\,b\,x}{a^2}\right)\frac{1}{\sqrt{(a+b\,x^2)}}.$$

(83). $$\int \frac{dx}{x^3\,(1+x^2)^{\frac{3}{2}}} = -\left(\frac{1}{2\,x^2} + \frac{3}{2}\right)\frac{1}{\sqrt{(1+x^2)}}$$

$$- \frac{3}{2}\,.\,\log \frac{\sqrt{(1+x)} - 1}{x}.$$

(84). $$\int \frac{dx}{(a+b\,x^2)^{\frac{5}{2}}} = \left(\frac{2\,b\,x^3}{3\,a^2} + \frac{x}{a}\right)\frac{1}{(a+b\,x^2)^{\frac{3}{2}}}.$$

(85). $$\int \frac{x\,dx}{(a+b\,x^2)^{\frac{5}{2}}} = -\frac{1}{3\,b\,(a+bx^2)^{\frac{3}{2}}}.$$

(86). $$\int \frac{dx}{x\,(a+b\,x^2)^{\frac{5}{2}}} = \left(\frac{4}{3\,a} + \frac{b\,x^2}{a^2}\right)\frac{1}{(a+b\,x^2)^{\frac{3}{2}}}$$

$$+ \frac{1}{a^2}\int \frac{dx}{x\,(a+b\,x^2)^{\frac{1}{2}}}. \qquad \text{(Ex. 7.)}$$

(87). $$\int \frac{dx}{(1+x^2)^{\frac{7}{2}}} = \left\{\frac{4\,.\,2}{5\,.\,3}\,x^5 + \frac{4}{3}\,x^3 + x\right\}\frac{1}{(1+x^2)^{\frac{5}{2}}}.$$

(88). $$\int dx\,\sqrt{(a+b\,x^2)} = \frac{x\,\sqrt{(a+b\,x^2)}}{2}$$

$$+ \frac{a}{2}\int \frac{dx}{\sqrt{(a+b\,x^2)}}. \qquad \text{(Ex. 3.)}$$

(89). $$\int \frac{dx\,\sqrt{(a+b\,x^2)}}{x} = \sqrt{(a+b\,x^2)}$$

$$+ a\int \frac{dx}{x\,\sqrt{(a+b\,x^2)}}. \qquad \text{(Ex. 7.)}$$

(90). $$\int \frac{dx\,(a+bx^2)^{\frac{3}{2}}}{x} = \left\{\frac{(a+bx^2)}{3} + a\right\}\sqrt{(a+bx^2)}$$

$$+\,a^2 \int \frac{dx}{x\sqrt{(a+bx^2)}}. \quad \text{(Ex. 7.)}$$

(91). $$\int \frac{dx}{(ax+bx^2)^{\frac{3}{2}}} = -\,\frac{2\,(2bx+a)}{a^2\sqrt{(ax+bx^2)}}.$$

(92). $$\int \frac{x\,dx}{\sqrt{(ax+bx^2)}} = \frac{\sqrt{(ax+bx^2)}}{b}$$

$$-\,\frac{a}{2b}\int \frac{dx}{\sqrt{(ax+bx^2)}}. \quad \text{(Ex. 11.)}$$

(93). $$\int \frac{x^3\,dx}{\sqrt{(2x-x^2)}} = -\left\{\frac{x^2}{3} + \frac{5}{3\,.\,2}\,x + \frac{5\,.\,3}{3\,.\,2}\right\} \times$$

$$\sqrt{(2x-x^2)} + \frac{5\,.\,3}{3\,.\,2\,.\,1}\ \text{ver}\sin^{-1} x.$$

(94). $$\int \frac{x^4\,dx}{\sqrt{(x+x^2)}} = \left(\frac{x^3}{4} - \frac{7}{4\,.\,3\,.\,2}\,x^2 + \frac{7\,.\,5}{4\,.\,3\,.\,2\,.\,2^2}\,x\right.$$

$$\left.-\,\frac{7\,.\,5\,.\,3}{4\,.\,3\,.\,2\,.\,1\,.\,2^3}\right)\sqrt{(x+x^2)} + \frac{7\,.\,5\,.\,3}{4\,.\,3\,.\,2\,.\,1\,.\,2^3} \times$$

$$\log\left\{2x + 1 + 2\sqrt{(x+x^2)}\right\}.$$

(95). $$\int \frac{x^6\,dx}{\sqrt{(2rx-x^2)}} = -\left(\frac{x^5}{6} + \frac{11\,r\,x^4}{6\,.\,5} + \frac{11\,.\,9\,r^2\,x^3}{6\,.\,5\,.\,4}\right.$$

$$+\,\frac{11\,.\,9\,.\,7\,r^3\,x^2}{6\,.\,5\,.\,4\,.\,3} + \frac{11\,.\,9\,.\,7\,.\,5\,r^4\,x}{6\,.\,5\,.\,4\,.\,3\,.\,2}$$

$$+\frac{11\,.\,9\,.\,7\,.\,5\,.\,3\,r^5}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}\Big)\sqrt{(2\,r\,x-x^2)}$$

$$+\frac{11\,.\,9\,.\,7\,.\,5\,.\,3\,r^6}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}\text{ versin}^{-1}\frac{x}{r}.$$

(96). $$\int\frac{d\,x}{x\sqrt{(a\,x+b\,x^2)}}=-\frac{2\sqrt{(a\,x+b\,x^2)}}{a\,x}.$$

(97). $$\int\frac{d\,x}{x^2\sqrt{(a\,x+b\,x^2)}}=-\left(\frac{1}{3\,a\,x^2}-\frac{2\,b}{3\,a^2\,x}\right)2\sqrt{(a\,x+b\,x^2)}.$$

(98). $$\int\frac{d\,x}{x^5\sqrt{(2\,r\,x-x^2)}}=-\Big(\frac{1}{9\,r\,x^5}+\frac{4}{9\,.\,7\,r^2\,x^4}$$

$$+\frac{4\,.\,3}{9\,.\,7\,.\,5\,r^3\,x^3}+\frac{4\,.\,3\,.\,2}{9\,.\,7\,.\,5\,.\,3\,.\,r^4\,x^2}+\frac{4\,.\,3\,.\,2\,.\,1}{9\,.\,7\,.\,5\,.\,3\,.\,1\,.\,r^5\,x}\Big\}\times$$

$$\sqrt{(2\,r\,x-x^2)}.$$

(99). $$\int\frac{x\,d\,x}{(a\,x+b\,x^2)^{\frac{3}{2}}}=\frac{2\,x}{a\sqrt{(a\,x+b\,x^2)}}.$$

(100). $$\int\frac{d\,x}{x\,(a\,x+b\,x^2)^{\frac{3}{2}}}=-\frac{2}{3\,a\,x\sqrt{(a\,x+b\,x^2)}}$$

$$-\frac{4\,b}{3\,a}\int\frac{d\,x}{(a\,x+b\,x^2)^{\frac{3}{2}}}.\quad\text{(Ex. 91.)}$$

(101). $$\int\frac{d\,x}{(a+b\,x+c\,x^2)^{\frac{3}{2}}}=\frac{2\,(2\,c\,x+b)}{(4\,a\,c-b^2)\sqrt{(a+b\,x+c\,x^2)}}.$$

(102). $$\int\frac{x\,d\,x}{\sqrt{(a+b\,x+c\,x^2)}}=\frac{\sqrt{(a+b\,x+c\,x^2)}}{c}$$

$$-\frac{b}{2\,c}\int\frac{d\,x}{\sqrt{(a+b\,x+c\,x^2)}}.\quad\text{(Ex. 15.)}$$

(103). $$\int \frac{dx}{x^2\sqrt{(a+bx+cx^2)}} = -\frac{\sqrt{(a+bx+cx^2)}}{ax}$$

$$-\frac{b}{2a}\int \frac{dx}{x\sqrt{(a+bx+cx^2)}}.\quad \text{(Ex. 15.)}$$

(104). $$\int \frac{x\,dx}{(a+bx+cx^2)^{\frac{3}{2}}}$$

$$= -\frac{2(2a+bx)}{(4ac-b^2)\sqrt{(a+bx+cx^2)}}.$$

(105). $$\int \frac{dx}{(a+bx)\sqrt{x}} = \pm\frac{2}{\sqrt{(ab)}}\tan^{-1}\sqrt{\left(\frac{bx}{a}\right)}$$

$$= \frac{1}{\sqrt{(-ab)}}\log\left\{\frac{a-bx+2\sqrt{x}\,.\sqrt{(-ab)}}{a+bx}\right\}.$$

(106). $$\int \frac{dx}{(1+x)^2\sqrt{x}} = \frac{\sqrt{x}}{1+x} \pm \tan^{-1}\sqrt{x}.$$

(107). $$\int \frac{dx\sqrt{x}}{1-x} = -2\sqrt{x} + \log\left\{\frac{1+x+2\sqrt{x}}{1-x}\right\}.$$

(108). $$\int \frac{dx}{(a+bx^2)\sqrt{x}} = \frac{1}{bk^3\sqrt{2}}\times$$

$$\left\{\log\left(\frac{x+k\sqrt{2x}+k^2}{\sqrt{(a+bx^2)}}\right) + \tan^{-1}\frac{k\sqrt{2x}}{k^2-x}\right\}$$

if $k = \sqrt[4]{\frac{a}{b}}$, or $= \frac{1}{2bk^3}\left\{\log\left(\frac{k-\sqrt{x}}{k+\sqrt{x}}\right) - 2\tan^{-1}\frac{\sqrt{x}}{k}\right\}$

if $k = \sqrt[4]{\left(-\frac{a}{b}\right)}$.

(109). $$\int \frac{dx}{(1+x^2)^2\sqrt{x}} = \frac{\sqrt{x}}{2(1+x^2)}$$

$$+\frac{3}{4\sqrt{2}}\left\{\log\frac{x^2+\sqrt{(2x)}+1}{\sqrt{(1+x^2)}}+\tan^{-1}\frac{\sqrt{(2x)}}{1-x}\right\}.$$

(110). $$\int \frac{dx}{(c+ex)^2\sqrt{(a+bx)}} = \frac{\sqrt{(a+bx)}}{k(c+ex)}$$

$$+\frac{b}{2k}\int\frac{dx}{(c+ex)\sqrt{(a+bx)}}.$$ (Ex. 26.)

where $k = bc - ae$.

(111). $$\int \frac{x\,dx}{(c+ex)\sqrt{(a+bx)}} = \frac{1}{e}\int\frac{dx}{\sqrt{(a+bx)}}$$

$$-\frac{c}{e}\int\frac{dx}{(c+ex)\sqrt{(a+bx)}}.$$ (Ex. 1. and 26.)

(112). $$\int \frac{x\,dx}{(c+ex^2)\sqrt{(a+bx^2)}} = \frac{1}{\sqrt{(ae^2-bce)}}\times$$

$$\log\left\{\frac{e\sqrt{(a+bx^2)}-\sqrt{(ae^2-bce)}}{\sqrt{(c+ex^2)}}\right\}$$

$$= \frac{1}{\sqrt{(bce-ae^2)}}\tan^{-1}\frac{e\sqrt{(a+bx^2)}}{\sqrt{(bce-ae^2)}}.$$

(113). $$\int \frac{x^2\,dx}{(c+ex^2)\sqrt{(a+bx^2)}} = \frac{1}{e}\int\frac{dx}{\sqrt{(a+bx^2)}}$$

$$-\frac{c}{e}\int\frac{dx}{(c+ex^2)\sqrt{(a+bx^2)}}.$$ (Ex. 3. and 33.)

(114). $$\int \frac{x\,dx}{(1+x)^{\frac{1}{3}}+(1+x)^{\frac{1}{2}}} = -\frac{3}{2}(1+x)^{\frac{2}{3}}$$

$$-\frac{6}{5}(1+x)^{\frac{5}{6}}-(1+x)-\frac{6}{7}(1+x)^{\frac{7}{6}}-\frac{3}{4}(1+x)^{\frac{4}{3}}-\frac{2}{3}(1+x)^{\frac{3}{2}}.$$

Assume $1+x=z^6$. Euler. *Cal. Int.* Tom. I. p. 54.

$$(115).\ \int\frac{dx}{\surd(1+x^2)-\surd(1-x^2)}=\frac{\surd(1+x^2)-\surd(1-x^2)}{2x}$$

$$+\frac{1}{2}\log\left\{x+\surd(1+x^2)\right\}-\frac{1}{2}\tan^{-1}\frac{x}{\surd(1-x^2)}.$$

Multiply numerator and denominator by $\surd(1+x^2)+\surd(1-x^2)$: Euler. *Ib.* p. 56.

$$(116).\ \int\frac{dx\surd(1-x)}{\surd(1+x)}=\surd(1-x^2)+2\tan^{-1}\sqrt{\frac{1+x}{1-x}}$$

$$=\surd(1-x^2)+\frac{\pi}{2}+\sin^{-1}x.$$

Euler. *Acta. Petrop.* Tom. IV. 1780, and *Calc. Integ.* Tom. IV. p. 10.

$$(117).\ \int\frac{(1+x^2)\,dx}{(1-x^2)\surd(1+x^4)}=\frac{1}{\surd 2}\log\frac{\surd(1+x^4)+\surd 2}{1+x^2}.$$

Assume $\dfrac{x\surd 2}{1-x^2}=u$. Euler. *Ib.* p. 22.

$$118).\ \int\frac{(1-x^2)\,dx}{(1+x^2)\surd(1+x^4)}=\frac{1}{\surd 2}\sin^{-1}\frac{x\surd 2}{1+x^2}.$$

Assume $\dfrac{x\surd 2}{1+x^2}=u$. Euler. *Ib.*

$$(119).\ \int\frac{dx\surd(1+x^4)}{1-x^4}=\frac{1}{2\surd 2}\log\frac{\surd(1+x^4)+x\surd 2}{1-x^2}$$

$$+\frac{1}{2\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}.$$

Assume $u=\dfrac{x\sqrt{2}}{\sqrt{(1+x^4)}}$. Euler. *Ib.*

(120). $$\int\frac{x^2\,dx}{(1-x^4)\sqrt{(1+x^4)}}=\frac{1}{4\sqrt{2}}\log\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-x^2}$$

$$-\frac{1}{4\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}.$$

Assume $\dfrac{x\sqrt{2}}{\sqrt{(1+x^4)}}=u$. Euler. *Ib.*

(121). $$\int\frac{dx\sqrt{(a+bx^2+cx^4)}}{a-cx^4}=-\int\frac{u^2\,du}{(u^2-b)^2-4ac};$$

where $u=\dfrac{\sqrt{(a+bx^2+cx^4)}}{x}$.

Euler. *Ib.* p. 41.

(122). $$\int\frac{x^{n-2}\,dx\sqrt[n]{(a+bx^n+cx^{2n})}}{a-cx^{2n}}$$

$$=-\int\frac{u^n\,du}{(u^n-b)^2-4ac};\text{ where }u=\frac{\sqrt[n]{(a+bx^n+cx^{2n})}}{x}.$$

Euler. *Ib.* p. 42.

(123). $$\int\frac{x^{4i-\lambda}\,dx\,(a+bx^2+cx^4)^{\frac{\lambda+i}{2}}}{(a-cx^4)^{2i+1}}.$$

$$=-\int\frac{u^{\lambda+2}\,du}{\{(u^2-b)^2-4ac\}^{i+1}},\text{ where}$$

$u=\dfrac{\sqrt{(a+bx^2+cx^4)}}{x}$. Euler. *Ib.* p. 46.

(124). $\int \frac{x\,dx\,\sqrt[3]{(a+c\,x^6)}}{a-c\,x^6} = -\int \frac{u^3\,d\,u}{u^6-4\,a\,c}$,

where $u = \frac{\sqrt[3]{(a+c\,x^6)}}{x}$. Euler. *Ib.* p. 47.

(125). $\int \frac{d\,x\,(a+c\,x^6)^{\frac{2}{3}}}{a-c\,x^6} = -\int \frac{u^4\,d\,u}{u^6-4\,a\,c}$

where $u = \frac{\sqrt[3]{(a+c\,x^6)}}{x}$. Euler. *Ib.* p. 48.

(126). $\int \frac{d\,x}{(a+b\,x^n)\,(a+2\,b\,x^n)^{\frac{1}{2\,n}}} = \int \frac{d\,u}{a+b^2\,u^{2\,n}}$,

where $u = \frac{x}{(a+2\,b\,x^n)^{\frac{1}{2\,n}}}$. Euler. *Ib.* p. 49.

(127). $\int \frac{d\,x}{(a+b\,x^n)\,(a^2+3\,a\,b\,x^n+3\,b^2\,x^{2n})^{\frac{1}{3\,n}}}$

$= \int \frac{d\,u}{a+b^3\,u^{3n}}$, where $u = \frac{x}{(a^2+3\,a\,b\,x^n+3\,b^2\,x^{2\,n})^{\frac{1}{3\,n}}}$

Euler. *Ib.* p. 56.

(128). $\int \frac{d\,x}{(a+b\,x^n)\,\{(a+b\,x^n)^\lambda - b^\lambda\,x^{\lambda n}\}^{\frac{1}{\lambda n}}}$

$$= \frac{1}{a}\int \frac{du}{(1+b^{\lambda} u^{\lambda n})}, \text{ where } u = \frac{x}{\{(a+bx^n)^{\lambda} - b^{\lambda} x^{\lambda n}\}^{\frac{1}{\lambda n}}}.$$

Euler. *Ib.* p. 53.

$$(129).\quad \int \frac{dx}{(1+x^4)\sqrt{\{1+x^4)^{\frac{1}{2}} - x^2\}}}$$

$$= \tan^{-1} \frac{x}{\sqrt{\{(1+x^4)^{\frac{1}{2}} - x^2\}}}.$$

Euler. *Ib.* p. 54.

$$(130).\quad \int \frac{dx}{(1+x^6)\sqrt{\{1+x^6)^{\frac{1}{3}} - x^2\}}}$$

$$= \tan^{-1} \frac{x}{\sqrt{\{(1+x^6)^{\frac{1}{3}} - x^2\}}}.$$

Euler. *Ib.* p. 55.

$$(131).\quad \int \frac{x^{m-1}\,dx}{(a+bx^n)\{(a+bx^n)^{\lambda} - b^{\lambda} x^{\lambda n}\}^{\frac{m}{\lambda n}}}$$

$$= \frac{1}{a}\int \frac{u^{m-1}\,du}{1+b^{\lambda} u^{\lambda n}} \text{ where } u = \frac{x}{\{(a+bx^n)^{\lambda} - b^{\lambda} x^{\lambda n}\}^{\frac{1}{\lambda n}}}.$$

Euler. *Ib.* p. 56.

$$132.\quad \int \frac{dx\,\{(a+bx^n)^{\lambda} - b^{\lambda} x^{\lambda n}\}^{\frac{1}{\lambda n}}}{x^2(a+bx^n)} = \frac{1}{a}\int \frac{du}{u^2(1+b^{\lambda} u^{\lambda n})}$$

$$\text{where } u = \frac{x}{\{(a+bx^n)^{\lambda} - b^{\lambda} x^{\lambda n}\}^{\frac{1}{\lambda n}}}.$$

(133). $$\int \frac{x^{m-1}dx}{(bx^n - a)\{b^\lambda x^{\lambda n} - (bx^n - a)^\lambda\}^{\frac{m}{\lambda n}}}$$

$$= \frac{1}{a}\int \frac{u^{m-1}du}{b^\lambda u^{\lambda n} - 1}$$

where $u = \dfrac{x}{\{b^\lambda x^{\lambda n} - (bx^n - a)^\lambda\}^{\frac{1}{\lambda n}}}$.

Euler, *Ib.* p. 58.

The fluents of many of these expressions may be found by methods explained in a valuable paper "*On the fluents of irrational functions*," by Edward Ffrench Bromhead, M. A. F. R. S. printed in the *Philosophical Transactions* for 1816.

On the Integration of Logarithmic Functions.

(1). $\int dx \log x = x \log x - x$.

(2). $\int x\,dx \log x = \dfrac{x^2}{2}\log x - \dfrac{1}{4}x^2$.

(3). $\int x^2\,dx \log x = \dfrac{x^3}{3}\log x - \dfrac{1}{9}x^3$.

(4). $\int x^n\,dx \log x = \dfrac{x^{n+1}}{n+1}\left\{\log x - \dfrac{1}{n+1}\right\}$.

(5). $\int \dfrac{dx}{x}\log x = \dfrac{1}{2}(\log x)^2$.

(6). $$\int x^m\,dx\,(\log x)^2 = \frac{x^{m+1}}{m+1}\left\{(\log x)^2 - \frac{2}{m+1}\log x + \frac{2\,.\,1}{(m+1)^2}\right\}.$$

(7). $$\int x^m\,dx\,(\log x)^3 = \frac{x^{m+1}}{m+1}\left\{(\log x)^3 - \frac{3}{m+1}(\log x)^2 + \frac{3\,.\,2}{(m+1)^2}\log x - \frac{3\,.\,2\,.\,1}{(m+1)^3}\right\}.$$

(8). $$\int \frac{dx}{(1-x)^2}\log x = \frac{x\log x}{1-x} + \log(1-x).$$

(9). $$\int \frac{dx}{x\sqrt{x}}\log\frac{1}{1-x} = 2\log\frac{1+\sqrt{x}}{1-\sqrt{x}} - \frac{2}{\sqrt{x}}\log\frac{1}{1-x}$$

(10). $$\int \frac{dx}{x\log x} = \log.\log x = \log^2 x.$$

(11). $$\int \frac{dx}{x(\log x)^2} = -\frac{1}{\log x}.$$

(12). $$\int \frac{dx}{a+bx}\log x = \frac{1}{b}\log x\,.\log\left(\frac{a+bx}{a}\right) - \frac{x}{a} + \frac{bx^2}{2^2a^2} - \frac{b^2x^3}{3^2a^3} + \text{\&c. } \textit{ad infinitum.}$$

$$= \frac{1}{b}\log x\,.\log(a+bx) - \frac{1}{2b}(\log bx)^2 + \frac{a}{b^2x} - \frac{a^2}{2^2b^3x^2} + \frac{a^3}{3^2b^4x^3} - \frac{a^4}{4^2b^5x^4} + \text{\&c. } \textit{in infinitum.}$$

(13). $$\int \frac{dx}{x} \log (a + bx) = \log a \,.\, \log x + \frac{bx}{a} - \frac{b^2 x^2}{2^2 a^2}$$

$$+ \frac{b^3 x^3}{3^2 a^3} - \text{\&c. } \textit{in infinitum.}$$

(14). $$\int \frac{dx}{\log x} = \log \log x + \frac{\log x}{1} + \frac{1}{2} \,.\, \frac{(\log x)^2}{1 \,.\, 2}$$

$$+ \frac{1}{3} \,.\, \frac{(\log x)^3}{1 \,.\, 2 \,.\, 3} + \text{\&c. } \textit{in infinitum.}$$

(15). $$\int \frac{x^m \, dx}{\sqrt{(\log x)}} = \frac{x^{m+1}}{(m+1) \sqrt{(\log x)}}$$

$$\left\{ 1 + \frac{1}{2\,(m+1) \log x} + \frac{1 \,.\, 3}{2^2\,(m+1)^2 (\log x)^2} \right.$$

$$\left. \frac{1 \,.\, 3 \,.\, 5}{2^3\,(m+1)^3 (\log x)^3} + \text{\&c. } \textit{in infinitum.} \right\}$$

(16). $$\int a^x \, dx = \frac{1}{\log a} \, a^x.$$

(17). $$\int a^x x \, dx = \frac{1}{\log a} \, a^x x - \frac{1}{(\log a)^2} \, a^x.$$

(18). $$\int a^x x^n \, dx = a^x \left\{ \frac{x^n}{\log a} - \frac{n\, x^{n-1}}{(\log a)^2} \right.$$

$$\left. + \frac{n.(n-1).\,x^{n-2}}{(\log a)^3} - \frac{n.(n-1).(n-2).\,x^{n-3}}{(\log a)^4} + \text{\&c. } \textit{in inf.} \right\}$$

(19). $$\int \frac{a^x \, dx}{x} = \log x + \frac{x \log a}{1 \,.\, 1} + \frac{x^2 (\log a)^2}{1 \,.\, 2 \,.\, 2}$$

$$+ \frac{x^3 (\log a)^3}{1 \,.\, 2 \,.\, 3 \,.\, 3} + \text{\&c. } \textit{in infinitum.}$$

(20). $$\int \frac{a^x\,d\,x}{x^2} = -\frac{a^x}{x} + \log a \int \frac{a^x\,d\,x}{x} \qquad \text{(Ex. 19.)}$$

(21). $$\int \frac{a^x\,d\,x}{x^3} = -\frac{a^x}{2\,x^2} - \frac{a^x \log a}{2\,.\,1\,.\,x} + \frac{(\log a)^2}{2\,.\,1} \times$$

$$\int \frac{a^x\,d\,x}{x}. \qquad \text{(Ex. 19.)}$$

(22). $$\int \frac{a^x\,d\,x}{\surd x} = \frac{a^x}{\surd x}\left\{\frac{1}{\log a} + \frac{1}{2\,x\,(\log a)^2} + \frac{1\,.\,3}{4\,x^2\,(\log a)^3}\right.$$

$$\left. + \text{ \&c. } \textit{in infinitum.}\right\}$$

(23). $$\int \frac{a^x\,d\,x}{1-x} = a^x\left\{\frac{1}{(1-x)\log a} - \frac{1}{(1-x)^2\,(\log a)^2}\right.$$

$$\left. + \frac{1\,.\,2}{(1-x)^3\,(\log a)^3} - \text{\&c. } \textit{in infinitum.}\right\}$$

(24). $$\int \frac{e^x\,d\,x\,(2-x^2)}{(1-x)^{\frac{3}{2}}\,\surd\,(1+x)} = \epsilon^x \sqrt{\left(\frac{1+x}{1-x}\right)}.$$

On the Integration of Circular Functions.

(1). $$\int X\,d\,x \sin^{-1} x = \sin^{-1} x\,.\int X\,d\,x$$

$$- \int \frac{d\,x \int X\,d\,x}{\surd(1-x^2)}.$$

(2). $$\int X\,d\,x \cos^{-1} x = \cos^{-1} x\,.\int X\,d\,x$$

$$+ \int \frac{d\,x \int X\,d\,x}{\surd(1-x^2)}.$$

(3). $\int X\,d\,x \tan^{-1}x = \tan^{-1}x \,.\int X\,d\,x$

$$-\int\frac{d\,x\int X\,d\,x}{1+x^2}.$$

(4). $\int X\,d\,x \sec^{-1}x = \sec^{-1}x\,.\int X\,d\,x$

$$-\int\frac{d\,x\int X\,d\,x}{x\sqrt{(x^2-1)}}.$$

(5). $\int d\,x \sin^{-1}x = x\sin^{-1}x + \sqrt{(1-x^2)}.$

(6). $\int x\,d\,x \sin^{-1}x = \left(\frac{x^2}{2}-\frac{1}{4}\right)\sin^{-1}x + \frac{1}{4}x\sqrt{(1-x^2)}.$

(7). $\int\frac{d\,x}{\sqrt{(1-x^2)}}\sin^{-1}x = \frac{1}{2}(\sin^{-1}x)^2.$

(8). $\int\frac{x\,d\,x}{\sqrt{(1-x^2)}}\sin^{-1}x = x - \sqrt{(1-x^2)}\sin^{-1}x.$

(9). $\int\frac{x^2\,d\,x}{\sqrt{(1-x^2)}}\sin^{-1}x = -\left(\frac{x}{2}\sqrt{(1-x^2)}-\frac{1}{4}\sin^{-1}x\right)\times$

$$\sin^{-1}x + \frac{x^2}{4}.$$

(10). $\int\frac{d\,x}{(1-x^2)^{\frac{3}{2}}}\sin^{-1}x = \frac{x\sin^{-1}x}{\sqrt{(1-x^2)}} + \log\sqrt{(1-x^2)}.$

(11). $\int\frac{d\,x}{1+x^2}\tan^{-1}x = \frac{1}{2}(\tan^{-1}x)^2.$

(12). $\int\frac{x^2\,d\,x}{1+x^2}\tan^{-1}x = (x-\frac{1}{2}\tan^{-1}x)\tan^{-1}x$

$$-\log\sqrt{(1+x^2)}.$$

(13). $$\int \frac{dx}{(1+x^2)^2} \tan^{-1}x = \left(\frac{x}{2\,(1+x^2)} + \frac{1}{4} \tan^{-1}x\right) \times$$

$$\tan^{-1}x + \frac{1}{4\,(1+x^2)}.$$

(14). $$\int \frac{d\,\theta}{\cos\,\theta} = \log \frac{\surd(1+\sin\theta)}{\surd(1-\sin\theta)} = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

(15). $$\int \frac{d\,\theta}{\sin\,\theta} = \log \frac{\surd(1-\cos\theta)}{\surd(1+\cos\theta)} = \log\tan\frac{\theta}{2}.$$

(16). $$\int \frac{d\,\theta\cos\theta}{\sin\theta} = \int \frac{d\,\theta}{\tan\theta} = \log\sin\theta.$$

(17). $$\int \frac{d\,\theta\sin\theta}{\cos\theta} = \int d\,\theta\tan\theta = -\log\cos\theta.$$

(18). $$\int \frac{d\,\theta}{\cos\theta\,\sin\theta} = \log\tan\theta.$$

(19). $$\int d\,\theta\sin\theta = -\cos\theta.$$

(20). $$\int d\,\theta\,(\sin\theta)^2 = -\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\,\theta.$$

(21). $$\int d\,\theta\,(\sin\theta)^3 = -\frac{1}{3}(\sin\theta)^2\cos\theta - \frac{2}{3}\cos\theta.$$

(22). $$\int d\,\theta\,(\sin\theta)^4 = -\frac{1}{4}(\sin\theta)^3\cos\theta - \frac{1\,.\,3}{2\,.\,4}\times$$

$$\sin\theta\cos\theta + \frac{1\,.\,3}{2\,.\,4}\,\theta.$$

(23). $$\int d\,\theta\,(\cos\theta) = \sin\theta.$$

(24). $$\int d\theta\,(\cos\theta)^2 = \frac{1}{2}\sin\theta\,\cos\theta + \frac{1}{2}\,\theta.$$

(25). $$\int d\theta\,(\cos\theta)^3 = \frac{1}{3}\sin\theta\,(\cos\theta)^2 + \frac{2}{3}\sin\theta.$$

(26). $$\int d\theta\,(\cos\theta)^4 = \frac{1}{4}\sin\theta\,(\cos\theta)^3 + \frac{1\,.\,3}{2\,.\,4}\times$$

$$\sin\theta\,\cos\theta + \frac{1\,.\,3}{2\,.\,4}\,\theta.$$

(27). $$\int d\theta\,\sin\theta\,(\cos\theta)^n = -\frac{1}{n+1}(\cos\theta)^{n+1}.$$

(28). $$\int d\theta\,\cos\theta\,(\sin\theta)^n = \frac{1}{n+1}(\sin\theta)^{n+1}.$$

(29). $$\int d\theta\,(\sin\theta)^2\,(\cos\theta)^2 = \frac{1}{4}(\sin\theta)^3\,(\cos\theta)$$

$$-\frac{1}{8}\sin\theta\,.\,\cos\theta + \frac{1}{8}\,\theta.$$

(30). $$\int d\theta\,(\sin\theta)^2\,(\cos\theta)^3 = \left(\frac{1}{5}(\cos\theta)^2 + \frac{2}{15}\right)(\sin\theta)^3.$$

(31). $$\int d\theta\,(\sin\theta)^3\,(\cos\theta)^2 = \left(\frac{1}{5}(\sin\theta)^4\right.$$

$$\left.-\frac{1}{15}(\sin\theta)^2 - \frac{2}{15}\right)\cos\theta.$$

(32). $$\int d\theta\,(\sin\theta)^3\,(\cos\theta)^3 = \left(\frac{1}{6}(\cos\theta)^2 + \frac{1}{12}\right)(\sin\theta)^4.$$

(33). $\int d\theta\,(\sin\theta)^4\,(\cos\theta)^2 = \left(\frac{1}{6}(\sin\theta)^5 - \frac{1}{24}(\sin\theta)^3\right.$

$\left. - \frac{1}{16}\sin\theta\right)\cos\theta + \frac{1}{16}\,\theta.$

(34). $\int d\theta\,(\sin\theta)^4\,(\cos\theta)^3 = \left(\frac{1}{7}(\cos\theta)^2 + \frac{2}{35}\right)(\sin\theta)^5.$

(35). $\int d\theta\,(\sin\theta)^5\,(\cos\theta)^5 = -\frac{1}{512}\left(\frac{1}{10}\cos 10\,\theta\right.$

$\left. - \frac{5}{6}\cos 6\,\theta + 5\cos 2\,\theta\right).$

(36). $\int d\theta\,(\sin\theta)^8\,(\cos\theta)^3 = \left(\frac{1}{11}(\cos\theta)^2 + \frac{2}{99}\right)(\sin\theta)^9.$

(37). $\int \frac{d\theta}{(\sin\theta)^2} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$

(38). $\int \frac{d\theta}{(\sin\theta)^3} = -\frac{\cos\theta}{2(\sin\theta)^2} + \frac{1}{2}\int\frac{d\theta}{\sin\theta}.$ (Ex. 15.)

(39). $\int \frac{d\theta}{(\sin\theta)^4} = -\left(\frac{1}{3(\sin\theta)^3} + \frac{2}{3\sin\theta}\right)\cos\theta$

$= \cot\theta - \frac{1}{3}(\cot\theta)^3.$

(40). $\int \frac{d\theta}{(\cos\theta)^2} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$

(41). $\int \frac{d\theta}{(\cos\theta)^3} = \frac{\sin\theta}{2(\cos\theta)^2} + \frac{1}{2}\int\frac{d\theta}{\cos\theta}.$ (Ex. 14.)

42). $$\int \frac{d\,\theta}{(\cos\,\theta)^4} = \left(\frac{1}{3\,(\cos\,\theta)^3} + \frac{2}{3\,\cos\,\theta}\right) \sin\,\theta$$

$$= \tan\,\theta + \frac{1}{3}\,(\tan\,\theta)^3.$$

(43). $$\int \frac{d\,\theta\,(\sin\,\theta)^2}{\cos\,\theta} = -\sin\,\theta + \int \frac{d\,\theta}{\cos\,\theta}.$$ (Ex. 14.)

(44). $$\int \frac{d\,\theta\,(\sin\,\theta)^3}{\cos\,\theta} = -\frac{(\sin\,\theta)^2}{2} + \log\sec\,\theta.$$

(45). $$\int \frac{d\,\theta\,(\sin\,\theta)^4}{\cos\,\theta} = -\frac{(\sin\,\theta)^3}{3} - \sin\,\theta + \int \frac{d\,\theta}{\cos\,\theta}.$$

(Ex. 15.)

(46). $$\int \frac{d\theta\,(\cos\theta)^2}{\sin\,\theta} = \cos\,\theta + \int \frac{d\,\theta}{\sin\,\theta}.$$ (Ex. 15.)

(47). $$\int \frac{d\,\theta\,(\cos\,\theta)^3}{\sin\,\theta} = -\frac{(\cos\,\theta)^2}{2} + \log\sin\,\theta.$$

(48). $$\int \frac{d\theta\,(\cos\theta)^4}{\sin\,\theta} = \frac{(\cos\,\theta)^3}{3} + \cos\,\theta + \int \frac{d\,\theta}{\sin\,\theta}.$$ (Ex. 15.)

(49). $$\int \frac{d\,\theta\,\sin\,\theta}{(\cos\,\theta)^2} = \frac{1}{\cos\,\theta} = \sec\,\theta.$$

(50). $$\int \frac{d\,\theta\,(\sin\,\theta)^2}{(\cos\,\theta)^2} = \int d\,\theta\,(\tan\,\theta)^2 = \frac{\sin\,\theta}{\cos\,\theta} - \theta = \tan\theta - \theta.$$

(51). $$\int \frac{d\,\theta\,(\sin\,\theta)^3}{(\cos\,\theta)^2} = -\{(\sin\,\theta)^2 - 2\}\frac{1}{\cos\,\theta} = \cos\,\theta + \sec\,\theta.$$

(52). $$\int \frac{d\,\theta\,(\sin\,\theta)^4}{(\cos\,\theta)^2} = -\left(\frac{1}{2}\,(\sin\,\theta)^3 - \frac{3}{2}\,\sin\,\theta\right) \times$$

$$\frac{1}{\cos\,\theta} - \frac{3}{2}\,\theta.$$

(53). $$\int \frac{d\,\theta\,\cos\,\theta}{(\sin\,\theta)^2} = -\frac{1}{\sin\,\theta} = -\operatorname{cosec}\,\theta.$$

(54). $$\int \frac{d\,\theta\,(\cos\,\theta)^2}{(\sin\,\theta)^2} = \int \frac{d\,\theta}{(\tan\,\theta)^2} = -\frac{\cos\,\theta}{\sin\,\theta}$$

$$-\,\theta = -\cot\,\theta - \theta.$$

(55). $$\int \frac{d\,\theta\,(\cos\,\theta)^3}{(\sin\,\theta)^2} = [(\cos\,\theta)^2 - 2]\,\frac{1}{\sin\,\theta}$$

$$= -\sin\,\theta - \operatorname{cosec}\,\theta.$$

(56). $$\int \frac{d\,\theta\,(\cos\,\theta)^4}{(\sin\,\theta)^2} = \left(\frac{1}{2}\,(\cos\,\theta)^3 - \frac{3}{2}\,\cos\,\theta\right)\frac{1}{\sin\,\theta} - \frac{3}{2}\,\theta.$$

(57). $$\int \frac{d\,\theta\,\sin\,\theta}{(\cos\,\theta)^3} = \frac{1}{2\,(\cos\,\theta)^2}.$$

(58). $$\int \frac{d\,\theta\,(\sin\,\theta)^2}{(\cos\,\theta)^3} = \frac{\sin\,\theta}{2\,(\cos\,\theta)^2} - \frac{1}{2}\int \frac{d\,\theta}{\cos\,\theta}. \quad \text{(Ex. 14.)}$$

(59). $$\int \frac{d\,\theta\,(\sin\,\theta)^3}{(\cos\,\theta)^3} = \int d\,\theta\,(\tan\,\theta)^3$$

$$= \frac{1}{2\,(\cos\,\theta)^2} + \log\,\cos\,\theta.$$

(60). $$\int \frac{d\theta\,(\sin\theta)^4}{(\cos\theta)^3} = -\left\{(\sin\theta)^3 - \frac{3}{2}\sin\theta\right\}\frac{1}{(\cos\theta)^2}$$

$$-\frac{3}{2}\int\frac{d\theta}{\cos\theta}. \quad \text{(Ex. 14).}$$

(61). $$\int \frac{d\theta\cos\theta}{(\sin\theta)^3} = -\frac{1}{2\,(\sin\theta)^2}.$$

(62). $$\int \frac{d\theta\,(\cos\theta)^2}{(\sin\theta)^3} = -\frac{\cos\theta}{2\,(\sin\theta)^2} - \frac{1}{2}\int\frac{d\theta}{\sin\theta}. \quad \text{(Ex. 15.)}$$

(63). $$\int \frac{d\theta\,(\cos\theta)^3}{(\sin\theta)^3} = \int\frac{d\theta}{(\tan\theta)^3} = -\frac{1}{2\,(\sin\theta)^2} - \log\sin\theta.$$

(64). $$\int \frac{d\theta\,(\cos\theta)^4}{(\sin\theta)^3} = \left\{(\cos\theta)^3 - \frac{3}{2}\cos\theta\right\}\frac{1}{(\sin\theta)^2}$$

$$-\frac{3}{2}\int\frac{d\theta}{\sin\theta}. \quad \text{(Ex. 15.)}$$

(65). $$\int \frac{d\theta\sin\theta}{(\cos\theta)^4} = \frac{1}{3}\,\frac{1}{(\cos\theta)^3}.$$

(66). $$\int \frac{d\theta\,(\sin\theta)^2}{(\cos\theta)^4} = \frac{(\sin\theta)^3}{3\,(\cos\theta)^3} = \frac{1}{3}(\tan\theta)^3.$$

(67). $$\int \frac{d\theta\,(\sin\theta)^3}{(\cos\theta)^4} = \left\{(\sin\theta)^2 - \frac{2}{3}\right\}\frac{1}{(\cos\theta)^3}.$$

(68). $$\int \frac{d\theta\,(\sin\theta)^4}{(\cos\theta)^4} = \int d\theta\,(\tan\theta)^4 = \left(\frac{4}{3}(\sin\theta)^3 - \sin\theta\right)\times$$

$$\frac{1}{(\cos\theta)^3} + \theta = \frac{1}{3}(\tan\theta)^3 - \tan\theta + \theta.$$

(69). $$\int \frac{d\theta\cos\theta}{(\sin\theta)^4} = -\frac{1}{3\,(\sin\theta)^3}.$$

(70). $$\int \frac{d\,\theta\,(\cos\,\theta)^2}{(\sin\,\theta)^4} = -\frac{(\cos\,\theta)^3}{3\,(\sin\,\theta)^3} = -\frac{1}{3}\,(\cot\,\theta)^3.$$

(71). $$\int \frac{d\,\theta\,(\cos\,\theta)^3}{(\sin\,\theta)^4} = -\left((\cos\,\theta)^2 - \frac{2}{3}\right)\frac{1}{(\sin\,\theta)^3}\,.$$

(72). $$\int \frac{d\,\theta\,(\cos\,\theta)^4}{(\sin\,\theta)^4} = \int \frac{d\,\theta}{(\tan\,\theta)^4}$$

$$= -\left(\frac{4}{3}\,(\cos\,\theta)^3 - \cos\,\theta\right)\frac{1}{(\sin\,\theta)^3} + \theta.$$

$$= -\frac{1}{3}\,(\cot\,\theta)^3 + \cot\,\theta + \theta.$$

(73). $$\int d\,\theta\,(\tan\,\theta)^5 = \frac{1}{4}\,(\tan\,\theta)^4 - \frac{1}{2}\,(\tan\,\theta)^2 - \log\cos\,\theta.$$

(74). $$\int \frac{d\,\theta}{(\tan\,\theta)^5} = -\frac{1}{4}\,(\cot\,\theta)^4 + \frac{1}{2}\,(\cot\,\theta)^2 + \log\sin\,\theta.$$

(75). $$\int d\,\theta\,(\tan\,\theta)^6 = \frac{1}{5}\,(\tan\,\theta)^5 - \frac{1}{3}\,(\tan\,\theta)^3 + \tan\,\theta - \theta.$$

(76). $$\int \frac{d\,\theta}{\sin\,\theta\,(\cos\,\theta)^2} = \frac{1}{\cos\,\theta} + \log\tan\frac{\theta}{2}\,.$$

(77). $$\int \frac{d\,\theta}{\sin\,\theta\,(\cos\,\theta)^3} = \frac{1}{2\,(\cos\,\theta)^2} + \log\tan\,\theta.$$

(78). $$\int \frac{d\,\theta}{\sin\,\theta\,(\cos\,\theta)^4} = \frac{1}{3\,(\cos\,\theta)^3} + \frac{1}{\cos\,\theta} + \log\tan\frac{\theta}{2}\,.$$

(79). $$\int \frac{d\,\theta}{(\sin\,\theta)^2\cos\,\theta} = -\frac{1}{\sin\,\theta} + \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

(80.) $$\int \frac{d\,\theta}{(\sin\,\theta)^2\,(\cos\,\theta)^2} = -2\cot 2\,\theta.$$

(81). $$\int \frac{d\,\theta}{(\sin\,\theta)^3 \cos\,\theta} = -\frac{1}{2\,(\sin\,\theta)^2} + \log \tan\,\theta.$$

(82). $$\int \frac{d\,\theta}{(\sin\,\theta)^3 (\cos\,\theta)^2} = \frac{1}{(\sin\,\theta)^2 \cos\,\theta} - \frac{3 \cos\,\theta}{2\,(\sin\,\theta)^2}$$

$$+ \frac{3}{2} \log \tan \frac{\theta}{2}\,.$$

(83). $$\int \frac{d\,\theta}{(\sin\,\theta)^3 (\cos\,\theta)^3} = -\frac{2 \cos 2\,\theta}{(\sin 2\,\theta)^2} + 2 \log \tan\,\theta.$$

(84). $$\int \frac{d\,\theta}{(\sin\,\theta)^4 \cos\,\theta} = -\frac{1}{3\,(\sin\,\theta)^3} - \frac{1}{\sin\,\theta}$$

$$+ \log \tan \left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

(85). $$\int \frac{d\,\theta}{(\sin\,\theta)^4 (\cos\,\theta)^2} = -\frac{1}{3 \cos\,\theta\,(\sin\,\theta)^3} - \frac{8}{3} \cot 2\,\theta.$$

(86). $$\int d\,\theta \cos\,(a\,\theta + b) = \frac{1}{a} \sin\,(a\,\theta + b).$$

(87). $$\int d\,\theta \sin\,(a\,\theta + b) = -\frac{1}{a} \cos\,(a\,\theta + b).$$

(88). $$\int \frac{d\,\theta}{[\sin\,(a\,\theta + b)]^3\,[\cos\,(a\,\theta + b)]^2}$$

$$= \frac{1}{a\,[\sin\,(a\,\theta + b)]^2 \cos\,(a\,\theta + b)} - \frac{3 \cos\,(a\,\theta + b)}{2\,a\,[\sin\,(a\,\theta + b)]^2}$$

$$+ \frac{3}{2\,a} \log \tan \left(\frac{a\,\theta + b}{2}\right).$$

(89). $$\int d\,\theta \sin\,(a\,\theta + b) \cos\,(a'\,\theta + b')$$

$$= -\frac{\cos\,[(a + a')\,\theta + b + b']}{2\,(a + a')} - \frac{\cos\,[(a - a')\,\theta + b - b']}{2\,(a - a')}\,.$$

(90). $\int d\theta \sin(a\theta + b)\sin(a'\theta + b')$

$$= \frac{\sin[(a-a')\theta + b - b']}{2(a-a')} - \frac{\sin[(a+a')\theta + b + b']}{2(a+a')}.$$

(91). $\int d\theta \cos(a\theta + b)\cos(a'\theta + b')$

$$= \frac{\sin[(a+a')\theta + b + b']}{2(a+a')} + \frac{\sin[(a-a')\theta + b - b']}{2(a-a')}.$$

(92). $\int \theta\, d\theta \cos\theta = \theta \sin\theta + \cos\theta.$

(93). $\int \theta^2 d\theta \cos\theta = \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta.$

(94). $\int \theta^3 d\theta \cos\theta = \theta^3\sin\theta + 3\theta^2\cos\theta - 6\theta\sin\theta - 6\cos\theta,$

(95). $\int \theta\, d\theta \sin\theta = -\theta\cos\theta + \sin\theta.$

(96). $\int \theta^2 d\theta \sin\theta = -\theta^2\cos\theta + 2\theta\sin\theta + 2\cos\theta.$

(97). $\int \theta^3 d\theta \sin\theta = -\theta^3\cos\theta + 3\theta^2\sin\theta$

$$+ 6\theta\cos\theta - 6\sin\theta.$$

(98). $\int \epsilon^{a\theta} d\theta \sin\theta = \dfrac{\epsilon^{a\theta}(a\sin\theta - \cos\theta)}{a^2+1}.$

(99). $\int \epsilon^{a\theta} d\theta (\sin\theta)^2 = \dfrac{\epsilon^{a\theta}\sin\theta(a\sin\theta - 2\cos\theta)}{a^2+4}$

$$+ \frac{1 \,.\, 2}{a(a^2+4)}\epsilon^{a\theta}.$$

(100). $\int \epsilon^{a\theta} d\theta (\sin\theta)^3 = \dfrac{\epsilon^{a\theta}(\sin\theta)^2(a\sin\theta - 3\cos\theta)}{a^2+9}$

$$+ \frac{2 \,.\, 3\,\epsilon^{a\theta}(a\sin\theta - \cos\theta)}{(a^2+1)(a^2+9)}.$$

(101). $$\int \epsilon^{a\theta}\, d\theta \cos\theta = \frac{\epsilon^{a\theta}(a\cos\theta + \sin\theta)}{a^2+1}.$$

(102). $$\int \epsilon^{a\theta}\, d\theta (\cos\theta)^2 = \frac{\epsilon^{a\theta}\cos\theta\,(a\cos\theta + 2\sin\theta)}{a^2+4}$$

$$+ \frac{1\,.\,2}{a\,(a^2+4)}\,\epsilon^{a\theta}.$$

(103). $$\int \epsilon^{a\theta}\, d\theta (\cos\theta)^3 = \frac{\epsilon^{a\theta}(\cos\theta)^2\,(a\cos\theta + 3\sin\theta)}{a^2+9}$$

$$+ \frac{2\,.\,3\,.\,\epsilon^{a\theta}\,(a\cos\theta + \sin\theta)}{(a^2+1)\,(a^2+9)}.$$

(104). $$\int \epsilon^{a\theta}\, d\theta \cos b\,\theta = \frac{\epsilon^{a\theta}(a\cos b\,\theta + b\sin b\,\theta)}{a^2+b^2}.$$

(105). $$\int \epsilon^{a\theta}\, d\theta \sin b\,\theta = \frac{\epsilon^{a\theta}(a\sin b\,\theta - b\cos b\,\theta)}{a^2+b^2}.$$

(106). $$\int \frac{d\theta}{a + b\cos\theta} = \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\frac{(a-b)\tan\frac{\theta}{2}}{\sqrt{(a^2-b^2)}}$$

$$= \frac{1}{\sqrt{(a^2-b^2)}}\tan^{-1}\frac{\sqrt{(a^2-b^2)}\sin\theta}{b + a\cos\theta}.$$

$$= \frac{1}{\sqrt{(a^2-b^2)}}\cos^{-1}\frac{b + a\cos\theta}{a + b\cos\theta}$$

$$= \frac{1}{\sqrt{(b^2-a^2)}}\log\left\{\frac{b + a\cos\theta + \sqrt{(b^2-a^2)}\sin\theta}{a + b\cos\theta}\right\}.$$

(107). $$\int \frac{d\theta}{a + a\cos\theta} = \frac{1}{a}\tan\frac{\theta}{2}.$$

(108). $$\int \frac{d\theta\sin\theta}{a + b\cos\theta} = -\frac{1}{b}\log(a + b\cos\theta).$$

(109). $\int \frac{d\theta \cos \theta}{a+b \cos \theta} = \frac{\theta}{b} - \frac{a}{b} \int \frac{d\theta}{a+b \cos \theta}$. (Ex. 106.)

(110). $\int \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{(b^2-a^2)} \left\{ \frac{b \sin \theta}{a+b\cos\theta} - a \int \frac{d\theta}{a+b\cos\theta} \right\}$. (Ex. 106.)

(111). $\int \frac{d\theta \cos \theta}{(a+b\cos\theta)^2} = \frac{1}{(a^2-b^2)} \left\{ \frac{a \sin \theta}{a+b\cos\theta} - b \int \frac{d\theta}{a+b\cos\theta} \right\}$. (Ex. 106.)

Nearly all the integrals of logarithmic and circular functions which we have put down, are to be found in the fourth and fifth Chapters of the Integral Calculus of Euler.

◆

On the Integration of Differential Equations of the First Order and the First Degree.

I. Differential expressions or differential equations which result from the simple differentiation of a function or an equation involving two variables, and which answer the *test* called the *criterion of integrability*. See Ex. 13. p. 204.

(1). Let $du = \frac{xy\,dy - y^2\,dx}{x^2\sqrt{(x^2+y^2)}} = P\,dx + Q\,dy$

$$\frac{dP}{dy} = -\frac{(2x^2+y^2)y}{x^2(x^2+y^2)^{\frac{3}{2}}} = \frac{dQ}{dx},$$

and therefore

$$\int Q\,dy = \frac{\sqrt{(x^2+y^2)}}{x} + X = R + X, \text{ where } X = f(x):$$

$$\text{and } dX = \left(P - \frac{dR}{dx}\right)dx = 0, \text{ or } X = c;$$

consequently,

$$u = \frac{\sqrt{(x^2+y^2)}}{x} + c.$$

(2). Let $du = \dfrac{2(y\,dx - x\,dy)}{y\sqrt{(x^2-y^2)}}$;

$$\frac{dP}{dy} = \frac{2y}{(x^2-y^2)^{\frac{3}{2}}} = \frac{dQ}{dx}$$

$$\int P\,dx = 2\log\{x + \sqrt{(x^2-y^2)}\} + Y = R + Y,$$

$$\text{and } dY = \left(Q - \frac{dR}{dy}\right)dy = -\frac{2\,dy}{y}$$

and $Y = c - 2\log y$,

$$\therefore\ u = 2\log\left\{\frac{x+\sqrt{(x^2-y^2)}}{y}\right\} + c$$

$$= \log\left\{\frac{x+\sqrt{(x^2-y^2)}}{x-\sqrt{(x^2-y^2)}}\right\} + c.$$

(3). Let $du = \dfrac{y\,dx - x\,dy}{y\sqrt{(y^2-x^2)}}$;

$$\frac{dP}{dy} = \frac{-y}{(y^2-x^2)^{\frac{3}{2}}} = \frac{dQ}{dx},$$

$$\text{and } u = \sin^{-1}\frac{x}{y} + c.$$

(4). Let the equation be

$$\frac{dy}{y} = \frac{x\,dy - y\,dx}{y\sqrt{(x^2+y^2)}}, \text{ and if } P\,dx + Q\,dy = 0,$$

$$\frac{dP}{dy} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{dQ}{dx};$$

$$\therefore \log\left\{\frac{x+\sqrt{(x^2+y^2)}}{c}\right\} = 0.$$

Euler. *Calc. Integ. Tom.* I. *Art.* 455.

(5). Let $dx\,(ax+by+c) + dy\,(bx+ey+f) = 0$,

$$\frac{dP}{dx} = b = \frac{dQ}{dx},$$

$$\therefore \frac{1}{2}ax^2 + bxy + cx + \frac{1}{2}ey^2 + fy = c.$$

Euler. *Ib. Art.* 454.

(6). Let $2axy\,dx + ax^2\,dy - y^3\,dx - 3xy^2\,dy = 0$.

$$\frac{dP}{dy} = 2ax - 3y^2 = \frac{dQ}{dx};$$

$$\therefore ax^2y - y^3x = c,$$

(7). Let $\dfrac{y\,dy + x\,dx - 2y\,dx}{(y-x)^2} = 0$

$$\frac{dP}{dy} = \frac{2y}{(y-x)^3} = \frac{dQ}{dx};$$

$$\therefore \log(y-x) - \frac{y}{y-x} = C.$$

(8). Let $\dfrac{dx}{x} + \dfrac{y^2\,dx}{x^3} - \dfrac{y\,dy}{x^2}$

$$+ \frac{(y\,dx - x\,dy)\sqrt{(x^2+y^2)}}{x^3} = c;$$

$$\frac{dP}{dy} = \frac{2y}{x^3} + \frac{x^2+2y^2}{x^3\sqrt{(x^2+y^2)}} = \frac{dQ}{dx};$$

$$\therefore \log x - \frac{y^2}{2x^2} - \frac{y\sqrt{(x^2+y^2)}}{2x^2} + \frac{1}{2}\log\{\sqrt{(x^2+y^2)}-y\}$$

$$+ \frac{1}{2}\log.\, y = C.$$

Euler. *Nov. Comm. Petrop.* 1760.

(9). Let $\dfrac{y\,dx - x\,dy}{(x+y)\sqrt{\{2y(x-y)\}}} = 0,$

$$\frac{dP}{dy} = \frac{x^2 - xy + 2y^2}{2(x+y)^2(x-y)\sqrt{\{2y(x-y)\}}} = \frac{dQ}{dx}$$

$$\int P\,dx = \int \frac{y\,dx}{(x+y)\sqrt{\{2y(x-y)\}}}$$

$$= \tan^{-1}\sqrt{\left(\frac{x-y}{2y}\right)} = \sin^{-1}\sqrt{\left(\frac{x-y}{x+y}\right)}, \text{ and } dY = 0;$$

$$\therefore \sin^{-1}\sqrt{\left(\frac{x-y}{x+y}\right)} = c.$$

(10). Let $dx\sin y + x\,dy\cos y + dy\sin x + y\,dx\cos x = 0,$

$$\frac{dP}{dy} = \cos x + \cos y = \frac{dQ}{dx}$$

$$\int P\,dx = x\sin y + y\sin x, \text{ and } dY = 0;$$

$$\therefore x\sin y + y\sin x = c.$$

II. Equations which do not satisfy the *criterion of integrability*, but in which the variables admit of separation by different artifices.

(1). Homogeneous equations in which the variables may be separated by making $y = u\,x$, or $x = u\,y$, as may appear most convenient from the form of the equation.

(11). Let $x\,d\,x + y\,d\,y = x\,d\,y - y\,d\,x$. Make $y = u\,x$. We from hence get $\frac{d\,x}{x} = \frac{d\,u\,(1 - u)}{1 + u^2}$: integrating and putting $\frac{y}{x}$ for u, we get the primitive equation, which is

$$\log\left(\frac{\sqrt{\{x^2 + y^2\}}}{c}\right) = \tan^{-1}\frac{y}{x}.$$

Euler. *Calc. Integ. Tom.* I. *Art.* 413.

(12). Let $x\,d\,y - y\,d\,x = d\,x\sqrt{(x^2 + y^2)}$: make $y = u\,x$,

$$\therefore \frac{d\,x}{x} = \frac{d\,u}{\sqrt{(1 + u^2)}},$$

$$\text{and } x^2 = c^2 + 2\,c\,y.$$

Euler. *Ib. Art.* 415.

(13). Let $\frac{y\,d\,x}{d\,y} = x + y$: make $x = u\,y$;

$$\therefore \frac{d\,y}{y} = d\,u, \text{ and } \frac{y}{c} = \epsilon^{\frac{x}{y}}.$$

(14). Let $y^2\,d\,x + (x\,y + x^2)\,d\,y = 0$: make $x = u\,y$,

$$\therefore \frac{d\,y}{y} = -\frac{d\,u}{2\,u + u^2}, \text{ and } y\sqrt{\left(\frac{x}{2\,y + x}\right)} = c.$$

(15). Let $x y\, d y - y^2\, d x = (x + y)^2\, d x\, \epsilon^{-\frac{y}{x}}$: make $y = u x$,

$$\therefore \frac{d x}{x} = \frac{\epsilon^u\, d u}{(1 + u)^2}, \text{ and}$$

$$(x + y) \log \frac{x}{c} = x\, \epsilon^{\frac{y}{x}}. \quad \text{(Ex. 10. p. 73.)}$$

(16). Let $x\, d y - y\, d x = y \log \left(\frac{y}{x}\right) d x$.

$$\text{Make } y = u x, \; \therefore \frac{d x}{x} = \frac{d u}{u \log u},$$

$$\text{and } y = x\, \epsilon^{x \epsilon^c}. \quad \text{(Ex. 9. p. 73.)}$$

(17). Let $\int y\, d x = \frac{y^3}{x}$, or $(x^2 y + y^3)\, d x = 3 x y^2\, d y$.

$$\text{Make } y = u x, \; \therefore \frac{d x}{x} = \frac{3 u\, d u}{1 - 2 u^2}, \text{ and}$$

$$(x^2 - 2 y^2)^3 = c x^2$$

(18). Let $d y \sqrt{(x^2 + y^2)} = y\, d x$.

$$\text{Make } x = y u, \; \therefore \frac{d y}{y} = \frac{d u}{\sqrt{(1 + u^2)} - u}, \text{ and}$$

$$\log \sqrt{\left\{\frac{c y}{x + \sqrt{(x^2 + y^2)}}\right\}} = \frac{x}{2 y^2} \left\{ x + \sqrt{(x^2 + y^2)} \right\}.$$

(19). Let $a y\, d x = \left\{ a x - \sqrt{(x^2 + y^2)} \right\} d y$.

Make $y = u\,x$, $\therefore \dfrac{d\,x}{x} = \dfrac{\{(a - \sqrt{(1+u^2)}\}\,d\,u}{u\sqrt{(1+u^2)}}$,

and $\dfrac{y^{a-1}}{\{x + \sqrt{(x^2+y^2)}\}^a} = c$.

(20). Let $x^2\,d\,y = (x^2 - a\,y^2)\,d\,x$. Make $y = u\,x$,

$$\therefore \frac{d\,x}{x} = \frac{d\,u}{1 - u - a\,u^2}, \text{ and}$$

$$\left(\frac{x}{c}\right)^{\frac{\sqrt{(1+4a)}}{2}} = \frac{2\sqrt{a}\sqrt{(a y^2 + x\,y - x^2)}}{2\,a\,y + x\,\{1 - \sqrt{(1+4\,a)}\}}.$$

Euler. *Ib.* Art. 414.

(21). Let $d\,x\,(a\,x + b\,y) = d\,y\,(a'\,x + b'\,y)$. Make $y = u\,x$,

$$\therefore \frac{d\,x}{x} = \frac{d\,u\,(e+u)}{f + g\,u - u^2}, \text{ where } e = \frac{a'}{b'},$$

$$f = \frac{a}{b'}, \text{ and } g = \frac{b - a'}{b'};$$

$$\log\left(\frac{c}{x}\right) = \log\sqrt{(f + g\,u - u^2)} + \left(\frac{g}{2} + e\right)\int\frac{d\,u}{f + g\,u - u^2}.$$

There are three cases to consider, according as g is positive, nothing, or negative. See Ex. 36. p. 273. Joh. Bern. *Op. Tom.* III. p. 119. 1726. Euler. *Ib.* Art. 412.

(22). Let $d\,x\,(3\,x + 2\,y) = d\,y\,(2\,x + y)$. Make $y = u\,x$,

$$\frac{d\,x}{x} = \frac{d\,u\,(2+u)}{3 - u^2}, \text{ and}$$

$$\frac{x^2}{c} = \frac{(x\sqrt{3} + y)^{\frac{1}{\sqrt{3}} + \frac{1}{2}}}{(x\sqrt{3} - y)^{\frac{1}{\sqrt{3}} - \frac{1}{2}}}.$$

(23). Let $dx\,(a+bx+cy) = dy\,(a'+b'x+c'y)$.

This equation becomes homogeneous, by making

$$a+bx+cy=t, \text{ and } a'+b'x+c'y = u;$$

$$\text{therefore} \quad x = \frac{c't-cu+ac'-a'c}{bc'-cb'},$$

$$y = \frac{bu-b't+ab'-a'b}{bc'-cb'};$$

$$\text{and } dt\,(c't+b'u) = du\,(bu+ct),$$

which coincides with Ex. 21.

If $bc' = cb'$, the expressions for x and y become infinite: in this case, if $\frac{b'}{b} = e$, and $ax+cy = t$, we find

$$dx = \frac{(a'+et)\,dt}{ac + a'b + (c+eb)\,t},$$

$$\text{and } x = \frac{(a'B - eA)}{B^2} \log (A+Bt) + \frac{et}{B} + C,$$

where $A = ac + a'b$, and $B = c + eb$.

If $c + eb = 0$, we readily find

$$x = \frac{(2at+et^2)}{2\,(ac+a'b)} + C,$$

If $a' = 0$, $b' = 0$, $c' = 0$, or if

$$dy = (a+bx+cy)\,dx, \text{ we find}$$

$$C\epsilon^{cx} = b + c\,(a+bx+cy).$$

Euler. *Ib.* Art. 417, 418, 419.

III. The variables in the general linear equation of the first order

$$dy + Py\,dx = Q\,dx.$$

where P and Q are functions of x, are separable, and

$$y = \epsilon^{-\int P\,dx}\{\int \epsilon^{\int P\,dx} Q\,dx + C\}.$$

This equation satisfies the *criterion* of *integrability*, when multiplied by $\epsilon^{\int P\,dx}$.

(24). Let $dy + y\,dx = a x^n\,dx$;

$$\therefore \int P\,dx = x, \text{ and}$$

$$y = C\epsilon^{-x} + x^n - n x^{n-1} + n(n-1)x^{n-2} - \&c.$$

Euler. *Ib.* Art. 424.

(25). Let $dy - x^2\,dy + xy\,dx = a\,dx$;

$$\therefore \int P\,dx = -\log\sqrt{(1-x^2)},\ \epsilon^{-\int P\,dx} = \sqrt{(1-x^2)}, \text{ and}$$

$$y = ax + c\sqrt{(1-x^2)}.$$

Euler. *Ib.* Art. 425.

(26). Let $dy + \dfrac{ny\,dx}{\sqrt{(1+x^2)}} = a\,dx$,

$$\therefore \int P\,dx = n \log\{x + \sqrt{(1+x^2)}\}, \text{ and}$$

$$y = \{x - \sqrt{(1+x^2)}\}^n \{\int a\,dx\,[x + \sqrt{(1+x^2)}]^n + C\}$$

$$= Cu^{-n} + \frac{au^{-1}}{2(n-1)} + \frac{au}{2(n+1)},$$

$$\text{where } u = \{x + \sqrt{(1+x^2)}\}.$$

Euler. *Ib.* Art. 428.

(27). Let $dy - x\,dy - a\,y\,dx - b\,dx + b\,x\,dx = 0$.

$$\int P\,dx = a \log (1 - x), \text{ and}$$

$$y = \frac{c}{(1 - x)^a} - \frac{b}{(a + 1)}(1 - x).$$

(28). Let $dy + \frac{(n - 1 - m\,x)}{x\,(1 - x)}\,y\,dx = \frac{(n - 1)\,d\,x}{x\,(1 - x)}$,

$$\therefore \epsilon^{\int P\,dx} = \frac{x^{n-1}}{(1 - x)^{n-1-m}}, \text{ and}$$

$$y = \frac{(1 - x)^{n-1-m}}{x^{n-1}} \left\{ \int \frac{(n-1)\,x^{n-2}\,d\,x}{(1-x)^{n-m}} + C \right\}.$$

This is the expression for the sum of the series

$$1 + \frac{m}{n}\,x + \frac{m\,(m+1)}{(n+1)}\,x^2 + \frac{m\,(m+1)\,(m+2)}{n\,(n+1)\,(n+2)}\,x^3 + \text{\&c. } \textit{in inf.}$$

Lagrange. *Théorie des Fonctions Analytiques*, p. 102.

(29). Let $dy + \frac{(n-1)\,y\,d\,x}{x} = \frac{(n-1)\,d\,x}{x}$

$$+ \frac{m\,d\,x}{1 - x} + \frac{x\,d\,x}{(1 - x)^2},$$

$$\therefore \epsilon^{\int P\,dx} = x^{n-1}, \text{ and}$$

$$y = x^{1-n} \left\{ C + x^{n-1} + m \int \frac{x^{n-1}\,d\,x}{1 - x} + \int \frac{x^n\,d\,x}{(1 - x)^2} \right\}.$$

This is the expression for the sum of the series

$$1 + \frac{m}{n}\,x + \frac{(m+1)}{(n+1)}\,x^2 + \frac{(m+2)}{(n+2)}\,x^3 + \text{\&c. } \textit{in infinitum.}$$

Lagrange. *Ib.* p. 103.

(30). **Problem.** To find the nature of the curve EM Fig. 117., so that if AQ be a line passing through the origin A, and making an angle of 45° with the axis AP, then the subtangent PT, is to the ordinate PM as a given line a, to the difference of PM and PQ.

If $AP = x$, and $PM = y$, the differential equation of the curve is

$$dy - \frac{y\,dx}{a} + \frac{x\,dx}{a} = 0\text{; which gives}$$

$$y = x + a + c\,\epsilon^{\frac{x}{a}}.$$

But $x = 0$, when $y = 0$, and therefore $c = -a$; the equation of the curve is,

$$y = x + a - a\,\epsilon^{\frac{x}{a}}.$$

This problem was proposed by a Mr. De Beaune to Descartes, and resolved by John Bernoulli*, who gave in common with Leibnitz and De l'Hôpital, methods of constructing the curve by points. It was afterwards proposed under a more general form by James Bernoulli†, who, in place of the line AQ, substituted any given curve: the differential equation then becomes

$$a\,dy - y\,dx = q\,dx,$$

where q is the function of x, which is equal to the ordinate PQ of the curve which is the locus of Q.

(31). Let $dy + Py\,dx = Qy^{n+1}\,dx$, where P and Q are functions of x.

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* Joh. Bern. *Opera*. Tom. I. p. 62, 1693.

† *Problema Beaunianum generalius conceptum. Opera*. Tom. II. p. 732. 1695.

Make $u = \frac{1}{y^n}$, and the equation becomes

$$du - n P u\, dx = - n Q\, dx.$$

Consequently

$$u = \frac{1}{y^n} = - \epsilon^{n \int P dx} \{ \int \epsilon^{-n \int P dx} n Q\, dx + C \}.$$

This is the form under which the problem in p. 329. was proposed by James Bernoulli[*]: it was solved nearly at the same time by Leibnitz[†] and John Bernoulli[‡].

(32). Let $dy + y\, dx = x y^3\, dx$;

$$\therefore \int P\, dx = x, \; n = 2, \text{ and}$$

$$u = \frac{1}{y^2} = x + \frac{1}{2} + c\, \epsilon^{2x}.$$

(33). Let $dy + \frac{x y\, dx}{(1 - x^2)} = x \sqrt{y}\, dx$,

$$\therefore \int P\, dx = - \log \sqrt{(1 - x^2)}, \; n = - \frac{1}{2}, \text{ and}$$

$$u = \sqrt{y} = c \sqrt[4]{(1 - x^2)} - \frac{(1 - x^2)}{3}.$$

IV. The equation

$$dy + y^2\, dx = a x^m\, dx$$

admits of the separation of its variables, when $m = - \frac{4 i}{2 i \pm 1}$, where i is any number from 0 to infinity.

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[*] Jac. Bern. *Opera*. Tom. I. p. 663. *Dec.* 1695.

[†] *Acta Leips. March.* 1696.

[‡] Joh. Bern. *Opera.* Tom. I. p. 175. *March* 1696.

(34). Let $dy + y^2 dx = a^2 dx$;

$$\therefore dx = \frac{dy}{a^2 - y^2}, \text{ and}$$

$$\epsilon^{-2ax}\left(\frac{y+a}{y-a}\right) = c.$$

(35). Let $dy + y^2 dx = \dfrac{a^2 dx}{x^4}$.

Make $y = \dfrac{1}{x} - \dfrac{z}{x^2}$, and $x = \dfrac{1}{t}$,

when it coincides with the last Ex. and

$$\epsilon^{\frac{2a}{x}}\left\{\frac{x^2 y - x + a}{x^2 y - x - a}\right\} = c.$$

(36). Let $dy + y^2 dx + \dfrac{a^2 dx}{x^4} = 0$.

The same process as in the last Example, gives

$$\frac{1}{a}\tan^{-1}\frac{(xy - 1)x}{a} - \frac{1}{x} = c.$$

(37). Let $dy + y^2 dx = \dfrac{a^2 dx}{x^{\frac{4}{3}}}$.

Make $x = t^{-3}$, and $y = -\dfrac{3a^2}{z}$, which gives

$$dz + z^2 dt = 9a^2 t^{-4} dt,$$

which coincides with Ex. 35; consequently

$$\epsilon^{-6ax^{\frac{1}{3}}}\left\{\frac{y(1 + 3ax^{\frac{1}{3}}) + 3a^2 x^{-\frac{1}{3}}}{y(1 - 3ax^{\frac{1}{3}}) + 3a^2 x^{-\frac{1}{3}}}\right\} = c.$$

(38). Let $dy+y^2dx=\frac{a^2dx}{x^{\frac{8}{3}}}$.

Make $x=\frac{1}{t}$, and $y=\frac{1}{x}-\frac{z}{x^2}=t-t^2z$,

and the equation becomes

$$dz+z^2dt=a^2t^{-\frac{4}{3}}dt, \text{ (Ex. 37.)}$$

and $$\epsilon^{6ax^{-\frac{1}{3}}}\left\{\frac{(x-3ax^{\frac{2}{3}})y-1+3ax^{-\frac{1}{3}}-3a^2x^{-\frac{2}{3}}}{(x+3ax^{\frac{2}{3}})y-1-3ax^{-\frac{1}{3}}-3a^2x^{-\frac{2}{3}}}\right\}=c.$$

(39). Let $dy-y^2dx=\frac{2dx}{x^{\frac{8}{3}}}$.

In the same manner, as in Ex. 38, we get

$$\frac{1}{a^{\frac{1}{3}}}=\frac{1}{3\sqrt{2}}\tan^{-1}\left\{\frac{x^{\frac{2}{3}}+yx^{\frac{5}{3}}-6}{3\sqrt{2}\,x^{\frac{1}{3}}(1+xy)}\right\}+c.$$

(40). Let $dy+y^2dx=\frac{a^2dx}{x^{\frac{8}{5}}}$.

Make $x=t^{-\frac{5}{3}}$, and $y=-\frac{5a^2}{3z}$, which gives

$$dz+z^2dt=\frac{25a^2}{9}t^{-\frac{8}{3}}dt,$$

which coincides with Ex. 38., and

$$\epsilon^{-10ax^{\frac{1}{5}}}\left\{\frac{(3+15ax^{\frac{1}{5}}+25a^2x^{\frac{2}{5}})y+5a^2x^{-\frac{2}{5}}+25a^2x^{-\frac{2}{5}}}{(3-15ax^{\frac{1}{5}}+25a^2x^{\frac{2}{5}})y+5a^2x^{-\frac{2}{5}}+25a^3x^{-\frac{2}{5}}}\right\}$$

$$=c.$$

(41). Let $dy + A y^2 t^{\mu} dt = B t^{\lambda} dt$.

Make $A t^{\mu} dt = dx$, $\therefore A t^{\mu+1} = (\mu + 1) x$,

and the equation becomes

$$dy + y^2 dx = \frac{B}{A}\left(\frac{\mu+1}{A} x\right)^{\frac{\lambda-\mu}{\mu+1}} dx = a x^m dx.$$

This is the form under which the equation in page 332. was proposed by Count James Riccati in the Leipsic Acts for 1732.

V. Equations whose variables are separable by particular substitutions, not included under any of the preceding cases.

(42). Let $a^2 dx = (x+y)^2 dx$.

$$\text{Make } x+y=u, \therefore dy = \frac{a^2 du}{a^2+u^2},$$

$$\text{and } \tan^{-1}\frac{(y+x)}{a} = \frac{y+c}{a}.$$

(43). Let $dy - \dfrac{x^2 dy}{\sqrt{(a+y)}} = x^3 dx$.

Make $\sqrt{(a+y)} = z$, and $x^2 = u$:

we thus get the homogeneous equation

$$u\,du + 4u\,dz = 4z\,dz,$$

$$\therefore \frac{a+y}{c} = \left\{\left(\frac{x^2}{\sqrt{(a+y)}} + 2\right)^2 - 8\right\} \times$$

$$\left\{\frac{x^2 + 2(1+\sqrt{2})\sqrt{(a+y)}}{x^2 + 2(1-\sqrt{2})\sqrt{(a+y)}}\right\}^{\frac{1}{\sqrt{2}}}.$$

(44). Let $x\,dx + dy = m y^{\frac{1}{2}} dx$.

Make $y = z^2$: which gives us the homogeneous equation

$$x\,dx + 2\,z\,dz = m\,z\,dx,$$

$$\text{and } \log \frac{c}{\sqrt{\{x^2 - m\,x\sqrt{y} + 2\,y\}}}$$

$$= \frac{m}{\sqrt{(8 - m^2)}} \tan^{-1} \frac{4\sqrt{y} - m\,x}{x\sqrt{(8 - m^2)}}.$$

(45). Let $a\,y\,dx + b\,x\,dy + x^m y^n (a' y\,dx + b'\,x\,dy) = 0$.

Make $x^a y^b = t$, and $x^{a'} y^{b'} = u$: we hence get

$$\frac{t^{\frac{a'n - b'm}{a'b - b'a}}}{a'n - b'm} + \frac{u^{\frac{an - bm}{ab' - a'b}}}{a\,n - b\,m} = c.$$

If $a'\,n - b'\,m = 0$, or $a\,n - b\,m = 0$, we must put $\log t$ or $\log u$ in the place of either of the preceding expressions. Euler. *Calc. Integ.* Tom. I. Art. 431.

(46). Let $\int y\,dx = 2\,xy + \frac{3\,x^3}{y}$, or

$$y\,dx + 2\,x\,dy + \frac{9\,x^2\,dx}{y} - \frac{3\,x^3\,dy}{y^2} = 0.$$

Make $xy^2 = t$, and $\frac{x^9}{y^3} = u$,

$$\therefore t^{\frac{4}{7}} + 2\,u^{\frac{2}{7}} = c, \text{ or}$$

$$x^{\frac{4}{7}}\,y^{\frac{14}{7}} + 2\,x^{\frac{18}{7}} = c\,y^{\frac{6}{7}}.$$

(47). Let $\epsilon^x\,dx + \frac{y}{e^x}\,dy = dy - y\,dx$.

Make $y = u\,\epsilon^x$; we hence get

$$\tan^{-1} y\,\epsilon^{-x} = c + \log\sqrt{\{\epsilon^{2x} + y^2\}}.$$

(48). Let $\{(x-ay)\,dy + y\,dx\}(1+x^2y^2) = b\,y\,dx + b\,x\,dy$.

Make $xy = u$: consequently

$$du - a\,y\,dy = \frac{b\,du}{1+u^2},$$

or $xy - b\tan^{-1}xy - \frac{ay^2}{2} = c.$

(49). Let $y\,dy + dy\,(a+bx+cx^2) = y\,dx\,(e+cx)$.

Make $u = \frac{dy}{dx} = \frac{y\,(e+cx)}{y+a+bx+cx^2}$;

and therefore $y = \frac{u\,(a+bx+cx^2)}{e+cx-u}$.

Differentiating the logarithm of this expression for y, collecting together those terms in the result which involve dx and du and multiplying them by $e+cx-u$, we find

$$\frac{dx}{(a+bx+cx^2)(e+cx)} = \frac{du}{u\,\{ac+e^2-eb+(b-2e)\,u+u^2\}}$$

where the variables are separated. Euler. *Ib.* Art. 433.

(50). Let $(y-x)\,dy = \frac{n\,dx\,(1+y^2)^{\frac{3}{2}}}{\sqrt{(1+x^2)}}$;

Make $y = \frac{x-u}{1+xu}$, which gives

$$\frac{dx}{1+x^2} = \frac{u\,du}{(1+u^2)\,\{n\sqrt{(1+u^2)}+u\}},$$

and putting $1+u^2=t^2$, and $t = \frac{1+s^2}{2s}$, we get

$$\tan^{-1} x = 2 \tan^{-1} s - \frac{2n}{\sqrt{(n^2-1)}} \tan^{-1} s \sqrt{\left(\frac{n-1}{n+1}\right)} + c.$$

Euler. *Ib.* Art. 434.

(51). Let $y\,dx - x\,dy + a\,x^n\,y\,dy\,(x^n+b)^{\frac{1}{n}} = 0.$

$$\text{Make } \frac{x}{(x^n+b)^{\frac{1}{n}}} = u\,y,$$

when the equation becomes

$$\frac{du}{u^n\,(a\,b+u)} + y^{n-1}\,dy = 0,$$

where the variables are separated. Euler. *Ib.* Art. 490.

V. Every differential equation admits of factors or multipliers, which render it a complete differential and satisfy the *criterion* of *integrability:* no general method, however is known, by which these multipliers may be found for any assigned equation.

If $M\,dx + N\,dy = 0$, be homogeneous, a multiplier is $\frac{1}{M\,x + N\,y}$, as the reader may readily ascertain in any of the examples of homogeneous equations which are given above.

If $\frac{1}{N}\left(\frac{d\,M}{dy} - \frac{d\,N}{d\,x}\right) = f\,(x) = X$, a multiplier is $e^{\int X\,dx}$: this applies to all linear equations of the first order, see p. 333.

Whenever the variables can be separated, a multiplier may be found: thus if z and t be the new variables introduced, if we get

$$Z\,dz + T\,dt = U\,(M\,dx + N\,dy), \text{ where}$$

$Z = f(z)$ and $T = f(t)$; then U is the multiplier required.

Thus in Ex. 35, by making $x = \frac{1}{t}$, and $y = t - t^2 z$, we find

$$-t^2 (dz + z^2 dt - a^2 dt)$$

$$= dy + y^2 dx - \frac{a^2 dx}{x^4},$$

and dividing by $t^2 (z^2 - a^2)$, we get

$$\frac{dz}{z^2 - a^2} + dt = -\frac{1}{t^2 (z^2 - a^2)} \left\{ dy + y^2 dx - \frac{a^2 dx}{x^4} \right\}$$

$$= -\frac{x^2}{x^2 (1 - xy)^2 - a^2} \left\{ dy + y^2 dx - \frac{a^2 dx}{x^4} \right\};$$

where $-\frac{x^2}{x^2 (1 - xy)^2 - a^2}$ is the multiplier required to render the second member of this equation a complete differential.

Now $\int N dy = \frac{1}{2a} \cdot \log \left\{ \frac{a + x(1 - xy)}{a - x(1 - xy)} \right\} + X$:

$dX = -\frac{dx}{x^2}$ and $X = -\frac{1}{x} + C$; the complete integral therefore is

$$\log \left\{ \frac{a + x(1 - xy)}{a - x(1 - xy)} \right\} = \frac{2a}{x} + C^*.$$

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* Euler. *Calc. Integ.* Tom. I. Art. 491.

The multiplier in Ex. 37, in a similar manner is found to be

$$\frac{1}{(y + 3\,a^2 x^{-\frac{1}{3}})^2 - 9\,a^2 x^{\frac{2}{3}} y^2};$$

and the same method may be applied to its determination in any of the Examples which are given above.

Euler has reversed the problem and in several cases investigated the nature of the functions *, which must enter into a differential equation of given form, in order that it may become *integrable per se*, when multiplied by a given factor: thus suppose it was required to determine the functions P and Q of x in the equation

$$P\,y\,d\,x + (y + Q)\,d\,y = 0$$

in order that it may satisfy the *criterion* of *integrability* when multiplied by

$$\frac{1}{y^3 + M\,y^2 + N\,y},$$

where M and N are also functions of x: the condition will be answered, when M is any assigned function

$$\text{and } P = b\,(2\,a - M)\frac{d\,M}{d\,x}$$

$$Q = M - a + b\,(2\,a - M)^2$$

$$N = a\,(M - a) + a\,b\,(2\,a - M)^2,$$

where a and b are arbitrary constants.

These investigations, however, frequently involve differential equations, which cannot be integrated by any known method, and it cannot be said that the cases in which they are successful, are of very great importance or extent.

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* Euler. *Calc. Integ.* Art. 493. to 539.

In order to ensure this method all the success of which it is capable, it would require a very complete classification of the forms of differential equations of the first order, as well as a knowledge of the forms of the multipliers which are suited to each class. The immense extent, however, of this enquiry and the great difficulties which are met with even in the simplest cases, preclude all hopes of its proving of much service in the general integration of differential equations of the first order.

(52). Let $\frac{a\,d\,x}{x} + \frac{b\,d\,y}{y} = \frac{c\,x^m\,d\,x}{y^b}$.

The multiplier is $x^a\,y^b$, and

$$\therefore\ x^a\,b^b = \frac{c\,x^{m+a+1}}{m+a+1} + C.$$

(53). Let $x\,d\,y - 2\,y\,d\,x + a\,d\,x = 0$.

The multiplier is $\frac{1}{x^3}$, and

$$\therefore\ \frac{y}{x^2} - \frac{a}{2\,x^2} + c = 0,$$

$$\text{or } y + c\,x^2 - \frac{a}{2} = 0.$$

(54.) Let $a\,y^n d\,y = \frac{2\,x\,d\,y - y\,d\,x}{x^2}$.

The factor is y, and

$$\frac{a\,y^{n+2}}{n+2} = \frac{y^2}{x} + C.$$

(55). Let $a\,y^r\,d\,y = n\,x\,d\,y - y\,d\,x$.

The multiplier is $-\frac{y^{n-1}}{y^{2n}} = -\frac{1}{y^{n+1}}$, and

$$\therefore c - \frac{a\,y^{r-n}}{r-n} = \frac{x}{y^n}.$$

(56). Let $a\,x\,d\,y + 2\,a\,y\,d\,x = x\,y\,d\,y$.

The factor is $\frac{1}{x\,y}$, and

$$\therefore x^2 y = c\,\epsilon^{\frac{y}{a}}.$$

(57). Let $y\,x\,d\,x = (x^2\,y\,d\,y + y^3\,d\,y)\sqrt{y} - y^2\,d\,y$.

The factor is $\frac{1}{y\,(x^2 + y^2)}$, and

$$\therefore \log\sqrt{(x^2 + y^2)} = \frac{2}{3}\cdot y^{\frac{3}{2}} + C.$$

(58). Let $d\,x + (a\,d\,x + 2\,b\,y\,d\,y)\sqrt{(1 + x^2)} = 0$.

The factor is $\frac{1}{\sqrt{(1 + x^2)}}$, and

$$\therefore \log\{x + \sqrt{(1 + x^2)}\} + a\,x + b\,y^2 = C.$$

(59). Let $(2\,x - y)\,d\,y + (2\,a - y)\,d\,x = 0$.

The factor is $\frac{1}{(2\,a - y)^3}$, and

$$\therefore x = y - a + c\,(2\,a - y)^2.$$

Euler. *Calc. Integ.* Art. 493.

(60). Let $\frac{n\,x\,y\,d\,x}{\sqrt{(1 + x^2)}} - x\,d\,x + (n + 1)\,y\,d\,y = 0$.

The factor is $\{y + \sqrt{(1 + x^2)}\}^n$; and if $n = 1$, we have

$$\frac{2\,y^3}{3} + y^2\sqrt{(1 + x^2)} - \frac{(1 + x^2)^{\frac{3}{2}}}{3} = C.$$

(61). Let $a c x y d x + y d y (a + b x + c x^2)$

$$- a d y (a - c x^2) = 0.$$

The multiplier is $\dfrac{y}{(a + b x + c x^2) y^2 - a (2 a + b x) y + a^3}$

and $a \log \surd \{ a (y - a)^2 + x^2 (y - a) x + c y^2 b y \}$

$$- \frac{a c}{2 \surd (b^2 - 4 a c)} \log \left\{ \frac{\left(\frac{2 c y}{y - a}\right) x + b - \surd (b^2 - 4 a c)}{\left(\frac{2 c y}{y - a}\right) x + b + \surd (b^2 - 4 a c)} \right\} = C.$$

Euler. *Calc. Integ.* Art. 499.

(62). Let $x d x + a y d x + 2 x y d y - 2 a^2 y d y = 0.$

The factor is $\dfrac{1}{(x - a^2) \surd (y^2 + 2 a y + x)}$, and

$$\therefore \surd (y^2 + 2 a y + x) + a \log \frac{a + y - \surd (y^2 + 2 a y + x)}{\surd (a^2 - x)} = c.$$

Euler. *Ib.* Art. 520.

(63). Let $y d y - x d x - \dfrac{b d y}{x^5} + \dfrac{b y d x}{x^6} = 0.$

The factor is $(y^2 - x^2)^2$, and the integral

$$\frac{(y^2 - x^2)^3}{6} - \frac{b y^5}{5 x^5} + \frac{2 b y^3}{3 x^3} - \frac{b y}{x} = c.$$

Euler. *Ib.* Art. 524.

(64). Let $d y + y^2 d x + \dfrac{a^2 d x}{x^{4 n}} - \dfrac{n (n - 1) d x}{x^2}$ . .

The factor is

$$\frac{1}{(x^n y - n x^{n-1})^2 + \dfrac{a^2}{x^{2n}}},$$

and $\therefore \frac{1}{a} \tan^{-1} \left(\frac{x^{2n} y - n x^{2n-1}}{a}\right) - \frac{1}{(2n-1)x^{2n-1}} = c.$

Euler. *Ib.* Art. 533.

(65). Let $dy + y^2 dx + \dfrac{a^2 dx}{(1-x^2)^2} - \dfrac{(2x^2-1)}{(1-x^2)^2} = 0.$

The factor is $\dfrac{1-x^2}{\{y(1-x^2)+x\}^2+a^2}$,

and $\therefore \frac{1}{a} \tan^{-1} \left(\frac{y(1-x^2)+x}{a}\right) + \log \sqrt{\left(\frac{1+x}{1-x}\right)} = c.$

Euler. *Ib.* Art. 533.

(66). Let $a x\, dy - a y\, dx = (x\, dx + y\, dy) \sqrt{(x^2+y^2-a^2)}$.

If the first side of this equation be multiplied by $\frac{1}{x^2}$, its integral is $\frac{ay}{x}$: the second side of the equation is integrable *per se*: the factors of the first side of the equation are included under the formula $\frac{1}{x^2} f\left(\frac{y}{x}\right)$, and those of the second under the formula $\phi\,(x^2+y^2-a^2)$: also since

$$\frac{1}{x^2+y^2} = \{a^2 + (x^2+y^2-a^2)\}^{-1} = \phi\{x^2+y^2-a^2)\}$$

$$= \frac{1}{x^2} \times \frac{1}{1+\dfrac{y^2}{x^2}} = \frac{1}{x^2} f\left(\frac{y}{x}\right),$$

it is a factor common to both sides of the equation: consequently

$$a \tan^{-1} \frac{y}{x} = \frac{\sqrt{(x^2+y^2-a^2)}}{2} - \frac{a}{2} \sec^{-1} \frac{\sqrt{(x^2+y^2)}}{2} + C.$$

By a similar process, we shall be enabled to find a factor common to both sides of the equation

$$a y\, d x + b x\, d y = x^m y^n (a' y\, d x + b' x\, d y);$$

which is $x^{\left(\frac{a'n - b'm}{ab' - a'b}\right) a - 1} \times y^{\left(\frac{a'n - b'm}{ab' - a'b}\right) b - 1}$.

See. Ex. 45. p. 336. Paoli. *Elementi d'Algebra*. Tom. II. p. 175.

V. The following equations, in which the variables are separated, admit of algebraical integrals, although the integral of each part considered separately, is transcendental.

(67). Let $\dfrac{dx}{\sqrt{(a+bx+cx^2+ex^3+fx^4)}}$

$$+ \frac{dy}{\sqrt{(a+by+cy^2+ey^3+fy^4)}} = 0.$$

Consider x and y as functions of t, and make

$$\frac{dx}{dt} = \sqrt{(a+bx+cx^2+ex^3+fx^4)} = \sqrt{X},$$

$$\frac{dy}{dt} = \sqrt{(a+by+cy^2+ey^3+fy^4)} = \sqrt{Y}:$$

Make also $x+y=p$, and $x-y=q$, $\therefore$ since

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = X + Y, \text{ and } \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = \frac{dX}{2dx} + \frac{dY}{2dy},$$

we have

$$\frac{d^2p}{dt^3} = b + cp + \tfrac{3}{4} e (p^2+q^2) + \tfrac{1}{2} fp (p^2+3q^2) \quad (\alpha).$$

$$\frac{dp\,dq}{dt^2} = X - Y = bq + cpq + \tfrac{1}{4}eq(3p^2+q^2)$$
$$+ \tfrac{1}{2}fpq(p^2+q^2). \quad (\beta).$$

Multiplying equation (α) by q, subtracting equation (β) from the result, multiplying the remainder by $\frac{2dp}{q^3}$, and integrating, we shall get

$$\frac{dp^2}{q^2dt^2} = C + ep + fp^2$$

$$\frac{dp}{dt} = \sqrt{X} + \sqrt{Y} = q\sqrt{(C+ep+fp^2)}$$

$$= (x-y)\sqrt{\{C+e(x+y)+f(x+y)^2\}}.$$

The integral of the equation

$$\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0$$

is $\sqrt{X} - \sqrt{Y} = (x-y)\sqrt{\{C+e(x+y)+f(x+y)^2\}}$

$$\text{or } \sqrt{X} - \sqrt{Y} = \frac{X-Y}{\sqrt{X}+\sqrt{Y}}$$

$$= \frac{b+c(x+y)+e(x^2+xy+y^2)+f(x^3+x^2y+xy^2+y^3)}{\sqrt{\{C+e(x+y)+f(x+y)^2\}}}.$$

This is the process given by Lagrange in the *Memoires de Turin* Tom. IV. p. 108; the integral had been previously determined by Euler under an equivalent, though much less simple form*.

The transcendent $\frac{dx}{\sqrt{X}}$ is always reducible to the form

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* *Novi Comment. Petrop.* Tom. VI. et VII: *Calc. Integ.* Tom. I. Cap. 5. 6.

$\frac{A\,d\theta}{\sqrt{(1-c^2\sin^2\theta)}}$, where A and c are constant quantities and c always less than 1*: the equation becomes by this transformation

$$\frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}+\frac{d\phi}{\sqrt{(1-c^2\sin^2\phi)}}=0:$$

and its integral found by the same method is

$$\sqrt{(1-c^2\sin^2\theta)}+\sqrt{(1-c^2\sin^2\phi)}=C\sin(\theta\pm\phi).$$

In the same manner, we may find the integral of

$$\frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}-\frac{d\phi}{\sqrt{(1-c^2\sin^2\phi)}}=0,$$

which is

$$\sqrt{(1-c^2\sin^2\theta)}-\sqrt{(1-c^2\sin^2\phi)}=C\sin(\theta\pm\phi).$$

(68). Let $\frac{dx}{\sqrt{(a+bx+cx^2)}}+\frac{dy}{\sqrt{(a+by+cy^2)}}=0.$

Make $\frac{dx}{dt}=\sqrt{X}$, $\frac{dy}{dt}=\sqrt{Y}$, $x+y=p$,

and we find as before

$$\sqrt{X}+\sqrt{Y}=\sqrt{\{C+2b(x+y)+c(x+y)^2\}}.$$

This result is obviously not deducible from the integral of Ex. 67, by making $e=0$ and $f=0$.

The integral of $\frac{dy}{\sqrt{X}}-\frac{dy}{\sqrt{Y}}=0$, is

$$\sqrt{X}-\sqrt{Y}=\sqrt{\{C+2b(x+y)+c(x+y)^2\}}.$$

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[*] Legendre *Exercices du Calc. Integ.* Tom. I. p. 10.

(69). Let $\frac{dx}{\surd(1-x^2)} = \frac{dy}{\surd(1-y^2)}$.

The integral is

$$x\surd(1-y^2) = y\surd(1-x^2) + C,$$

or deducing it from Ex. 68,

$$\surd(1-x^2) - \surd(1-y^2) = \surd\{C + (x+y)^2\}.$$

Lagrange. *Memoires de Turin.* Tom. IV.

(70). Let $\frac{dx}{\alpha+\beta x} + \frac{dy}{\alpha+\beta y} = 0$.

This is a case of Ex. 68. where $\surd(a+bx+cx^2) = \alpha+\beta x$, and therefore $a=\alpha^2$, $b=2\alpha\beta$, and $c=\beta^2$: consequently

$$C = \alpha(x+y) + \beta xy.$$

Euler. *Acta. Petrop.* Tom. II.

(71). Let $\frac{dx}{a+2bx+cx^2} + \frac{dy}{a+2by+cy^2} = 0$.

The integral of this equation is not deducible from that of Ex. 67, by supposing $\surd X = a+2bx+cx^2$; for in that case, if we exterminate the radical in the final equation, it reduces itself to $C = \frac{a^2}{4}$, from which no inference can be drawn.

Make therefore $\frac{dx}{dt} = a + 2bx + cx^2 = X$,

$\frac{dy}{dt} = -(a+2by+cy^2) = -Y$, $u = xy$, and $q = x - y$;

$$\therefore \frac{du}{dt} = \frac{y\,dx}{dt} + \frac{x\,dy}{dt} = yX - xY = q(cu-a).$$

Also $\frac{d^2 x}{d t d x} + \frac{d^2 y}{d t d y} = 2 c q = \frac{2 c d u}{(c u - a) d t}$,

$$\text{or } \frac{d^2 x}{d x} + \frac{d^2 y}{d y} = \frac{2 c d u}{c u - a},$$

$$\therefore \log d x + \log d y = 2 \log (c u - a) + \log C,$$

$$\text{or } \frac{d x d y}{(c u - a)^2} = C d t^2, \text{ since } d t \text{ is constant;}$$

$$\text{or } -\frac{X Y}{(c x y - a)^2} = C = \frac{(a + 2 b x + c x^2)(a + 2 b y + c y^2)}{(c x y - a)^2},$$

or subtracting unity from both sides of equation

$$\frac{2 a b (x + y) + a c (x + y)^2 + 4 b^2 x y + 2 b c x y (x + y)}{(c x y - a)^2} = C.$$

Euler. *Ib.*

(72). Let $\frac{d x}{1 + x^2} + \frac{d y}{1 + y^2} = 0.$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - x y} = C,$$

$$\text{or } \frac{x + y}{1 - x y} = C,$$

the same result as is given by the second of the formulæ in Ex. 71. Euler. *Ib.*

(73). Let $\frac{d x}{1 + x + x^2} + \frac{d y}{1 + y + y^2} = 0,$

$$\therefore \frac{2}{\sqrt{3}} \tan^{-1} \frac{x \sqrt{3}}{2 + x} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{y \sqrt{3}}{2 + y}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left\{ \frac{2\sqrt{3}\,(x+y+xy)}{4+2(x+y)-2xy} \right\} = C,$$

$$\text{or } \frac{x+y+xy}{2+x+y-xy} = c.$$

The second formula in Ex. 71. gives

$$\frac{(x+y)+(x+y)^2+xy+xy\,(x+y)}{(xy-1)^2} = c.$$

$$\text{or } \frac{(1+x+y)\,(x+y+xy)}{(xy-1)^2} = c,$$

one of whose factors $\frac{x+y+xy}{xy-1}$, coincides with the solution deduced by the ordinary method, as may be readily seen, by subtracting unity from its reciprocal. Euler. *Ib.*

(74). Let $\frac{dx}{a+2bx+cx^2} = \frac{dy}{a+2by+cy^2}$.

Let $\frac{dx}{dt} = X$, $\frac{dy}{dt} = Y$, $q = x-y$;

$$\therefore \frac{d^2x}{dt\,dx} + \frac{d^2y}{dt\,dy} = \frac{2\,dq}{q\,dt},$$

$$\text{or } \log dx + \log dy = 2 \log q + C,$$

$$\text{or } \frac{dx\,dy}{q^2} = C\,dt^2;$$

$$\therefore \frac{XY}{(x-y)^2} = C' = \frac{(a+2bx+cx^2)\,(a+2by+cy)^2}{(x-y)^2},$$

or adding $b^2 - ac$, and extracting the square root of the sum, we find

$$\frac{a+b\,(x+y)+cxy}{(x-y)} = C.$$

Euler. *Ib.*

(75). Let $$\frac{dx}{\sqrt{(b+cx^2+ex^4+fx^6)}} \pm \frac{dy}{\sqrt{(b+cy^2+cy^4+fy^6)}} = 0.$$

This example coincides with Ex. 67, if we make $a=0$, and put x^2 and y^2 in the place of x and y. Euler. *Ib.*

(76). Let $$\frac{dx}{\sqrt{(c+ex^4+fx^8)}} \pm \frac{dy}{\sqrt{(c+ey^4+fy^8)}} = 0.$$

This example coincides with Ex. 67, if we make $a=0$, and $b=0$, and put x^4 and y^4 in the place of x and y. Euler. *Ib.*

On the Integration of Differential Equations of the First Order, but not of the First Degree.

I. By resolving the equation with respect to $\frac{dy}{dx}$ or $\frac{dx}{dy}$, and thus reducing it to an equation of the first order and first degree, which must be treated by the methods in the preceding section.

(1). Let $dy^2 - a^2 dx^2 = 0$:

$$\therefore \frac{dy}{dx} = \pm a, \text{ and } y = ax + c, \text{ or } y = -ax + c';$$

both of which values of y satisfy the equation.

The equation is also satisfied by the product of these integrals, or

$$(y - ax - c)(y + ax - c') = 0,$$

which is equivalent to the equation

$$(y-c)^2 = a^2 x^2.$$

(2). Let $dy^2 - a x dx^2 = 0$,

$$\therefore \frac{dy}{dx} = \pm \sqrt{a x},\ y = \frac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} + c, \text{ or } y = -\frac{2}{3} a^{\frac{1}{2}} x + c',$$

or combining these integrals together, we get

$$(y-c)^2 = \frac{4}{9} a x^3.$$

The two first integrals represent two separate branches of a cubical parabola, which are both represented by the third.

(3). Let $dy^3 + dx^3 = 0$;

$$\therefore y = -x + c, \text{ or } y = \frac{(1+\sqrt{-3})}{2} x + c', \text{ or } y = \frac{(1-\sqrt{-3})}{2} x + c'';$$

or combining them, we get

$$y^3 + x^3 - (c + c' + c'') y^2 + \left\{ c - \frac{(1-\sqrt{-3})}{2} c' - \frac{(1+\sqrt{-3})}{2} c'' \right\} \times$$

$$(xy - x^2) + (cc' + cc'' + c'c'') y$$

$$+ \left\{ c'c'' - \frac{(1-\sqrt{-3})}{2} cc' - \frac{(1+\sqrt{-3})}{2} cc'' \right\} x - cc'c'' = 0,$$

$$\text{or } y^3 + x^3 - a y^2 - b x y - c x^2 + A y + B x + C = 0,$$

where the constants A, B, C must be so related to each other that the equation may admit of resolution into simple factors. Euler. *Calc. Integ.* Tom. I. Art. 680.

(4). Let $\frac{dy^3}{dx^3} - (xy + x^2 + y^2) \frac{dy^2}{dx^2}$

$$+ (x^3 y + y^3 x + x^2 y^2) \frac{dy}{dx} - x^3 y^3 = 0.$$

Resolving the equation and integrating its factors, we find

$$y = \epsilon^{\frac{x^2}{2}} + c,\ y = \frac{x^3}{3} + c',\ y = -\frac{1}{x} + c'', \text{ or}$$

$$(y - \epsilon^{\frac{x^2}{2}} - c)(y - \frac{x^3}{3} - c')(y + \frac{1}{x} - c'') = 0.$$

(5). Let $y\,dy^2 + 2x\,dy.dx = y\,dx^2$,

$$\therefore \frac{dy}{dx} = \frac{-x \pm \sqrt{(x^2 + y^2)}}{y}, \text{ or}$$

$$\frac{y\,dy + x\,dx}{\sqrt{(x^2 + y^2)}} = \pm\,dx : \text{ consequently}$$

$$\sqrt{(x^2 + y^2)} = x + c, \text{ or } \sqrt{(x^2 + y^2)} = -x + c';$$

which combined, produce

$$y^2 = 2cx + c^2.$$

(6). Let $\sqrt{(dx^2 + dy^2)} = a\,dx + b\,dy$.

Make $dy = e\,dx$, $\therefore y = ex + c$: consequently

$$\sqrt{\left\{1 + \left(\frac{y-c}{x}\right)^2\right\}} = a + \frac{b(y-c)}{x}.$$

(7). Let $\sqrt{\left\{dy^2 + \frac{a^4\,dx^2}{(a^2 + x^2)^{\frac{3}{2}}}\right\}} = \frac{a^2\,dx}{(a^2 + x^2)^{\frac{3}{2}}} + b\,dy.$

Make $du = \frac{a^2\,dx}{(a^2 + x^2)^{\frac{3}{2}}}$, and therefore $u = \frac{x}{\sqrt{(a^2 + x^2)}}$;

by making $dy = e\,du$, we find as before

$$\sqrt{\{x^2 + (y-c)^2 (a^2 + x^2)\}} = x + b(y-c)\sqrt{(a^2 + x^2)}.$$

II. When the equation involves $d\,x$, $d\,y$, and x only.

(8). Let $d\,x^2+d\,y^2=\frac{d\,x^2}{x}$. Make $\frac{d\,y}{d\,x}=p$,

$$\therefore\ x=\frac{1}{1+p^2},\ \text{and } y=\int p\,d\,x=x\,p-\int x\,d\,p$$

$$=\frac{p}{1+p^2}-\int\frac{dp}{1+p^2}=\frac{p}{1+p^2}-\tan^{-1}p+c\,;$$

and eliminating p, we find

$$y=\sqrt{(x-x^2)}-\tan^{-1}\sqrt{\left(\frac{1-x}{x}\right)}+c.$$

(9). Let $x\,d\,x+a\,d\,y=b\sqrt{(d\,x^2+d\,y^2)}$:

$$x=-\,a\,p+b\sqrt{(1+p^2)},$$

$$y=-\frac{a\,p^2}{2}+\frac{b\,p}{2}\sqrt{(1+p^2)}-\frac{b}{2}\log\{p+\sqrt{(1+p^2)}\}+c,$$

and the elimination of p would give the primitive relation between x and y. Euler. *Ib.* Art. 672.

(10). Let $y\frac{d\,x}{d\,y}=\sqrt{\left(1+\frac{d\,x^2}{d\,y^2}\right)}$.

$$\text{Make } \frac{d\,x}{d\,y}=p;\ \therefore y=\frac{\sqrt{(1+p^2)}}{p},$$

$$\frac{x}{c}=\log\left\{\frac{\sqrt{(1+p^2)}+1}{p}\right\}.$$

(11). Let $d\,y^4+y\,d\,y^3\,d\,x-d\,x^4=0$. Make $\frac{d\,x}{d\,y}=p$;

$$\therefore y=\frac{p^4-1}{p},$$

$$x=\frac{3}{4}p^4+\log p+c.$$

(12). Let $x^3\,d\,x^3+d\,y^3=a\,x\,d\,x^2\,d\,y$,

$$\text{or } x^3+p^3=a\,p\,x.$$

We must resolve a cubic equation, in order to express x in terms of p: make therefore $p=u\,x$, which gives us

$$x=\frac{a\,u}{1+u^3},\ p=\frac{a\,u^2}{1+u^3},\ \text{and}$$

$$y=\int p\,d\,x=a^2\int\frac{u^2\,d\,u\,(1-2\,u^3)}{(1+u^3)^3}$$

$$=\frac{1}{6}\,a^2\cdot\frac{2\,u^3-1}{(1+u^3)^2}+\frac{1}{3}\,a^2\cdot\frac{1}{1+u^3}+c.$$

Euler. *Ib.* p. 441.

III. When the equation is homogeneous with respect to x and y.

(13). Let $y\,d\,x-x\,d\,y=x\,\surd(d\,x^2+d\,y^2)$.

$$\text{Make } y=u\,x,\ \therefore u=p+\surd(1+p^2),$$

$$\frac{d\,x}{x}=\frac{d\,u}{p-u},\ \text{and } \log x=-\log(p-u)+\int\frac{d\,p}{p-u}$$

$$=-\log\surd(1+p^2)-\log\{p+\surd(1+p^2)\}+\log c,$$

$$\text{or } x=\frac{c}{\surd(1+p^2)}\{p-\surd(1+p^2)\}$$

$$y=\frac{c\,p}{\surd(1+p^2)}\{p-\surd(1+p^2)\};$$

and eliminating p, we find

$$x=0,\ \text{and } x^2+y^2+2\,c\,x=0.$$

The value $x=0$, results from the second of these solutions, by making $c=\infty$: it is not, therefore, a particular solution.

This equation becomes, by resolving it with respect to $\frac{dy}{dx}$,

$$2xy\,dy + x^2\,dx - y^2\,dx = 0,$$

the integral of which is very easily found. Euler. *Ib.* Art. 683.

(14). Let $x\,dy^3 + y\,dx^3 = dy\,dx\sqrt{\{xy(dx^2+dy^2)\}}$,

or $p^2 + u = p\sqrt{\{u(1+p^2)\}}$, making $y = ux$;

$$\therefore u = \tfrac{1}{2}p^2 - p^3 + \tfrac{1}{2}p^4 + \tfrac{1}{2}p^2\sqrt{(1+p^2)(1-4p+p^2)},$$

$$\text{and } \log x = -\log(p-u) + \tfrac{1}{2}\int\frac{dp\,(2-p)}{p\,(1-p+p^2)}$$

$$+\,2\int\frac{q^2\,dq}{(3+q^4)\sqrt{\{4-(1-q^2)^2\}}}$$

$$\text{where } q = \sqrt{\left(\frac{1-4p+p^2}{1+p^2}\right)}.$$

Euler. *Ib.* Art. 684.

(14). Let $y^2\,dx^6 - yx\,dx\,dy^5 - 2yx\,dy\,dx^5 + x^2\,dy^6 + x^2\,dy^2\,dx^4 = 0$,

$$\therefore u = p, \text{ or } u = p\,(p^4+1),$$

the second of which gives

$$x = \frac{c\,\epsilon^{\frac{1}{4p^4}}}{p^6},\quad y = \frac{(p^4+1)\,c\,\epsilon^{\frac{1}{4p^4}}}{p^4}.$$

The other value of u, gives $x = y$, which is a *particular* solution.

(15). Let $\int\sqrt{(dx^2+dy^2)} = \sqrt{(2xy)}$, or

$$\sqrt{(1+p^2)} = \frac{xp+y}{\sqrt{(2xy)}};$$

$$\therefore u = 1 - p + p^2 + (1 - p)\sqrt{(1 + p^2)};$$

and making $p = \frac{1 - v^2}{2v}$ in the expression for $\frac{dp}{p - u}$, we find

$$\log x = \log c - \log \{2 - (1 + v)^2\} - \frac{1}{\sqrt{2}} \log \left\{\frac{\sqrt{2} + 1 + v}{\sqrt{2} - 1 - v}\right\},$$

and $u = \frac{y}{x} = \frac{1}{2}(1 + q)^2$: consequently

$$x = \frac{cx}{x - y}\left\{\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right\}^{\frac{1}{\sqrt{2}}}, \text{ or}$$

$$(\sqrt{x} + \sqrt{y})^{1 + \frac{1}{\sqrt{2}}} = c(\sqrt{x} - \sqrt{y})^{\frac{1}{\sqrt{2}} - 1}.$$

Euler. *Ib.* Art. 685.

(16). Let $\int \sqrt{(dx^2 + dy^2)} = \sqrt{(y^2 + ex^2)}$,

or $\sqrt{(1 + p^2)} = \frac{py + ex}{\sqrt{(y^2 + ex^2)}}$. Make $y = ux$,

from whence we get

$$\frac{x}{c} = \left\{\frac{y + \sqrt{(y^2 + ex^2)}}{x}\right\}^{\pm\sqrt{\frac{e}{e-1}}}.$$

If $e = 1$, the original equation becomes $y\,dx - x\,dy = 0$, or $y = cx$, the equation of a straight line passing through the origin of x and y. Euler. *Ib.* Art. 693.

IV. Equations included under *Clairaut's* formula

$$y = px + f(p),$$

which are integrable by differentiation.

(17). Let $y\,dx - x\,dy = a\sqrt{(dx^2 + dy^2)}$,

$$\text{or } y = px + a\sqrt{(1+p^2)}:$$

differentiating, we find

$$-x\,dp = \frac{ap\,dp}{\sqrt{(1+p^2)}};$$

$$\text{consequently } dp = 0, \text{ or } x = -\frac{ap}{\sqrt{(1+p^2)}}.$$

The first of these values gives $p=c$, and

$$y = cx + a\sqrt{(1+c^2)},$$

which is the *general solution.*

The other gives, by the elimination of p

$$x^2 + y^2 = a^2,$$

an equation involving no arbitrary constant, and not deducible from the former, unless by making c variable and equal to

$$\frac{x}{\sqrt{(a^2 - x^2)}};$$

it is therefore a *particular solution.*

If we had resolved the equation with respect to $\frac{dy}{dx}$, we should have found, $(x^2-a^2)\,dy - xy\,dx = a\,dx\,\sqrt{(x^2+y^2-a^2)}$ which might be treated by the same method as that used in Ex. 66. p. 344. We thus find the multiplier

$$\frac{1}{y\,(x^2-a^2)} \times \frac{y}{\sqrt{(x^2+y^2-a^2)}} = \frac{1}{(x^2-a^2)\sqrt{(x^2+y^2-a^2)}},$$

which makes both the sides of the equation complete differentials. This process however will furnish us with the general solution only.

This is the solution of the following problem: "To find the nature of the curve, so that the perpendiculars upon its tangent from a given point, may be constantly equal to a given line."

The circle which results from the particular solution, is the locus of the intersections of the straight lines which are expressed by giving every possible value to c, in the general solution, and which are likewise tangents to the circle.

In general, the curve expressed by the particular solution is the locus of the intersections of the curves which are expressed by giving every possible value to the constant in the general solution.

Euler, *Hist. de l'Acad. de Berlin.* 1758. p. 301. Lagrange, *Mémoires de Berlin.* 1774. p. 220.

(18). Let $y\,dx - x\,dy = a\sqrt[3]{(dx^3 + dy^3)}$

The *general* solution is

$$y = cx + \sqrt[3]{(1 + c^3)}.$$

The *particular* solution is

$$y = \frac{(a^3 + x^3 + y^3)^{\frac{2}{3}}}{a\sqrt[3]{4}}.$$

Euler, *Ib.* p. 306.

(19). Let $x\,dy\,dx - dy^2 = y\,dx^2 - dy\,dx$.

The general solution is

$$y - xp = p - p^2,$$

The *particular* solution is

$$4y = (1 + x)^2,$$

the equation of a parabola.

Clairaut, *Mémoires de l' Acad. des Sciences*, p. 210. 1734.

(20). PROBLEM. To find the nature of the curve *BMC*, of which the hypothenuse *T t* of the right-angled triangle *ATt*, the sum of whose sides *AT* and *A t* is a constant quantity, is always a tangent. Fig. 118.

Make $AP = x$, $PM = y$, and $AT + At = a = AC$: then we have

$$x - \frac{y\,dx}{dy} = y + (a - x)\frac{dy}{dx},$$

$$\text{or } px - y = py + (a - x)p^2.$$

The *general* solution is

$$cx = y + \frac{ac^2}{1 + c};$$

the *particular* solution is

$$(x + y)^2 = 4ay$$

the equation of a parabola. Simpson's *Fluxions*. vol. II. p. 514. Clairaut, *Ib.*

(21). PROBLEM. To find the nature of the curve *AMA'*, so that the rectangle under the lines *A T* and *A' T*, which are perpendicular to *A A'*, and meet the tangent *TMT'* in *T* and *T'*, may be constantly of the same magnitude. Fig. 119.

Make $AA' = 2a$, $AP = x$, $PM = y$: then we have

$$AT = y - \frac{x\,dy}{dx}, \quad A'T' = y + (2a - x)\frac{dy}{dx},$$

and therefore

$$\left(y - \frac{x\,d\,y}{d\,x}\right)\left(y + (2\,a - x)\frac{d\,y}{d\,x}\right) = b^2,$$

or

$$y = p\,x - a\,p \pm \sqrt{(b^2 + a^2 p^2)}.$$

The general solution is

$$y = c\,x - a\,c \pm \sqrt{(b^2 + a^2 c^2)}.$$

The *particular* solution is

$$y = \frac{b}{a}\sqrt{(2\,a\,x - x^2)},$$

the equation of an ellipse, of which it is a well known property. Euler. *Ib.* 308.

(22). PROBLEM. To find the nature of the curve *A M A′*, so that if *B V* be a perpendicular upon its tangent *V M* from a given point *B*, the line *B′ V* drawn from another given point *B′*, may be constantly of the same length. Fig. 119.

Let $B'V = a$, $B\,B' = a\,e$, $B\,P = x$, and $P\,M = y$;

then we have $B'\,V^2 = V\,L^2 + B'\,L^2$ or

$$a^2 = a^2 e^2 + 2\,a\,e\,d\,y\,.\,\frac{(y\,d\,x - x\,d\,y)}{d\,x^2 + d\,y^2} + \frac{(y\,d\,x - x\,d\,y)^2}{d\,x^2 + d\,y^2};$$

and therefore

$$y = p\,x - a\,e\,p + a\sqrt{(1 - e^2 + p^2)};$$

the particular solution of which equation gives

$$x = a\left(e - \frac{p}{\sqrt{(1 - e^2 + p^2)}}\right),$$

$$y = \frac{a(1-e^2)}{\sqrt{(1-e^2+p^2)}};$$

and eliminating p, we find

$$\frac{(ae-x)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1,$$ or making

$a^2(1-e^2) = b^2$ and $ae - x = x'$, we get

$$y^2 = \frac{b^2}{a^2}(a^2 - x'^2).$$ Euler, *Ib.*

(23). **Problem.** To find the nature of the curve EME', so that the rectangle contained by the perpendiculars BV and $B'V'$ upon the tangent VMV' may be constantly of the same magnitude. Fig. 120.

Let $BB' = 2ae$, $BP = x$, $PM = y$: then we have

$$BV = \frac{y\,dx - x\,dy}{\sqrt{(dx^2+dy^2)}}, \quad B'V' = \frac{y\,dx + (2ae - x)\,dy}{\sqrt{(dx^2+dy^2)}}$$

and $$\frac{(y\,dx - x\,dy)\{y\,dx + (2ae-x)\,dy\}}{dx^2+dy^2} = b^2,$$

or

$$y = px - aep + \sqrt{\{b^2 + (b^2 + a^2e^2)p^2\}}.$$

The particular solution gives, if $b^2 + a^2e^2 = a^2$,

$$\frac{ae-x}{a} = \frac{ap}{\sqrt{(b^2+a^2p^2)}}, \quad \frac{y}{b} = \frac{b}{\sqrt{(b^2+a^2p^2)}}$$

and $\therefore\ y^2 = \frac{b^2}{a^2}\{a^2 - (ae-x)^2\}$,

$$= \frac{b^2}{a^2}(a^2 - x'^2), \text{ if } ae - x = x'.$$

Euler, *Ib.* Lagrange, *Calcul des Fonctions*, p. 280.

(24). To find the nature of the curve, so that the locus of a perpendicular drawn from a given point upon its tangent, may be a given straight line. Fig. 121.

Let S be the given point, ST a perpendicular upon the tangent TM, meeting the given line AE in T; and let $SA = a$, be a perpendicular upon AE: let $SP = x$, $PM = y$; then we find

$$y = px + \frac{a}{p}(1 + p^2);$$

a particular solution of which gives

$$y^2 = 4a(a + x),$$

the equation of a parabola, whose vertex is A and focus S.

V. Equations, which are integrable by differentiation, though not included under the preceding formula.

(25). Let $y\,dx - nx\,dy = a\sqrt{(dx^2 + dy^2)}$; or

$$y - nxp = a\sqrt{(1 + p^2)}.$$

Differentiating, we get

$$dx - \frac{nx\,dp}{(1 - n)p} = \frac{a\,dp}{(1 - n)\sqrt{(1 + p^2)}}$$

a linear equation of the first order.

Consequently,

$$p^{\frac{n}{n-1}}x = \frac{a}{1 - n}\int \frac{p^{\frac{n}{n-1}}\,dp}{\sqrt{(1 + p^2)}}$$

If $n = \frac{2\lambda + 1}{2\lambda}$, we have

$$y = px + \frac{px}{2\lambda} + a\sqrt{(1 + p^2)},$$

$$x = \frac{c}{p^{2\lambda+1}} - \frac{2\lambda a}{(2\lambda+1)p}\left\{1 - \frac{2\lambda}{(2\lambda-1)p^2}\right.$$

$$\left. + \frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4} - \&c.\right\} \surd(1+p^2).$$

Euler. *Calc. Integ.* Art. 703.

VI. Equations which may be treated by particular methods, not included under any of those above-mentioned.

(26). Let $dy^2 = (2x^2 + by)\,dx^2$.

Make $y = z^2$ and the equation becomes

$$4z^2 dz^2 = (2x^2 + bz^2)\,dx^2,$$

which is homogeneous.

Consequently if $z = ux$ and $v = \surd(2 + bu^2)$, we find

$$\frac{dx}{x} = \frac{2v\,dv}{bv - 2v^2 + 4} \text{ (Ex. 68. p. 280.)}$$

Euler. *Ib.* Art. 704.

(27). Let $\dfrac{x\,dy - y\,dx}{\surd(dx^2 + dy^2)} = \surd(x^2 + y^2 - a^2)$.

Make $x = uz$ and $y = u\surd(1 - z^2)$, when the equation becomes

$$\frac{-u^2 dz}{\surd\{u^2 dz^2 + (1 - z^2)\,du^2\}} = \surd(u^2 - a^2),$$

or

$$\frac{dz}{\surd(1 - z^2)} = \frac{du\,\surd(u^2 - a^2)}{au}$$

where the variables are separated.

Consequently

$$\sin^{-1}\frac{x}{\sqrt{(x^2+y^2)}}+\sin^{-1}\frac{\sqrt{(x^2+y^2-a^2)}}{\sqrt{(x^2+y^2)}}$$
$$=\frac{\sqrt{(x^2+y^2-a^2)}}{a}+\text{C}.$$

(28). Let $\dfrac{x\,d\,y-y\,d\,x}{\sqrt{(x^2+y^2)}\sqrt{(d\,x^2+d\,y^2)}}=\sqrt{\left\{\dfrac{x^2+y^2}{2\,x^2+y^2}\right\}}$.

Make $x=u\,z$ and $y=u\sqrt{(1-z^2)}$, which gives

$$\frac{z\,d\,z}{\sqrt{(1-z^2)}}=\frac{d\,u}{u}.$$

Consequently

$$\log\frac{\sqrt{(x^2+y^2)}}{\text{C}}=\frac{y}{\sqrt{(x^2+y^2)}}.$$

(29). Problem. To find the nature of the curve whose normal bears a constant ratio to the part of the axis intercepted between the origin and normal.

The differential equation of the curve is

$$y\sqrt{\left(1+\frac{d\,y^2}{d\,x^2}\right)}=e\left(x+\frac{y\,d\,y}{d\,x}\right)$$

which is homogeneous and may be treated as the Examples given in p. 356. : or more simply, by resolving the equation with respect to $\dfrac{y\,d\,y}{d\,x}$, and multiplying the result by

$\dfrac{1}{\sqrt{\left(\frac{e^2}{4}+e\,x-y^2\right)}}$, when it becomes integrable *per se*:

we thus get

$$\sqrt{\left(\frac{e^2}{4}+e\,x-y^2\right)}+x=C:$$

or making $c = C + \frac{e}{2}$,

$$y^2 + (x - c)^2 = e\,c,$$

which is the equation to a circle.

The particular solution deduced from this, by considering c as variable, is

$$y^2 - e\,x - \frac{e^2}{4} = 0,$$

the equation of a parabola, which obviously resolves the problem.

The reader will find a most instructive discussion of this Example and of the different solutions of it given by Leibnitz and John Bernoulli, in the *Calcul des Fonctions* of Lagrange. p. 263.

On the Integration of Differential Equations of the second and higher orders.

I. Equations, in which if we make

$\frac{dy}{dx} = p$ and $\frac{d^2y}{dx^2} = q$, we find q a function of p.

(1). Let $a\,d^2y = dx\,dy$; or $q = \frac{p}{a}$;

$$\therefore dx = \frac{dp}{P} = \frac{a\,dp}{p}\text{; and } x = c + a \log p:$$

$$dy = \frac{p\,dp}{P} = a\,dp\text{; and } y = c' + a\,p.$$

Consequently, by eliminating p, we find

$$x = c + a \log \left(\frac{y - c'}{a}\right).$$

Euler. *Calc. Integ.* Tom. II. Art. 725.

(2). Let $d^2 y = d x \sqrt{(d x^2 + d y^2)}$ or $q = \sqrt{(1 + p^2)}$:

$$x = c \log \left\{\frac{y + \sqrt{(y^2 - c'^2)}}{c'}\right\}.$$

(3). Let $\dfrac{(d x^2 + d y^2)^{\frac{3}{2}}}{-d x\, d^2 y} = a$;

$$\therefore\ x = c - \frac{a p}{\sqrt{(1 + p^2)}},\ y = c' + \frac{a}{\sqrt{(1 + p^2)}},$$

$$\text{and } (x - c)^2 + (y - c')^2 = a^2.$$

Euler. *Ib.* Art. 726.

(4). Let $\dfrac{d s\, d y}{d^2 x} = \dfrac{a\, d x}{d y}$, where $d s = \sqrt{(d x^2 + d y^2)}$ is constant;

$$\therefore\ d^2 s = 0 = d^2 x \sqrt{(1 + p^2)} + \frac{p\, d x\, d p}{\sqrt{(1 + p^2)}};\ \text{therefore}$$

$$d^2 x = -\frac{p\, d x\, d p}{1 + p^2};\ \text{and the equation becomes}$$

$$\frac{p\, d x (1 + p^2)^{\frac{3}{2}}}{-p\, d p} = \frac{a}{p};\ \text{consequently}$$

$$x = c - \frac{a}{\sqrt{(1 + p^2)}} + a \log \left\{\frac{1 + \sqrt{(1 + p^2)}}{p}\right\},$$

$$y = c' - \frac{a p}{\sqrt{(1 + p^2)}}.$$

Euler. *Ib.* Art. 728.

(5). Let $\frac{d\,s\,d\,y}{d^2\,x} = a \tan^{-1} \frac{d\,y}{d\,x}$, where $d\,s = \sqrt{(d\,x^2 + d\,y^2)}$ is constant. Consequently, as in the last Example, we get

$$d\,x = \frac{-a\,d\,p}{(1+p^2)^{\frac{3}{2}}} \tan^{-1} p,$$

$$x = c - \frac{a\,p}{\sqrt{(1+p^2)}} \tan^{-1} p - \frac{a\,p}{\sqrt{(1+p^2)}},$$

$$y = c' + \frac{a}{\sqrt{(1+p^2)}} \tan^{-1} p - \frac{a}{\sqrt{(1+p^2)}}.$$

This is the solution of the following problem: "To find the curve, whose radius of curvature is proportional to the angle, which its tangent makes with the axis of the abscissæ." Euler. *Ib.* Art. 729.

II. Differential equations of higher orders, where the highest differential coefficient is a function of the one next inferior to it.

(6). Let $a\,d^2\,y\,d^3\,y = d\,x^3 \sqrt{(d\,x^4 + d^2\,y^2)}$,

$$\text{or } a\,q\,d\,q = d\,x\,\sqrt{(1+q^2)}, \text{ where } q = \frac{d^2\,y}{d\,x^2};$$

$$\therefore\ x = c + a\sqrt{(1+q^2)},$$

$$\frac{d\,y}{d\,x} = \int q\,d\,x = \int \frac{a\,q^2\,d\,q}{\sqrt{(1+q^2)}}$$

$$= \frac{a\,q}{2}\sqrt{(1+q^2)} - \frac{a}{2}\log\{\,q + \sqrt{(1+q^2)}\,\} + c',$$

$$\text{and } y = \frac{a^2\,q^3}{6} - \frac{a^2}{2}\sqrt{(1+q^2)}\log\{\,q+\sqrt{(1+q^2)}\,\}$$

$$+ \frac{a^2}{2}\,q + a\,c'\sqrt{(1+q^2)} + c''.$$

The elimination of q from these expressions, will give the primitive relation between x and y.

(7). Let $\frac{a\,d^4 y}{d\,x^4} = \frac{d^3 y}{d\,x^3}$; or if $\frac{d^3 y}{d\,x^3} = r$, let $a\frac{d\,r}{d\,x} = r$;

$$\therefore\ x = a \log r + c.$$

$$y = a^3 r + \frac{a^2 c'}{2} (\log r)^2 + a\,c'' \log r + c'''.$$

(8). Let $d^4 y = a\,d\,x^2 \sqrt{(d^3 y\,d\,x)}$;

$$x = \frac{2\sqrt{r}}{a} + c,$$

$$y = \frac{2\,r^{\frac{5}{2}}}{3\,.\,5\,.\,a^3} + \frac{2\,c'\,r}{a^2} + \frac{2\,c''\sqrt{r}}{a} + c'''.$$

III. Where q is a function of x.

(10). Let $d^2 y = \frac{a\,d\,x^2}{x}$;

$$\therefore\ d\,p = \frac{a\,d\,x}{x},\ p = a \log x + c,$$

$$\text{and } y = a\,x \log x - a\,x + c\,x + c'$$

$$= a\,x \log x + c'\,x + c''.$$

Euler. *Ib.* Art. 739.

(11). Let $\frac{d\,s^2\,d^2 y}{d\,x^4} = \frac{1}{a} \cos \frac{x}{b}$, where

$d\,s = \sqrt{(d\,x^2 + d\,y^2)}$, is constant.

Consequently $d\,s\,d^2 s = d\,x\,d^2 x + d\,y\,d^2 y = 0$;

and since $d^2 x = \frac{-p\,d x\,d p}{1+p^2}$, Ex. 4. p. 367. we have

$$d p = (1 + p^2)\frac{d^2 y}{d x} = \frac{d s^2\, d^2 y}{d x^3} = \frac{d x}{a}\cos\frac{x}{b}.$$

$$\therefore p = \frac{b}{a}\sin\frac{x}{b} + c$$

$$y = \frac{-b^2}{a}\cos\frac{x}{b} + c x + c'.$$

Euler, *Ib.* Art. 740.

(12.) PROBLEM. A body T moves uniformly along a straight line BC, whilst another body M moves uniformly in *pursuit* of it: to find the nature of the curve described by M. Fig. 122.

Let $BP = x$, $PM = y$, $AM = s$; then we have $\frac{y\,d x}{d y} - x = e s$, where e expresses the ratio of the velocities of the bodies: differentiating this equation, considering $d y$ as constant, we find

$$\frac{d^2 x}{\sqrt{(d x^2 + d y^2)}} = \frac{e\,d y}{y}$$

$$= \frac{d p}{\sqrt{(1 + p^2)}}, \text{ where } p = \frac{d x}{d y};$$

$$\therefore p + \sqrt{(1 + p^2)} = \left(\frac{y}{c}\right)^e,$$

$$\text{and } x = \frac{y^{e+1}}{2(e+1)c^e} + \frac{c^e}{2(e-1)y^{e-1}} + C.$$

This is one of the simplest cases of the problem for determining *curves* of *pursuit*. See *Bouguer* and *Maupertuis* *Mémoires de l' Acad. des Sciences.* 1735. Simpson's *Fluxions.* vol. II. p. 516.

IV. Where q is a function of y.

(13). Let $a^2 d^2 y = y\, d x^2$;

$$\therefore\ d y . d^2 y = \frac{y\, d y\, d x^2}{a^2}, \text{ and}$$

$$\frac{d y^2}{d x^2} = \frac{y^2}{a^2} + c = \frac{y^2 + c}{a^2};$$

$$\therefore\ d x = \frac{a\, d y}{\sqrt{(y^2 + c)}} \text{ and } x = a \log \left\{ \frac{y + \sqrt{(y^2 + c)}}{c'} \right\};$$

or expressing y in terms of x, we find

$$y = C\, \epsilon^{\frac{x}{a}} + C'\, \epsilon^{-\frac{x}{a}}.$$

Euler, *Ib.* Art. 745.

(14). Let $a^2 d^2 y + y\, d x^2 = 0$.

$$x = a \sin^{-1} \frac{y}{c} + C',$$

$$\text{or } y = C \sin \frac{x}{a} + C' \cos \frac{x}{a}.$$

Euler, *Ib.* Art. 746.

(15). Let $d^2 y \sqrt{(a y)} = d x^2$.

$$\therefore\ \frac{d y^2}{d x^2} = \frac{4 (\sqrt{y} + c \sqrt{a})}{\sqrt{a}},$$

$$\text{and } \frac{x}{\sqrt[4]{a}} = \frac{2}{3} (\sqrt{y} - 2 \sqrt{c'}) (\sqrt{y} + \sqrt{c'}) + C',$$

where $c' = c \sqrt{a}$. Euler, *Ib.* Art. 748.

VI. Where a differential coefficient of y is a function of one whose order is inferior by two unities.

(16). Let $\frac{a^2 d^4 y}{d x^4} = \frac{d^2 y}{d x^2}$; or $\frac{a^2 d^2 q}{d x^2} = q$;

$$\therefore \frac{d q \, d^2 q}{d x^2} = \frac{q \, d q}{a^2} \text{ and } \frac{d q^2}{d x^2} = \frac{q^2}{a^2} + c = \frac{q^2 + c}{a^2};$$

consequently $x = \int \frac{a \, d q}{\sqrt{(q^2 + c)}} = a \log \left\{ \frac{q + \sqrt{(q^2 + c)}}{c'} \right\}$

$$\frac{d y}{d x} = \int q \, d x = a \sqrt{(q^2 + c)} + c'',$$

and $y = \int Q \, d x = a^2 q + a c'' \log \{ q + \sqrt{(q^2 + c)} \} + c'''$

$$= \frac{a^2 c'}{2} . \epsilon^{\frac{x}{a}} - \frac{c a^2}{2 c'} . \epsilon^{-\frac{x}{a}} + c'' x + c'''$$

$$= C \epsilon^{\frac{x}{a}} + C' \epsilon^{-\frac{x}{a}} + C'' x + C'''.$$

VII. Where the equation involves x, p and q only.

(17). Let $\frac{(d x^2 + d y^2)^{\frac{3}{2}}}{d x \, d^2 y} = X = \frac{a^2}{2 x}$;

$$\therefore \frac{d p}{(1 + p^2)^{\frac{3}{2}}} = \frac{d x}{X} = \frac{2 x \, d x}{a^2}.$$

$$\therefore \frac{p}{\sqrt{(1 + p^2)}} = \frac{x^2 + a c}{a^2} \text{ and } p = \frac{(x^2 + a c)}{\sqrt{\{ a^4 - (x^2 + a c)^2 \}}}$$

and $y = \int \frac{(x^2 + a c) \, d x}{\sqrt{\{ a^4 - (x^2 + a c)^2 \}}}$, which is the general equation of the elastic curve. See Mr. Whewell's *Mechanics*, Vol. I. p. 210. Euler, *Ib.* p. 757.

(18). Let $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}=\frac{x^2}{a}$;

$$\therefore p=\frac{(cx-a)}{\sqrt{\{x^2-(cx-a)^2\}}} \text{ and}$$

$$y=\int\frac{(cx-a)\,dx}{\sqrt{(1-c^2)}\sqrt{\left\{x^2+\frac{2acx}{1-c^2}-\frac{a^2}{1-c^2}\right\}}}$$

$$=C\sqrt{\{x^2-(cx-a)^2\}}$$

$$-\frac{a}{(1-c^2)^{\frac{3}{2}}}\cdot\log\left\{\frac{x(1-c^2)+ac}{\sqrt{(1-c^2)}}+\sqrt{\{x^2-(cx-a)^2\}}\right\}+C'.$$

(19). Let $dx(dx^2+dy^2)+x\,dy\,d^2y=a\,d^2y\sqrt{(dx^2+dy^2)}$, or

$$dx(1+p^2)+xp\,dp=a\,dp\sqrt{(1+p^2)}$$

which is integrable when divided by $\sqrt{(1+p^2)}$.

Therefore

$$x=\frac{ap+c}{\sqrt{(1+p^2)}},\ y=\frac{cp-a}{\sqrt{(1+p^2)}}-c\log\left\{\frac{p+\sqrt{(1+p^2)}}{c'}\right\}$$

$$=\sqrt{(a^2+c^2-x^2)}-c\log\left\{\frac{c+\sqrt{(a^2+c^2-x^2)}}{c'(x-a)}\right\}.$$

Euler, *Ib.* Art. 759.

(20). Let $a^2d^2y\sqrt{(a^2+x^2)}+a^2dx\,dy=x^2dx^2$,

$$\text{or } a^2dp\sqrt{(a^2+x^2)}+a^2p\,dx=x^2dx,$$

$$\text{or } dp+\frac{p\,dx}{\sqrt{(a^2+x^2)}}=\frac{x^2dx}{a^2\sqrt{(a^2+x^2)}},$$

a linear equation of the first order.

Consequently

$$a^2 p = -\left\{\frac{x^2+2a^2-2x\sqrt{(a^2+x^2)}}{3}\right\} + c\sqrt{(a^2+x^2)} - cx,$$

and $a^2 y = -\frac{1}{9}x^3 - \frac{2}{3}a^2 x + \frac{2}{9}(a^2+x^2)^{\frac{3}{2}} - \frac{1}{2}cx^2$

$$+\frac{cx}{2}\sqrt{(a^2+x^2)} + \frac{a^2 c}{2}\log\left\{\frac{x+\sqrt{(a^2+x^2)}}{c'}\right\}.$$

Euler. *Ib.* Art. 761.

(21). Let $(a^2 dy^2 + x^2 dx^2)\,d^2 y = n x\,dx^3\,dy$,

or $(a^2 p^2 + x^2)\,dp = n p x\,dx$,

a homogeneous equation of the first order, with respect to x and p.

Therefore, if $x = pu$, we find

$$p = c\left\{a^2+(1-n)u^2\right\}^{\frac{n}{2(1-n)}},$$

$$x = pu = cu\left\{a^2+(1-n)u^2\right\}^{\frac{n}{2(1-n)}}$$

$$y = px - \int x\,dp$$

$$= c^2 u\left\{a^2+(1-n)u^2\right\}^{\frac{n}{1-n}} -$$

$$n c^2 \int u^2 du\left\{a^2+(1-n)u^2\right\}^{\frac{2n-1}{2(1-n)}}.$$

If $n=1$, we find

$$x = ap\sqrt{\left(2\log\frac{p}{c}\right)},\ y = ap^2\log\left(2\log\frac{p}{c}\right)$$

$$-a\int p\,dp\sqrt{\left(2\log\frac{p}{c}\right)}.$$

Euler. *Ib.* Art. 762.

(22). Let $a\,dx\,dy^2 + x^2\,dx\,d^2 y = n x\,dy\sqrt{(dx^4 + a^2 d^2y^2)}$,

$$\text{or } a p^2 d x + x^2 d p = u x p \sqrt{(d x^2 + a^2 d p^2)},$$

which is homogeneous with respect to x and p;

$$\therefore \frac{d p}{d x} = q = \frac{a u^2 + n u \sqrt{(1 - n^2 a^2 u^2 + a^4 u^4)}}{n^2 a^2 u^2 - 1}, \text{ and}$$

$$\frac{d x}{x} = \frac{d u}{q - u}$$

$$= \frac{d u}{u} \cdot \frac{n^2 a^2 u^2 - 1}{1 + a u - n^2 a^2 u^2 + n \sqrt{(1 - n^2 a^2 u^2 + a^4 u^4)}}.$$

If $n = \sqrt{2}$, we have

$$\frac{d x}{x} = \frac{d u (1 - \sqrt{2})}{u} + \frac{a d u (3 - 2 \sqrt{2})}{1 - a u}.$$

Euler. *Ib.* Art. 764.

(23). Let $d x^3 d y - x d s^2 d^2 y = a d x d s \sqrt{(d^2 x^2 + d^2 y^2)}$,

where $d s = \sqrt{(d x^2 + d y^2)}$, is constant.

Since $d^2 s = 0$, we have

$$d^2 x = - \frac{p q d x^2}{1 + p^2}, \text{ and } d^2 y = \frac{q d x^2}{1 + p^2};$$

and the equation becomes

$$p - x q = a q,$$

which comes under Clairaut's formula, p. 357.

If $d q = 0$, or $q = \frac{1}{c}$, we have

$$p = \frac{x + a}{c}, \text{ and } y = \frac{x^2 + 2 a x}{2 c} + c'.$$

If $x = -a$, we have $p = 0$, and $y = c'$, which is a particular *integral* arising from making $c = \infty$. Euler. *Ib.* Art. 767.

(24). Let $dx^3\,dy - x\,ds^2\,d^2y = \dfrac{b\,dx^4\,ds^2\,d^2y}{\sqrt{(dx^8 + a^2\,ds^4\,d^2y^2)}}$,

where $ds = \sqrt{(dx^2 + dy^2)}$, is constant.

In the same manner as in the last Example, we find

$$p - qx = \frac{bq}{\sqrt{(1 + a^2 q^2)}}:$$

Consequently $q = \dfrac{1}{c}$, $p = \dfrac{x}{c} + \dfrac{b}{\sqrt{(c^2 + a^2)}}$,

$$\text{and } y = \frac{x^2}{2c} + \frac{bx}{\sqrt{(c^2 + a^2)}} + c',$$

is the general solution.

The elimination of q from the equations

$$x = -\frac{b}{(1 + a^2 q^2)^{\frac{3}{2}}}$$

$$y = -\frac{b^2 q\,(1 + 2a^2 q^2)}{2\,(1 + a^2 q^2)^3} + \frac{b^2 q}{8\,(1 + a^2 q^2)^2}$$

$$+ \frac{3b^2 q}{16\,(1 + a^2 q^2)} + \frac{3b^2}{16a} \tan^{-1} aq + c',$$

will furnish a particular solution of the equation. Euler. *Ib.* Art. 768.

VII. When the equation is between y, p and q.

(25). Let $ab\,d^2y = dx\,\sqrt{(y^2\,dx^2 + a^2\,dy^2)}$,

$$\text{or } abq = \sqrt{(y^2 + a^2 p^2)} = \frac{ab\,p\,dp}{dy},$$

an equation which is homogeneous with respect to y and p.

Consequently, if $y = p\,u$, we get

$$\frac{dy}{y} = \frac{a\,b\,d\,u}{a\,b\,u - u^2\sqrt{(a^2+u^2)}},$$

and by making $\sqrt{(a^2+u^2)} = s\,u$, and $\frac{a}{b} = e$, we find

$$y^{2\sqrt{(e^2+1)}} = c\frac{\{s-e+\sqrt{(e^2+1)}\}^{e-\sqrt{(e^2+1)}}}{\{s-e-\sqrt{(e^2+1)}\}^{e+\sqrt{(e^2+1)}}},$$

$$\text{and } d\,x = -\frac{a\,s\,d\,s}{(s^2-2\,e\,s-1)\sqrt{(s^2-1)}}.$$

Euler. *Ib.* Art. 780.

(26). Let $\frac{(d\,y^2 + y^2\,d\,x^2)^{\frac{3}{2}}}{2\,d\,y^2 d\,x + y^2\,d\,x^3 - y\,d^2 y\,d\,x} = n\,y$, or

$$\frac{(p^2+y^2)^{\frac{3}{2}}}{2\,p^2+y^2-q\,y} = n\,y, \text{ or}$$

$$d\,y\,(p^2+y^2)^{\frac{3}{2}} = 2\,n\,p^2\,y\,d\,y + n\,y^3\,d\,y - n\,y^2\,p\,d\,p,$$

which is homogeneous with respect to p and y.

Therefore

$$\frac{dy}{y} = \frac{n\,u\,d\,u}{(u^2+1)\{n-\sqrt{(u^2+1)}\}},$$

$$d\,x = \frac{n\,d\,u}{(u^2+1)\{n-\sqrt{(u^2+1)}\}}$$

Euler. *Ib.* Art. 781.

(27). Let $\frac{(p^2+y^2)^{\frac{3}{2}}}{2\,p^2+y^2-q\,y} = a$.

Let $p^2+y^2=z^2$; then we have $p\,d\,p + y\,d\,y$

$$= q\,dy + y\,dy = z\,dz, \text{ and } q + y = \frac{z\,dz}{dy};$$

∴ the equation reduced becomes

$$z^2\,dy = 2az\,dy - ay\,dz.$$

Consequently,

$$y^2 = \frac{cz}{2a - z}, \text{ and } dx = \frac{(c+u)\,du}{2u\sqrt{\{4a^2u - (c+u)^2\}}}, \text{ if } u = y^2.$$

Euler, *Ib.* Art. 785.

(28). Let $d^2y\,(y\,dy + a\,dx) = dy\,(dx^2 + dy^2)$,

$$\text{or } dp\,(py + a) = dy\,(1 + p^2),$$

$$\text{or } dy - \frac{py\,dp}{1+p^2} = \frac{a\,dp}{1+p^2};$$ Ex. 24. p. 329.

$$\therefore y = ap + c\sqrt{(1+p^2)},$$

$$x = a \log p + c \log \{p + \sqrt{(1+p^2)}\} + c'.$$

Euler. *Ib.* Art. 788.

(29). Let $dy^2 - y\,d^2y = n\sqrt{(dx^2\,dy^2 + a^2\,d^2y^2)}$

or making $q = \dfrac{p\,dp}{dy} = pu$, and therefore $dp = u\,dy$,

$$p - uy = n\sqrt{(1 + a^2u^2)},$$

which comes under Clairaut's formula.

If $du = 0$, we have $u = c$, $p = cy + n\sqrt{(1 + a^2c^2)}$,

and $cx = \log\{cy + n\sqrt{(1 + a^2c^2)}\} + c'$.

If $y = -\dfrac{na^2u}{\sqrt{(1 + a^2u^2)}}$, we have

$$p = \frac{n}{\sqrt{(1 + a^2 u^2)}}, \; x = c - a \tan^{-1} a u,$$

and $\frac{c - x}{a} = \sin^{-1} \frac{y}{n a}$, a particular solution.

Euler. *Ib.* Art. 789.

VIII. Differential equations which become homogeneous, by considering the dimension of x or y as 1, of p as 0, and of q as -1.

(30). Let $x^2 d^2 y = x\, d\, x\, dy + 3 y\, d x^2$, or

$$x^2 q = x p + 3 y.$$

$$\text{Make } y = u x, \; q = \frac{v}{x}.$$

Consequently $\frac{d x}{x} = \frac{d p}{v} = \frac{d p}{p + 3 u} = \frac{d u}{p - u}$;

$$\text{or } (p + 3 u)\, d u = p\, d p - u\, d p,$$

which is homogeneous with respect to u and p.

Therefore

$$p = u + \sqrt{(c + 4 u^2)},$$

$$\log x = \int \frac{d x}{x} = \frac{1}{2} \log \left\{ \frac{2 u + \sqrt{(c + 4 u^2)}}{C'} \right\}$$

$= \frac{1}{2} \log \left\{ \frac{u + \sqrt{(c + u^2)}}{C'} \right\}$, putting $4 c$ for c;

$$\therefore x^2 = \frac{u + \sqrt{(c + u^2)}}{C'} = \frac{y + \sqrt{(c x^2 + y^2)}}{C' x},$$

$$\text{or } y = C' x^3 - \frac{c}{x}.$$

Euler. *Ib.* Art. 802.

(31). Let $\frac{x^2\,d^2y}{dx} = \sqrt{(m\,x^2\,dy^2 + n\,y^2\,dx^2)}$.

Making the same substitutions as before, we find $du\,\sqrt{(m\,p^2 + n\,u^2)} = (p-u)\,dp$; and making $p = u\,s$, we get

$$\frac{du}{u} = \frac{(s-1)\,ds}{\sqrt{(m\,s^2+n)} - s^2 + s},$$

$$\text{and } \frac{dx}{x} = \frac{du}{(s-1)\,u} = \frac{ds}{\sqrt{(m\,s^2+n)} - s^2 + s}.$$

Euler. *Ib.* Art. 803.

(32). Let $n\,x^3\,d^2y = (y\,dx - x\,dy)^2$.

Make $y = u\,x$, and $q = \frac{v}{x}$, which gives

$$n\,dp = (p-u)\,du, \text{ or if } p-u = s,$$

$$du - \frac{n\,ds}{s-n} = 0: \text{ consequently}$$

$$x = \frac{s-n}{c\,s},\quad y = n\,x\log\left(\frac{s-n}{c'}\right)$$

$$= n\,x\log\frac{n\,c\,x}{c'\,(1-c\,x)}.$$

Euler. *Ib.* Art. 804.

(33). Let $(dx^2 + dy^2)^{\frac{3}{2}} = n\,dx\,d^2y\,\sqrt{(x^2+y^2)}$.

Making $y = u\,x$, and $q = \frac{v}{x}$, we get

$$(1+p^2)^{\frac{3}{2}}\,du = n\,(p-u)\,dp\,\sqrt{(1+u^2)}.$$

In order to integrate this equation, make $p = \tan\theta$, $u = \tan\phi$: this gives

$$d\theta = \frac{d\theta - d\phi}{1 - n\sin(\theta - \phi)} = \frac{d\psi}{1 - n\sin\psi}, \text{ if } \psi = \theta - \phi:$$

$$\text{and } \frac{dx}{x} = \frac{du}{p-u} = \frac{d\phi\cos\theta}{\cos\phi\sin\psi}$$

$$= \frac{d\phi(\cos\phi\cos\psi - \sin\phi\sin\psi)}{\cos\phi\sin\psi} = \frac{d\phi\cos\psi}{\sin\psi} - \frac{d\phi\sin\phi}{\cos\phi}$$

$$= \frac{n\,d\psi\cos\psi}{1 - n\sin\psi} - \frac{d\phi\sin\phi}{\cos\phi}, \text{ since } d\phi = \frac{n\sin\psi\,d\psi}{1 - n\sin\psi};$$

$$\therefore x = \frac{c\cos\phi}{1 - n\sin\psi},\ y = \frac{c\sin\phi}{1 - n\sin\psi},$$

$$\text{and } \phi = \int \frac{n\,d\psi}{1 - n\sin\psi}.$$

Euler. *Ib.* Art. 806.

VIII. Differential equations which become homogeneous by considering y, p and q as of n, $n-1$, and $n-2$ dimensions respectively.

(34). Let $x^2 d^2y = a y\, dx^2 + b x\, dx\, dy$.

This equation becomes homogeneous, by making $p = \frac{t}{x}$, $q = \frac{v}{x^2}$, n being equal to 0; consequently $v = ay + bt$;

$$\text{but } dp = q\,dx, \text{ or } x\,dt - t\,dx = v\,dx;$$

$$\therefore \frac{dx}{x} = \frac{dt}{v+t}, \text{ and } dy = p\,dx = \frac{t\,dx}{x},$$

$$\therefore \frac{dx}{x} = \frac{dy}{t},\ \therefore \frac{dt}{v+t} = \frac{dy}{t};$$

$$\text{or } t\,dt = (v+t)\,dy = \{ay + (b+1)t\}\,dy,$$

which is homogeneous and of the first order.

Consequently

$$y=c'\left\{\frac{c^f}{x^f}+\frac{x^g}{c^g}\right\},$$

where $a=fg$, and $b+1=g-f$.

Euler. *Ib.* Art. 817.

(35). Let $x^4\,d^2y=x^3\,dx\,dy+2\,x\,y\,dx\,dy-4\,y^2\,dx^2$,

$$\text{or } x^2\,q=x^3\,p+2\,x\,y\,p-4\,y^2.$$

Make $y=x^2\,u$, $p=t\,x$, $q=v$, and therefore

$$v=t+2\,u\,t-4\,u^2.$$

Also $dy=x^2\,du+2\,u\,x\,dx=p\,dx=t\,x\,dx$,

$$\text{and } \therefore \frac{dx}{x}=\frac{du}{t-2\,u}$$

Also $dp=q\,dx=v\,dx$,

$$\text{or } x\,dt+t\,dx=v\,dx:$$

$$\therefore \frac{dx}{x}=\frac{dt}{v-t}=\frac{dt}{2\,u\,t-4\,u^2},$$

$$\therefore \frac{du}{t-2\,u}=\frac{dt}{2\,u\,t-4\,u^2},$$

$$\text{or } 2\,u\,du\,(t-2\,u)=dt\,(t-2\,u).$$

Let $2\,u\,du=dt$, and $\therefore t=u^2+c$;

$$\therefore \frac{dx}{x}=\frac{du}{u^2-2\,u+c}.$$

If $c=1$, $x^2=(x^2-y)\log\frac{x}{c'}$.

If $c = 1 - e^2$, $x^2 = c' \left\{ \frac{(e+1)\,x^2 - y}{(e-1)\,x^2 + y} \right\}^{\frac{1}{e}}$.

If $c = 1 + e^2$, $\frac{y - x^2}{e\,x^2} = \tan \left\{ e \log \frac{x}{c'} \right\}$.

If $t - 2\,u = 0$, we have $\frac{d\,x}{x} = \frac{d\,u}{t - 2\,u}$, and $\therefore\ d\,u = 0$;

$$\therefore\ u = c, \text{ and } y = c\,x^2:$$

this is a particular solution. Euler. *Ib.* Art. 820.

IX. Equations in which y and its differentials amount to the same dimension in every term.

(36). Let $a\,y\,d^2 y + b\,d\,y^2 = \frac{y\,d\,x\,d\,y}{\sqrt{(e^2 + x^2)}}$;

Make $y = \epsilon^{\int u\,d\,x}$: and the equation becomes

$$d\,u + u^2\,d\,x = \frac{u\,d\,x}{a\sqrt{(e^2 + x^2)}} - \frac{b\,u^2\,d\,x}{a};$$

by making $u = \frac{1}{t}$, this becomes

$$d\,s + \frac{s\,d\,x}{a\sqrt{(e^2 + x^2)}} = \left(1 + \frac{b}{a}\right) d\,x,$$

a linear equation of the first order, whose integral is

$$c'\,y^{\frac{a+b}{a}} = c + \frac{1}{(a+1)} \left\{ x + \sqrt{(e^2 + x^2)} \right\}^{\frac{a+1}{a}}$$

$$+ \frac{e^2}{1-a} \cdot \left\{ x + \sqrt{(e^2 + x^2)} \right\}^{\frac{1-a}{a}}.$$

Euler. *Ib.* Art. 828.

(37). Let $x y d^2 y = y d x d y + x d y^2 + \dfrac{b x d y^2}{\sqrt{(a^2-x^2)}}$.

Make $y = \epsilon^{\int u d x}$, and the equation becomes

$$d u + u^2 d x = \frac{u d x}{x} + u^2 d x + \frac{b u^2 d x}{\sqrt{(a^2-x^2)}};$$

$$\therefore \frac{x d u - u d x}{u^2} = \frac{b x d x}{\sqrt{(a^2-x^2)}};$$

$$\therefore u = \frac{x}{c + b\sqrt{(a^2-x^2)}}, \int u d x = \int \frac{x d x}{c + b\sqrt{(a^2-x^2)}}$$

$$= -\frac{\sqrt{(a^2-x^2)}}{b} + \frac{c}{b^2} \log \left\{\frac{c+b\sqrt{(a^2-x^2)}}{c'}\right\} = \log y.$$

Euler. *Ib.* Art. 830.

X. When the equations are linear with respect to one of the variables and its differentials.

(38). Let $d^2 y + A d y d x + B d x^2 = 0$, where A and B are constant.

Make $y = \epsilon^{\int u d x}$, from whence we get

$$d u + u^2 d x + A u d x + B d x = 0.$$

Consequently

$$d x = -\frac{d u}{u^2 + A u + B}.$$

(1). Let $u^2 + A u + B = (u-a)(u-b)$, then

if $u = a$, we have $y = c\,\epsilon^{a x}$;
if $u = b$, we have $y = c'\,\epsilon^{b x}$;

and the complete integral is

$$y = c\,\epsilon^{a x} + c'\,\epsilon^{b x}.$$

(2). If $a = b$, then we have

$$y = \epsilon^{ax}(c + c' x).$$

(3). If $a = \alpha + \beta \sqrt{(-1)}$, and $b = \alpha - \beta \sqrt{(-1)}$, we have

$$y = \epsilon^{\alpha x} \{ c \epsilon^{\beta x \sqrt{(-1)}} + c' \epsilon^{-\beta x \sqrt{(-1)}} \}$$

$$= \epsilon^{\alpha x}(c \cos \beta x + c' \sin \beta x),$$

where $C = c + c'$, and $C' = (c - c')\sqrt{(-1)}$

Euler. *Ib.* Art. 775.

(39). Let $d^2 y + 5\, dy\, dx + 4\, y\, dx^2 = 0$,

$$\therefore y = c\,\epsilon^{-x} + c'\,\epsilon^{-4x}.$$

(40). Let $d^2 y - 4\, dy\, dx + 13\, y\, dx^2 = 0$,

$$y = \epsilon^{2x} \{ c \cos 3x + c' \sin 3x \}$$

$$= C \epsilon^{2x} \sin(3x + C');$$

making $c = C \sin C'$, and $c' = C \cos C'$.

(41). Let $d^2 y - 10\, dy\, dx + 25\, y\, dx^2 = 0$,

$$y = \epsilon^{5x}(c + c' x).$$

(42). Let $d^2 y + \dfrac{A\, dy\, dx}{x} + \dfrac{B\, y\, dx^2}{x^2} = 0.$

Make $\dfrac{dx}{x} = dt$; then we have

$$\frac{dy}{dx} = \frac{dy}{x\, dt},$$

$$\frac{d^2 y}{dx^2} = \frac{d^2 y}{x^2\, dt^2} - \frac{dy}{x^2\, dt}.$$

Substituting these expressions, we find

$$\frac{d^2 y}{d t^2} + (A-1)\frac{d y}{d t} + B y = 0,$$

the integral of which is given in Ex. 38. Euler, *Ib.* Art. 847.

(43). Let $d^2 y + \frac{A\,dy\,dx}{a + b x} + \frac{B\,y\,dx^2}{(a+bx)^2} = 0.$

Make $\frac{dx}{a + bx} = dt$, and therefore

$$\frac{dy}{dx} = \frac{dy}{(a+bx)\,dt}, \quad \frac{d^2 y}{dx^2} = \frac{d^2 y}{(a+bx)^2\,dt^2} - \frac{b\,dy}{(a+bx)^2\,dt};$$

and the equation becomes

$$\frac{d^2 y}{d t^2} + (A - b)\frac{d y}{d t} + B y = 0. \qquad \text{(Ex. 38).}$$

(44). Let $d^2 y - \frac{n\,dy\,dx}{x} + A x^n\,dy\,dx + B x^{2n} y\,dx^2 = 0.$

Make $x^n\,dx = dt$, and the equation becomes

$$\frac{d^2 y}{d t^2} + \frac{A\,d y}{d t} + B y = 0 \qquad \text{(Ex. 38.)}$$

Euler, *Ib.* Art. 850.

(45). Let $\frac{d^2 y}{d x^2} + y = 0.$

$$\therefore\ y = c \sin x + c' \cos x.$$

Woodhouse. Physical Astronomy, p. 23.

(46). Let $\frac{d^2 y}{dx^2} + y = X$, where X is a function of x.

$$\therefore\ y = c \sin x + c' \cos x + \sin x \int X \cos x\, d x$$
$$- \cos x \int X \sin x\, d x.$$

Woodhouse. *Ib.* p. 97.

(47). Let $\frac{d^2 y}{d x^2} + y + A \cos m x = 0.$

$$y = c \sin x + c' \cos x + \frac{A \cos m x}{m^2 - 1} - \frac{A \cos x}{m^2 - 1}.$$

Woodhouse, *Ib.* p. 99.

(48). Let $\frac{d^2 y}{d x^2} + y + A \cos x = 0.$

$$\therefore\ y = c \sin x + c' \cos x - \frac{A x}{2} \cos x.$$

Woodhouse, *Ib.* p. 107.

(49). Let $\frac{d^2 y}{d x^2} + y + A \cos m x + B \cos n x = 0.$

$$\therefore\ y = c \sin x + c' \cos x + \frac{A \cos m x}{m^2 - 1} + \frac{B \cos n x}{n^2 - 1}$$
$$- \left(\frac{A}{m^2 - 1} + \frac{B}{n^2 - 1}\right) \cos x.$$

Woodhouse, *Ib.* p. 100.

(50). Let $\frac{d^2 y}{d x^2} + A \frac{d y}{d x} + B y = X$

$$y = \frac{\epsilon^{a x} \{ c + \int \epsilon^{-a x} X\, d x \}}{a - b} + \frac{\epsilon^{b x} \{ c' + \int \epsilon^{-b x} X\, d x \}}{b - a},$$

where a and b are roots of the equation

$$u^2 + A u + B = 0.$$

(51). Let $\frac{d^2 y}{d x^2} + \frac{A\,d y}{d x} + B y = n\,(n-1)\,x^{n-2}$

$$+ n A x^{n-1} + B x^n.$$

$$y = \frac{\epsilon^{ax} \{ c - \epsilon^{-ax} (b x^n - n x^{n-1}) \}}{a-b}$$

$$+ \frac{\epsilon^{bx} \{ c' - \epsilon^{-bx} (a x^n - n x^{n-1}) \}}{b-a}$$

$$= \frac{c\,\epsilon^{ax}}{a-b} - \frac{c'\,\epsilon^{bx}}{a-b} + x^n$$

$$= C\,\epsilon^{ax} + C'\,\epsilon^{bx} + x^n.$$

Euler, *Ib.* Art. 859.

(52). Let $\frac{d^2 y}{d x^2} + \frac{2\,d y}{d x} + 2 y = x.$

The roots of the equation $u^2 + 2u + 2 = 0$, are $-1 + \sqrt{(-1)}$, and $-1 - \sqrt{(-1)}$: consequently

$$y = \epsilon^{-x} \cos x \{ c - \textstyle\int \epsilon^x x\,d x \cos x \}$$

$$+ \epsilon^{-x} \sin x \{ c' + \textstyle\int \epsilon^x x\,d x \sin x \}.$$

$$= \epsilon^{-x} \cos x \{ c - \frac{\epsilon^x}{2} (x \sin x - x \cos x + \cos x) \}$$

$$+ \epsilon^{-x} \sin x \{ c' - \frac{\epsilon^x}{2} (x \cos x + x \sin x - \sin x) \}$$

$$= \epsilon^{-x} (c \cos x + c' \sin x) + \frac{1}{2} (x - 1).$$

(53). Let $\frac{d^2 y}{d x^2} - \frac{6\,d y}{d x} + 13\,y = A \cos m x.$

The roots of the equation $u^2 - 6u + 13 = 0$

are $3+2\sqrt{(-1)}$ and $3-2\sqrt{(-1)}$; therefore

$$y=\epsilon^{3x}(c\cos 2x+c'\sin 2x)$$

$$+\frac{A}{4}\cdot\frac{\{(m+5)\sin(m+2)x+(m-1)\cos(m+2)x\}}{m^2+4m+13.}$$

$$+\frac{A}{4}\cdot\frac{\{(m-5)\sin(m-2)x-(m+1)\cos(m-2)x\}}{m^2-4m+13}.$$

(54). Let $\dfrac{d^2y}{dx^2}-\dfrac{4\,dy}{dx}+4y=x^2.$

The roots of the equation $u^2-4u+4=0$, are 2 and 2.

$$\therefore\ y=\epsilon^{2x}\{x\textstyle\int\epsilon^{-2x}x^2\,dx-\int\epsilon^{-2x}x^3\,dx\}$$

$$=\epsilon^{2x}(c+c'x)+\frac{x^2}{2^2}+\frac{4x}{2^3}+\frac{6}{2^4}.$$

(55). Let $\dfrac{d^2y}{dx^2}+\dfrac{dy}{x\,dx}-\dfrac{y}{x^2}=\dfrac{1}{x^2-1}.$

A particular integral of the equation

$$\frac{d^2v}{dx^2}+\frac{dv}{x\,dx}-\frac{v}{x^2}=0,$$

is $y=\dfrac{1}{x}$: consequently

$$y=v\int\frac{\epsilon^{-\int\frac{dx}{x}}}{v^2}dx\ \{\textstyle\int\epsilon^{\int\frac{dx}{x}}v\,X\,dx\}$$

$$=\frac{1}{x}\left(\frac{x^2}{2}+c\right)\ \left\{c'+\frac{1}{2}\log\frac{x-1}{x+1}\right\}$$

(56). Let $\dfrac{d^2y}{dx^2}-\dfrac{3y}{(1-x^2)^2}=\dfrac{2}{(1-x^2)^{\frac{3}{2}}}.$

A particular integral of the equation

$$\frac{d^2 v}{d x^2} - \frac{3 v}{(1 - x^2)^2} = 0,$$

is $v = \frac{x}{\sqrt{(1 - x^2)}}$: therefore

$$y = v \int \frac{d x}{v^2} \int \frac{2 v \, d x}{(1 - x^2)^{\frac{3}{2}}}$$

$$= \frac{x}{\sqrt{(1 - x^2)}} \left\{ c - \left(x + \frac{1}{x} \right) \right\} \left\{ c' + \frac{1}{1 - x^2} \right\}$$

$$= \frac{\{ c' (1 - x^2) + 1 \} \{ c x - (1 + x^2) \}}{(1 - x^2)^{\frac{3}{2}}}.$$

(57). Let $d^2 y (1 - a x^2) = b x \, d x \, d y$.

Make $d y = p \, d x$. See Ex. 17. p. 372.: we thus get

$$p = c (1 - a x^2)^{-\frac{b}{2a}}, \text{ and } y = c \int d x \, (1 - a x^2)^{-\frac{b}{2a}}.$$

Euler, *Calc. Integ. Tom.* IV. p. 534.

(58). Let $d^2 y (1 - a x^2) - a x \, d x \, d y - c y \, d x^2 = 0$.

Multiply by $d y$ and integrate: we thus get

$$d y^2 (1 - a x^2) - c y^2 d x^2 = C \, d x^2; \text{ or}$$

$$\frac{d x}{\sqrt{(1 - a x^2)}} = \frac{d y}{\sqrt{(C + c y^2)}}, \text{ the integral of which is}$$

$$2 \gamma y = C' \{ \alpha x + \sqrt{(1 + \alpha^2 x^2)} \}^{\frac{\gamma}{\alpha}}$$

$$- \frac{C}{C'} \cdot \{ \alpha x + \sqrt{(1 + \alpha^2 x^2)} \}^{-\frac{\gamma}{\alpha}}$$

where $\alpha^2 = - a$, and $\gamma^2 = C$.

There are many cases in which it is possible to transform the equation

$$d^2 y\,(1-a\,x^2)-b\,x\,d\,x\,d\,y-c\,y\,d\,x^2=0,$$

so that it may coincide with one of the two preceding examples: thus, if we assume $y=\frac{d\,v}{d\,x}$, the equation becomes

$$d^3 v\,(1-a\,x^2)-b\,x\,d\,x\,d^2 v-c\,d\,x^2\,d\,v=0;$$

and by integrating every term and collecting the results, we find

$$d^2 v\,(1-a\,x^2)-(b-2\,a)x\,d\,x\,d\,v-(c-b+2\,a)\,v\,d\,x^2=0,$$

an equation in every respect similar to the original and admitting of integration in the same cases.

Thus if $c-b+2\,a=0$, or if $b=3\,a$, the equation admits of integration.

By making $v=\frac{d\,v'}{d\,x}$, and repeating the same process, we find

$$d^2 v'\,(1-a\,x^2)-(b-4\,a)\,x\,d\,x\,d\,v'+(c-2\,b+6\,a)\,v'\,d\,x^2=0;$$

which admits of integration, when $b-4\,a=a$, or when

$$c-2\,b+6\,a=0.$$

By continuing this process, we should find that the equation admits of integration, whenever $b=2\,i\,a+a$, or when $c=i\,b-i(i+1)\,a$, i being a positive whole number.

Another transformation of this equation may be effected, by supposing $y=\int z\,d\,x$, which gives us

$$d\,z\,(1-a\,x^2)-b\,x\,z\,d\,x-c\,d\,x\int z\,d\,x=0,$$

which becomes, by differentiation,

$$d^2 z\,(1-a\,x^2)-(b+2\,a)\,x\,d\,x-(c+b)\,z\,d\,x^2=0;$$

which admits of integration, when $b+2\,a=a$ or when $c+b=0$: a repetition of this process will shew, that the integration may be effected, whenever

$b = -2\,i\,a + a$ or when $c = -i\,b - i\,(i-1)\,a$.

Euler, *Ib.* p. 534.

(59). Let $\frac{d^3 y}{d x^3} + A \frac{d^2 y}{d x^2} + B \frac{d y}{d x} + C y = 0$.

If a, a', a'' be the roots of the equation

$$u^3 + A u^2 + B u + C = 0,$$

then we have

$$y = c\,\epsilon^{a x} + c'\,\epsilon^{a' x} + c''\,\epsilon^{a'' x}.$$

If $a' = \alpha + \beta\sqrt{(-1)}$ and $a'' = \alpha - \beta\sqrt{(-1)}$, then

$$y = c\,\epsilon^{a x} + C' \cos \beta x + C'' \sin \beta x.$$

If $a = a' = a''$, then we have

$$y = \epsilon^{a x}\,(C + C' x + C'' x^2)$$

Euler, *Calc. Integ.* Tom. II. Art. 1117.

(60.) Let $y - \frac{2\,d y}{d x} + \frac{2\,d^2 y}{d x^2} - \frac{2\,d^3 y}{d x^3} + \frac{d^4 y}{d x^4} = 0$.

The roots of the equation

$$1 - 2\,u + 2\,u^2 - 2\,u^3 + u^4 = 0$$

are 1, 1, $\sqrt{(-1)}$, $-\sqrt{(-1)}$: therefore

$$y = \epsilon^x\,(c + c' x) + c'' \cos x + c''' \sin x.$$

Euler, *Ib.* Art. 1131.

(61). Let $a^2 b^2 y + (a^2 + b^2) \frac{d^2 y}{d x^2} + \frac{d^4 y}{d x^4} = 0$.

$$y = c \cos a x + c' \sin a x + c'' \cos b x + c''' \sin b x.$$

Euler, *Ib.* Art. 1132.

(60). Let $a^4 y + \frac{2 a^2 d^2 y}{d x^2} + \frac{d^4 y}{d x^4} = 0$.

$$y = (c + c' x) \cos a x + (C + C' x) \sin a x.$$

Euler, *Ib.* Art. 1132.

(61). Let $a^4 y - \frac{d^4 y}{d x^4} = 0$.

$$y = c\, \epsilon^{ax} + c'\, \epsilon^{-ax} + C \cos (a x + C').$$

Euler, *Ib.* Art. 1136.

(62). Let $a^5 y - \frac{d^5 y}{d x^5} = 0$.

$$y = c\, \epsilon^{ax} + \epsilon^{a x \cos \frac{2 \pi}{5}} \left\{ c' \cos a x \sin \frac{2 \pi}{5} + c'' \sin a x \sin \frac{2 \pi}{5} \right\}$$

$$+ \epsilon^{a x \cos \frac{4 \pi}{5}} \left\{ C' \cos a x \sin \frac{4 \pi}{5} + C'' \sin a x \sin \frac{4 \pi}{5} \right\}$$

$$= c\, \epsilon^{ax} + c_{,}\, \epsilon^{ax \cos 72^\circ} \cos (a x \sin 72^\circ + c_2)$$

$$+ C_{,}\, \epsilon^{-ax \cos 36^\circ} \cos (a x \sin 36^\circ + C_2).$$

Euler. *Ib.* Art. 1136.

(63). Let $a^6 y + \frac{d^6 y}{d x^6} = 0$.

$$y = c\, \epsilon^{\frac{ax\sqrt{3}}{2}} \cos \left(\frac{1}{2} a x + c' \right)$$

$$+ c_1 \cos (a x + c'_1)$$

$$+ c_2\, \epsilon^{-\frac{ax\sqrt{3}}{2}} \cos \left(\frac{1}{2} a x + c'_2 \right).$$

Euler, *Ib.* Art. 1136.

(64). Let $\frac{d^3 y}{d x^3} + \frac{A\, d^2 y}{(a + b x)\, d x^2} + \frac{B\, d y}{(a + b x)^2 d x} + \frac{C y}{(a + b x)^3} = 0$.

Make $\frac{d\,x}{a+b\,x} = d\,t$, and the equation becomes

$$\frac{d^3\,y}{d\,t^3} + (A-3\,b)\frac{d^2\,y}{d\,t^2} + (B-A\,b+2\,b^2)\frac{d\,y}{d\,t} + C\,y = 0\,;$$

the solution of which is given above. Ex. 57. p. 392.

(65). Let $\frac{d^3\,y}{d\,x^3} - 19\frac{d\,y}{d\,x} + 30\,y = x^2 \cos x$.

The roots of the equation $u^3 - 19\,u + 30 = 0$, are 2, 3 and -5: therefore

$$\begin{aligned}
y = {} & c\epsilon^{2x} + c'\,\epsilon^{3x} + c''\,\epsilon^{-5x} \\
& + \frac{1}{7 \times 5}(2\cos x - \sin x)\left(x^2 + \frac{4\,x}{5} + \frac{6}{25}\right) \\
& - \frac{1}{8 \times 10}(3\cos x - \sin x)\left(x^2 + \frac{3\,x}{5} + \frac{4}{25}\right) \\
& + \frac{1}{7 \times 8 \times 26}(5\cos x + \sin x)\left(x^2 - \frac{5\,x}{13} + \frac{12}{169}\right) \\
& - \frac{2}{7 \times 25}(\cos x + 2\sin x)\left(x + \frac{4}{5}\right) \\
& + \frac{2}{8 \times 100}(\cos x + 3\sin x)\left(x + \frac{3}{5}\right) \\
& + \frac{2}{7 \times 8 \times 26^2}(\cos x - 5\sin x)\left(x - \frac{5}{13}\right):
\end{aligned}$$

the integrations being performed by means of the formula,

$$\begin{aligned}
\int \epsilon^{-ax} X \cos m\,x\,d\,x = {} & -\frac{\epsilon^{-a\,x}}{a^2+m^2}\Big\{(a\cos m\,x \\
& - m\sin m\,x)\left(X + \frac{a}{a^2+m^2}\cdot\frac{d\,X}{d\,x} + \frac{a^2-m^2}{(a^2+m^2)^2}\cdot\frac{d^2\,X}{d\,x^2}\right. \\
& \left. + \frac{a\,(a^2-3\,m^2)}{(a^2+m^2)^3}\cdot\frac{d^3\,X}{d\,x^3} + \&c.\right) - \frac{m}{a^2+m^2}(m\cos m\,x \\
& + a\sin m\,x)\left(\frac{d\,X}{d\,x} + \frac{2\,a}{a^2+m^2}\cdot\frac{d^2\,X}{d\,x^2}\right. \\
& \left. + \frac{3\,a^2-m^2}{(a^2+m^2)^2}\cdot\frac{d^3\,X}{d\,x^3} + \&c.\right)\Big\}.
\end{aligned}$$

(66). Let $\frac{d^4 y}{d x^4} - \frac{8 d^3 y}{d x^3} + \frac{23 d^2 y}{d x^2} - \frac{28 d y}{d x} + 12 y = x.$

The roots of the equation $u^4 - 8 u^3 + 23 u^2 - 28 u + 12 = 0$, are 2, 2, 1, 3 : therefore

$$y = \frac{\epsilon^{2x} \{ x \int \epsilon^{-2x} x \, d x - \int \epsilon^{-2x} x^2 \, d x \}}{(2-1)(2-3)}$$

$$+ \frac{\epsilon^{x} \int \epsilon^{-x} x \, d x}{(1-2)^2 (1-3)} + \frac{\epsilon^{3x} \int \epsilon^{-3x} x \, d x}{(3-2)^2 (3-1)}$$

$$= \epsilon^{2x} (c + c' x) + C \epsilon^{x} + C' \epsilon^{3x} + \frac{x}{12} + \frac{7}{36}.$$

(67). Let $\frac{d^4 y}{d x^4} - 54 \frac{d^2 y}{d x^2} - 216 \frac{d y}{d x} - 240 y = x.$

The roots of the equation $u^4 - 54 u^2 - 216 u - 240 = 0$, are -3, -3, -3, and 9 : consequently, making $a = -3$, we have

$$y = \frac{\epsilon^{-3x} \{ x^2 \int \epsilon^{3x} x \, d x - 2 x \int \epsilon^{3x} x^2 \, d x + \int \epsilon^{3x} x^3 \, d x \}}{2 (a-9)}$$

$$+ \epsilon^{-3x} \{ x \int \epsilon^{3x} x \, d x - \int \epsilon^{3x} x^2 \, d x \} \frac{d \left(\frac{1}{a-9} \right)}{d a}$$

$$+ \frac{\epsilon^{-3x} (\int \epsilon^{3x} x \, d x)}{2} \frac{d^2 \left(\frac{1}{a-9} \right)}{d a^2} + \frac{\epsilon^{9x} \int \epsilon^{-9x} x \, d x}{(9-a)^3};$$

and since $\frac{d . \frac{1}{(a-9)}}{d a} = - \frac{1}{12^2}$, and $\frac{d^2 . \frac{1}{(a-9)}}{d a^2} = - \frac{2}{12^3}$,

we have

$$y = \epsilon^{-3x} (c + c' x + c'' x^2) + C \epsilon^{9x} - \frac{64 x}{12^3 \times 9} + \frac{512}{12^3 \times 9^2}.$$

(68). Let $\frac{d^4 y}{d x^4} - 10 \frac{d^3 y}{d x^3} + 62 \frac{d^2 y}{d x^2} - 210 \frac{d y}{d x} + 261 y = \epsilon^x$.

The roots of the equation

$$u^4 - 10 u^3 + 62 u^2 - 210 u + 261 = 0$$

are $2 + 5 \sqrt{(-1)}$, $2 - 5 \sqrt{(-1)}$, 3, 3, or of the form

$\alpha + \beta \sqrt{(-1)}$, $\alpha - \beta \sqrt{(-1)}$, a and a: therefore

$$y = \frac{\epsilon^{a x} \{ x \int \epsilon^{-a x} X d x - \int \epsilon^{-a x} X x d x \}}{(a - \alpha)^2 + \beta^2}$$

$$- \frac{2 (a - \alpha) \epsilon^{a x} \int \epsilon^{-a x} X d x}{\{ (a - \alpha)^2 + \beta^2 \}^2}$$

$$+ \frac{\epsilon^{\alpha x} \{ 2 (a - \alpha) \beta^2 \cos \beta x - [\beta^2 - (a - \alpha)^2] \beta \sin \beta x \} \int \epsilon^{-\alpha x} X d x \cos}{4 (a - \alpha)^2 \beta^4 + [(a - \alpha)^2 - \beta^2]^2 \beta^2}$$

$$+ \frac{\epsilon^{\alpha x} \{ 2 (a - \alpha) \beta^2 \sin \beta x + [\beta^2 - (a - \alpha)^2] \beta \cos \beta x \} \int \epsilon^{-\alpha x} X d x \sin}{4 (a - \alpha)^2 \beta^4 + [(a - \alpha)^2 - \beta^2]^2 \beta^2}$$

$$+ (c + c' x) \epsilon^{3x} + (C \cos 5 x + C' \sin 5 x) \epsilon^{2x}$$

$$+ \frac{\epsilon^x}{104}.$$

(69). Let $6 y - 6 x \frac{d y}{d x} + 3 x^2 \frac{d^2 y}{d x^2} - x^3 \frac{d^3 y}{d x^3} + x = 0$.

Make $\frac{d x}{x} = d t$, and the equation becomes

$$\frac{d^3 y}{d t^3} - 6 \frac{d^2 y}{d t^2} + 11 \frac{d y}{d t} - 6 y = \epsilon^t:$$

consequently

$$y = \frac{\epsilon^t \int \epsilon^{-t} \times \epsilon^t d t}{2} - \epsilon^{2t} \int \epsilon^{-2t} \times \epsilon^t d t$$

$$+\frac{\epsilon^{3t}\int\epsilon^{-3t}\times\epsilon^{t}\,dt}{2}$$

$$=c\,\epsilon^{t}+c'\,\epsilon^{2t}+c''\,\epsilon^{3t}+\frac{t\,\epsilon^{t}}{2}+\frac{3\,\epsilon^{t}}{4}$$

$$=C\,x+c'\,x^{2}+c''\,x^{3}+x\log\sqrt{x}.$$

(70). Let $\frac{d^n y}{d x^n}+a^n y=X$, where n is an odd number.

One root of $u^n+a^n=0$, is $-a$, and the rest are included in the quadratic formula,

$$u^2-2\,a\,u\cos\theta+a^2=0,$$

where $\theta=(2i+1)\frac{\pi}{n}$, admits of every *different* value which can arise from integral values of i.

If $P=u^n+a^n$, we have $\frac{dP}{du}=n\,u^{n-1}$, which becomes, when $u=a\cos\theta\pm a\sqrt{(-1)}\sin\theta$,

$=n a^{n-1}\{\cos(n-1)\theta\pm\sqrt{(-1)}\sin(n-1)\,\theta\}=A\pm B\sqrt{(-1)}$:

consequently

$$y.=\frac{\epsilon^{-ax}}{n\,a^{n-1}}\int\epsilon^{ax}\,X\,dx$$

$$-\frac{\epsilon^{\ \cos\theta}\{2A\cos(ax\sin\theta)+2B\sin(ax\sin\theta)\}\int\epsilon^{-ax\cos\theta}X\,dx\cos(ax\sin\theta)}{A^2+B^2}$$

$$-\frac{\epsilon^{\ \cos\theta}\{2A\sin(ax\sin\theta)-2B\cos(ax\sin\theta)\}\int\epsilon^{-ax\cos\theta}X\,dx\sin(ax\sin\theta)}{A^2+B^2}$$

c.

and since $A=n\,a^{n-1}\cos(n-1)\,\theta=-n\,a^{n-1}\cos v$,

$$B = n\,a^{n-1} \sin (n-1)\,\theta = n\,a^{n-1} \sin \theta, \text{ and}$$

$$A^2 + B^2 = n^2\,a^{2n-2}, \text{ we find}$$

$$y = \frac{\epsilon^{-ax}}{n\,a^{n-1}} \int \epsilon^{ax}\,X\,d\,x$$

$$- \frac{2\,\epsilon^{a\,x \cos \theta}}{n\,a^{n-1}} \left\{ \cos (\theta + a\,x \sin \theta) \int \epsilon^{-a\,x \cos \theta}\,X d\,x \cos (a\,x \sin \theta) \right.$$

$$\left. + \sin (\theta + a\,x \sin \theta) \int \epsilon^{-a\,x \sin \theta}\,X\,d\,x \sin (a\,x \sin \theta \right\} - \&\text{c.} :$$

the number of terms, including the first, being $\frac{n+1}{2}$.

Euler. *Calc. Integ.* Tom. II. p. 1189.

(71). Let $\frac{d^n\,y}{d\,x^n} - a^n\,y = X$, where n is an odd number.

In this case

$$y = - \frac{\epsilon^{ax}}{n\,a^{n-1}} \int \epsilon^{-ax}\,X\,d\,x$$

$$- \frac{2\,\epsilon^{a\,x \cos \theta}}{n\,a^{n-1}} \left\{ \cos (\theta + a\,x \sin \theta) \int \epsilon^{-a\,x \cos \theta}\,X\,d\,x \cos (a\,x \sin \theta) \right.$$

$$\left. + \sin (\theta + a\,x \sin \theta) \int \epsilon^{-a\,x \cos \theta}\,X\,d\,x \sin (a\,x \sin \theta) + \&\text{c.} \right\}$$

where $\theta = \frac{2\,i\,\pi}{n}$, and the number of terms including the first is $\frac{n+1}{2}$. Euler. *Ib.* Art. 1192.

(72). Let $\frac{d^4\,y}{d\,x^4} - y = \cos x$.

$$\therefore\ y = - \frac{\epsilon^{x}}{4} \int \epsilon^{-x} \cos x\,d\,x + \frac{\epsilon^{-x}}{4} \int \epsilon^{x} \cos x\,d\,x$$

$$+\frac{1}{2}\{\sin x\int(\cos x)^2\,dx-\cos x\,.\int\cos x\sin x\,dx\}$$

$$=c\,\epsilon^{x}+c'\,\epsilon^{-x}+\frac{1}{4}\cos x$$

$$+\frac{\sin x}{2}\left(C+\frac{x}{2}+\frac{\sin 2x}{4}\right)-\frac{\cos x}{2}\left(C'-\frac{\cos 2x}{4}\right)$$

$$=c\,\epsilon^{x}+c'\,\epsilon^{-x}+c_1\sin x+c_2\cos x+\frac{x\sin x}{4}.$$

(73). Let $X=y+\frac{dy}{dx}+\frac{d^2y}{dx^2}+\&c.\ldots\ldots+\frac{d^ny}{dx^n}$.

The factors of

$$1+u+u^2+u^3+\ldots\ldots\ldots\ldots+u^n=\frac{1-u^{n+1}}{1-u}=P,$$

are included in the formula

$$1-2u\cos\theta+u^2,$$

where $\theta=\frac{2i\pi}{n+1}$: by substituting $u=\cos\theta\pm\sqrt{(-1)}\sin\theta$, in $\frac{dP}{du}$, and reducing the result by making $\sin(n+1)\theta=0$, $\cos(n+1)\theta=1$, and therefore $\sin n\theta=-\sin\theta$, and $\cos n\theta=\cos\theta$, we get

$$A\pm B\sqrt{(-1)}=-\frac{1}{2}(n+1)(1+2\cos\theta)\pm\frac{1}{2}(n+1)\times$$

$$\frac{\sin\theta\,(1-2\cos\theta)}{1-\cos\theta}\sqrt{(-1)},\text{ and }A^2+B^2=\frac{(n+1)^2}{2\,(1-\cos\theta)}:$$

consequently

$$y=-\frac{4}{n+1}\,\epsilon^{x\cos\theta}\sin\frac{\theta}{2}\left\{\sin\frac{1}{2}(3\theta+2x\sin\theta)\times\right.$$

$$\int \epsilon^{-x\cos\theta} X\,dx \cos(x\sin\theta) - \cos\frac{1}{2}(3\,\theta + 2\,x\sin\theta)\,X$$

$$\left.\int \epsilon^{-x\cos\theta} X\,dx \sin(x\sin\theta)\right\} - \&c.,$$

making θ successively equal to $\frac{2\,\pi}{n+1}$, $\frac{4\,\pi}{n+1}$, $\frac{6\,\pi}{n+1}$, &c., as long as it continues less than π.

If $n+1$ be an even number, $1+u$ is a factor of P', and we must add $\frac{2}{n+1}\,\epsilon^{-x}\int \epsilon^{x} X\,dx$ to complete the integral. Euler. *Ib.* Art. 1194.

(74). Let $X = y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \&c.$ *in infinitum.*

where $X = a + b\,x + c\,x^2 + d\,x^3 + e\,x^4 + \&c.$

Let $v = A\,\epsilon^{x\cos\theta}\cos(x\sin\theta + \phi)$ be an integral of the equation

$$0 = v + \frac{dv}{dx} + \frac{d^2v}{dx^2} + \&c.$$

where θ and ϕ are any angles whatever: then we have

$$y = (a-b) + (b-2\,c)\,x + (c-3\,d)\,x^2 + (d-4\,e)\,x^3 + \&c. + v$$

$$= X - \frac{dX}{dx} + A\,\epsilon^{x\cos\theta}\cos(x\sin\theta + \phi).$$

Euler. *Ib.* 1198.

(75). Let $X = y + \frac{n\,dy}{dx} + \frac{n\,(n-1)}{1\,.\,2}\,.\,\frac{dy^2}{dx^2}$

$$+ \frac{n\,(n-1)\,(n-2)\,d^2y}{1\,.\,2\,.\,3\,dx^3} + \&c.$$

In this case, the factors of

$$1+nu+\frac{n(n-1)}{1.2}u^2+\&c. = (1+u)^n$$

are equal to each other: therefore

$$y=\frac{\epsilon^{-x}}{1.2\ldots.(n-1)}\left\{x^{n-1}\int\epsilon^x X\,dx-(n-1)\,x^{n-2}\int\epsilon^x X\,x\,dx\right.$$

$$+\frac{(n-1)(n-2)}{1.2}x^{n-3}\int\epsilon^x X\,x^2\,dx$$

$$\left.-\frac{(n-1)(n-2)(n-3)}{1.2.3}x^{n-4}\int\epsilon^x X\,x^3\,dx+\&c.\right\}.$$

If $X=\epsilon^{-x}$, the integral becomes,

$$y=\frac{x^n\epsilon^{-x}}{1.2.3..n}+\frac{\epsilon^{-x}}{1.2\ldots(n-1)}\{c_1x^{n-1}+c_2x^{n-2}$$

$$+\&c.\ldots\ldots\ldots\ldots c_{n-1}x+c_n\}.$$

Euler. *Ib.* Art. 1200.

(76). Let $X=\frac{ny}{a}-\frac{n(n-1)}{1.2a^2}\frac{dy}{dx}$

$$+\frac{n(n-1)(n-2)}{1.2.3a^3}\frac{d^2y}{dx^2}-\&c.$$

$$\therefore P=\frac{a^n-(a-u)^n}{a^nu},$$

whose factors are contained in the formula

$$a^2-2a(a-u)\cos 2\theta+(a-u)^2,$$

where $2\theta=\frac{2i\pi}{n}$: the integral is

$$y=\frac{4a^2\sin\theta}{n}\epsilon^{2ax(\sin\theta)^2}\times$$

$$\{ \sin (3\,\theta - a\,x \sin 2\,\theta) \int \epsilon^{-2ax(\sin\theta)^2} X\,dx \cos (a\,x \sin 2\,\theta)$$

$$+ \cos(3\,\theta - a\,x \sin 2\,\theta) \int \epsilon^{-2ax(\sin\theta)^2} X\,dx \sin (a x \sin 2\theta) \} + \&c.$$

putting for θ the values $\frac{\pi}{n}$, $\frac{2\,\pi}{n}$, $\frac{3\,\pi}{n}$, &c. as long as they continue less than $\frac{\pi}{2}$.

If x be an even number, we must add the integral

$$-\frac{2\,a^2}{n}\,\epsilon^{2ax} \int \epsilon^{-2ax} X\,dx.$$

If $n = \infty$, and $a = n$, the equation becomes

$$X = y - \frac{d\,y}{1\,.\,2\,.\,d\,x} + \frac{d^2\,y}{1\,.\,2\,.\,3\,.\,d\,x^2} - \frac{d^3\,y}{1\,.\,2\,.\,3\,.\,4\,.\,dx^3} + \&c.;$$

and its integral, when $\theta = \frac{i\,\pi}{n}$ is indefinitely small, and therefore $a\,x \sin 2\,\theta = 2\,i\,\pi\,x$, is

$$y = 4\,i\,\pi \{ \cos (2\,i\,\pi\,x) \int X\,d\,x \sin (2\,i\,\pi\,x)$$

$$- \sin (2\,i\,\pi\,x) \int X\,d\,x \cos (2\,i\,\pi\,x) \} + \&c.$$

omitting those parts of the expression which are indefinitely small, and putting 1, 2, 3, &c. successively for i. Euler. *Ib.* Art. 1206, 1209.

(77). Let $X = y + n\,\frac{(n-1)}{1\,.\,2}\,.\,\frac{d^2\,y}{a^2\,dx^2}$

$$+ \frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,a^4}\,.\,\frac{d^4\,y}{dx^4} + \&c.$$

$$\therefore\ P = \frac{(a+u)^n + (a-u)^n}{2\,a^n},$$

and $y = + \frac{2a (\cos\theta)^{n-2}}{n} \{ \sin(a x \tan\theta) \int X dx \cos(a x \tan\theta)$

$- \cos(a x \tan\theta) \int X d x \sin(a x \tan\theta \} - \&c.$

writing successively for θ the arcs $\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}$, &c. as long as they continue less than $\frac{\pi}{2}$; the terms being alternately + and —.

If $n = \infty$, and $a = e n$, the equation becomes

$$X = y + \frac{d^2 y}{1 . 2 e^2 d x^2} + \frac{d^4 y}{1 . 2 . 3 . 4 e^4 d x^4} + \&c.$$

Since θ is in this case indefinitely small, we have $\cos\theta = 1$, and $\tan\theta = \theta$: consequently

$$\frac{y}{2e} = \sin\phi \int X d x \cos\phi - \cos\phi \int X d x \sin\phi$$

$$- \sin 3\phi \int X d x \cos 3\phi + \cos 3\phi \int X d x \sin 3\phi$$

$$+ \sin 5\phi \int X d x \cos 3\phi - \cos 3\phi \int X d x \cos 5\phi$$

$-$ &c. *in infinitum.*

where $\phi = \frac{\pi e x}{2}$.

If $e = b \sqrt{(-1)}$, we have

$$X = y - \frac{d^2 y}{1 . 2 . b^2 d x^2} + \frac{d^4 y}{1 . 2 . 3 . 4 b^4 d x^4} - \&c.$$

and the integral gives

$$\frac{y}{b} = \epsilon^{-\psi} \int \epsilon^{\psi} X d x - \epsilon^{\psi} \int \epsilon^{-\psi} X d x$$

$- \epsilon^{-3\psi} \int \epsilon^{3\psi} X d x + \epsilon^{3\psi} \int \epsilon^{-3\psi} X d x +$ &c. *in infinitum.*

where $\psi = \frac{\pi b x}{2}$. Euler. *Ib.* Arts. 1211, 1214.

XI. On the integration of equations of the second and higher orders, which satisfy the criteria of integrability.

If u be the complete differential of u', and if we put x_1, x_2, x_3, &c. y_1, y_2, y_3, &c. for dx, d^2x, d^3x, &c, dy, d^2y, d^3y, &c. we shall find

$$\frac{du}{dx} - d\frac{du}{dx_1} + d^2\frac{du}{dx_2} - d^3\frac{du}{dx_3} + \&c. = 0. \qquad (1).$$

$$\frac{du}{dy} - d\frac{du}{dy_1} + d^2\frac{du}{dy_2} - d^3\frac{du}{dy_3} + \&c. = 0; \qquad (2).$$

the number of equations being equal to the number of variables in u.

If u result from *two successive* differentiations of u', whose order is therefore inferior by two, we shall likewise find

$$\frac{du}{dx_1} - 2d\frac{du}{dx_2} + 3d^2\frac{du}{dx_3} - 4d^3\frac{du}{dx_4} + \&c. = 0. \qquad (3).$$

$$\frac{du}{dy_1} - 2d\frac{du}{dy_2} + 3d^2\frac{du}{dx_3} - 4d^3\frac{du}{dy_4} + \&c. = 0. \qquad (4).$$

If u result from *three successive differentiations* of u', we shall find, in addition to equations (1), (2), (3), (4),

$$\frac{du}{dx_2} - 3d\frac{du}{dx_3} + 6d^2\frac{du}{dx_4} - 10d^3\frac{du}{dx_5} + \&c. = 0. \qquad (5).$$

$$\frac{du}{dy_2} - 3d\frac{du}{dy_3} + 6d^2\frac{du}{dy_4} - 10d^3\frac{du}{dy_5} + \&c. = 0. \qquad (6).$$

and so on.

These equations, which are denominated *equations of condition*, were discovered by Euler.

(78). Let $u = x\,d^2 y - y\,d^2 x$;

$$\therefore \frac{d\,u}{d\,x} = y_2,\quad \frac{d\,u}{d\,x_1} = 0,\quad \frac{d\,u}{d\,x_2} = y,$$

$$\frac{d\,u}{d\,y} = -x_2,\quad \frac{d\,u}{d\,y_1} = 0,\quad \frac{d\,u}{d\,y_2} = x.$$

Consequently

$$\frac{d\,u}{d\,x} - d\,\frac{d\,u}{d\,x_1} + d^2\,\frac{d\,u}{d\,x_2} = y_2 - d^2 y = 0.$$

$$\frac{d\,u}{d\,y} - d\,\frac{d\,u}{d\,y_1} + d^2\,\frac{d\,u}{d\,y_2} = -x_2 + d^2 x = 0.$$

The function u is therefore a complete differential,

and $u' = x\,dy - y\,d\,x$.

(79). Let $u = x^2\,d^2 y + (a+2)\,x\,d\,y\,d\,x + (a\,y + 2\,x)\,d\,x^2$
$+ (a\,x\,y + x^2)\,d^2 x$.

$$\therefore \frac{d\,u}{d\,x} = 2\,x\,y_2 + (a+2)\,y_1\,x_1 + 2\,x_1^{\,2} + (a\,y + 2\,x)\,x_2$$

$$d\,\frac{d\,u}{d\,x_1} = (a+2)\,(x\,d\,y_1 + y\,d\,x) + 2\,(a\,y + 2\,x)\,d\,x_1$$
$$+ 2\,x_1\,(a\,d\,y + 2\,d\,x)$$

$$d^2\,\frac{d\,u}{d\,x_2} = a\,(x\,d^2 y + 2\,d\,x\,d\,y + y\,d^2 x) + 2\,x\,d^2 x + 2\,d\,x^2;$$

$$\frac{d\,u}{d\,y} = a\,x_1^{\,2} + a\,x\,x_2$$

$$d\,\frac{d\,u}{d\,y_1} = (a+2)\,(x\,d\,x_1 + x_1^{\,2}) = (a+2)\,(x\,x_2 + x_1^{\,2})$$

$$d^2\,\frac{d\,u}{d\,y_2} = 2\,(x\,d\,x_1 + x_1^{\,2}) = 2\,(x\,x_2' + x_1^{\,2}).$$

It is readily shewn that, by the substitution of these quantities, the general equations (1) and (2) are satisfied: and

$$u' = x^2\, dy + a\,x\,y\,dx + x^2\,dx + C.$$

(80). Let $u = (a\,x - 2\,y)\,d^2 y - 2\,dy^2 + 2\,a\,dy\,dx$

$$+ a\,y\,d^2 x.$$

In this case, we shall find

$$\frac{du}{dx} - d\frac{du}{dx_1} + d^2\frac{du}{dx_2} = 0.$$

$$\frac{du}{dx_1} - 2\,d\frac{du}{dx_2} = 0:$$

$$\frac{du}{dy} - d\frac{du}{dy_1} + d^2\frac{du}{dy_2} = 0.$$

$$\frac{du}{dy_1} - 2\,d\frac{du}{dy_2} = 0:$$

consequently u results from two differentiations of u'', which

$$= a\,x\,y - y^2 + C.$$

The test furnished by these equations is of universal application, but the laborious operations which it usually requires, render it expedient to seek for other conditions of greater simplicity, at least for those cases which are of most frequent occurrence. Thus, if we take the general equation of the second order,

$$u = P\,dx^2 + Q\,dx\,dy + R\,dy^2 + S\,d^2 x + T\,d^2 y,$$

we shall find, from the preceding *general equations* of *condition*, whenever $u = d\,u'$, that

$$P = \frac{dS}{dx},\ R = \frac{dT}{dy} \text{ and } Q = \frac{dT}{dx} + \frac{dS}{dy}. \qquad (a.)$$

If $S=0$, or if dx be constant, the equations of condition are

$$R=\frac{d\,T}{d\,y} \text{ and } \frac{d\,Q}{d\,x}=\frac{d\,P}{d\,y}+\frac{d^2\,T}{d\,x^2}. \qquad (b).$$

(81). Let $u=(x^2+x\,y)\,d^2\,y-y^2\,d^2\,x+x\,d\,y^2$

$$+\,(2\,x-y)\,d\,y\,d\,x.$$

In this case $S=-y^2$ and $P=\frac{d\,S}{d\,x}=0$: the other conditions (a) are also verified, and

$$u'=(x^2+x\,y)\,d\,y-y^2\,d\,x+C.$$

(82). Let $u=2\,x^3\,y^2\,d^2\,x+\frac{3\,y\,d^2\,y}{x}+6\,x^2\,y^2\,d\,x^2$

$$+\frac{3\,d\,y^2}{x}+4\,x^3\,y\,d\,x\,d\,y-\frac{3\,y\,d\,x\,d\,y}{x^2}.$$

All the conditions (a) are satisfied,

$$\text{and } u'=2\,x^3\,y^2\,d\,x+\frac{3\,y}{x}\,d\,y+C.$$

Any equation of the second order, and of the first degree with respect to d^2y, in which $d\,x$ is constant, is reducible to the form,

$$(P\,dp+Q\,d\,x)\,d\,x^m=u\,d\,x^m,$$

by making $d\,y=p\,d\,x$ and therefore $d^2y=dp\,dx$: the function $P\,d\,p+Q\,d\,x$ will arise from the differentiation of $u'=\int P\,d\,p+V$, where V is a function of x and y only, whenever we have

$$\frac{d^2\,Q}{d\,p^2}=\frac{d^2\,P}{d\,x\,d\,p}+\frac{2\,d\,P}{d\,y}+\frac{d^2\,P}{d\,y\,d\,p}\,p. \qquad (\alpha.)$$

$$\frac{d\,Q}{d\,y}-\frac{d^2Q}{dp\,dx}-\frac{d^2Q}{dp\,dy}\,p+\frac{d^2P}{d\,x^2}+\frac{2\,d^2\,P}{d\,x\,d\,y}\,p+\frac{d^2P}{d\,y^2}\,p^2=0. \qquad (\beta.)$$

(83). Let $u\,dx^2 = (2\,xy\,dy + x^2y\,dx)\,d^2y + x\,dy^3 +$

$$(y+x^2)\,dy^2\,dx + (2+3\,y)\,xy\,dy\,dx^2 + y^3\,dx^3,$$

$$\therefore\ P = 2\,xyp + x^2y \text{ and } Q = x\,p^3 + (y+x^2)\,p^2$$

$$+ (2+3\,y)\,xy\,p + y^3.$$

The equations (α) and (β) are satisfied and therefore

$$u' = \int P\,dp + V$$

$$= xyp^2 + x^2yp + V:$$

Differentiating u' and comparing it with u, we find

$$\frac{dV}{dx}\,dx + \frac{dV}{dy}\,dy = y^3\,dx + 3\,xy^2\,dy, \text{ and}$$

$$\therefore\ V = xy^3 + C: \text{ consequently}$$

$$u'\,dx^2 = (xyp^2 + x^2yp + xy^3 + C)\,dx^2.$$

(84). Let $u = 3\,a\,x^3p^2\,dp - b\,x\,dp + 3\,a\,x^2p^3\,dx - b\,p\,dx$

$$+ c\,xp\,dx + c\,y\,dx.$$

The equations of condition are satisfied and

$$u' = \int P\,dp + V = a\,x^3p^3 - b\,xp + V, \text{ and}$$

$$\frac{dV}{dx}\,dx + \frac{dV}{dy}\,dy = c\,x\,dy + c\,y\,dy;\ \therefore\ V = c\,xy + C.$$

Consequently $u' = a\,x^3p^3 - b\,xp + c\,xy + C$.

Particular equations of condition have been investigated for the equation of the third order,

$$P\,dx^3 + Q\,dx^2\,dy + R\,dx\,d^2y + S\,dx\,dy^2$$

$$+ T\,dy\,d^2y + V\,dy^3 + Z\,d^3y = 0,$$

and formulæ given for its integration, when these conditions are satisfied: this subject however has already exceeded the limits consistent with the object of this work, and we are therefore compelled to omit them.

XII. On the Integration of Differential Equations of the second and higher Orders, by means of Multipliers.

The equation

$$d^2 y + Q\, d y^2 + R\, d x\, d y + S\, d x^2 = 0$$

where Q, R and S are functions of x and y only, is integrable when multiplied by the factor,

$$z = \epsilon^{(\int T d x + Q d y)},$$

where

$$T = \frac{\frac{d^2 Q}{d x^2} - R \frac{d Q}{d x} + S \frac{d Q}{d y} + Q \frac{d S}{d y} + \frac{d^2 S}{d y^2} - \frac{d^2 R}{d x\, d y}}{\frac{d R}{d y} - 2 \frac{d Q}{d x}},$$

if the following equations be satisfied:

1st, $\frac{d T}{d y} = \frac{d Q}{d x}$ and therefore $T d x + Q d y$ *integrable per se.*

2d, $Q S + \frac{d S}{d y} = T(R - T) + \frac{d R}{d x} - \frac{d T}{d x}$.

(85). Let $x^2 y\, d^2 y + 2 x^2 d y^2 + (2 x + 3 x y)\, d x\, d y$

$$+ 2 y\, d x^2 = 0;$$

$\therefore\ Q = \frac{2}{y}$, $R = \frac{2 + 3 y}{x y}$, $S = \frac{2}{x^2}$; consequently

$T = \frac{3}{x}$, $\int (T d x + Q d y) = \log x^3 y^2$ and $z = x^3 y^2$.

Multiplying the equation by $x^3 y^2$ and integrating as in Ex. 83, and 84, we find

$$x^3 y^2 \frac{dy}{dx} + x^2 y^3 = C.$$

(86). Required the factor necessary to make the equation

$$d^2 y + A\,dy\,dx + B\,y\,dx^2 = X\,dx^2$$

integrable *per se.*

Make $d^2 y = p\,dx$ and multiply by $V = f(x)$; then we have

$$V\,dp + A\,V\,p\,dx + B\,V\,y\,dx = V\,X\,dx;$$

the integral of which, by the method in p. 408, is

$$V p + S = \int V\,X\,dx: \text{ consequently}$$

$$dS = -p\,dV + A\,V\,p\,dx + B\,V\,y\,dx$$

$$= dy\left(A\,V - \frac{dV}{dx}\right) + B\,V\,y\,dx,$$

which is integrable, when $V = \epsilon^{ax}$, which gives

$$S = (A - a)\,\epsilon^{ax}\,y, \text{ if we likewise have}$$

$$(A - a)\,a = B \text{ or } a^2 - A\,a + B = 0.$$

We thus get the first integral of the equation, which is, making $A - a = b$,

$$dy + b\,y\,dx = \epsilon^{-ax} dx \int \epsilon^{ax}\,X\,dx,$$

which is linear and of the first order: consequently multiplying by ϵ^{bx}, where b is the second root of the equation $a^2 - A\,a + B = 0$, we find

$$y = \frac{1}{b-a}\,\epsilon^{-ax} \int \epsilon^{ax}\,X\,dx + \frac{1}{a-b}\,\epsilon^{-bx} \int \epsilon^{bx}\,X\,dx.$$

Euler. *Calc. Integ. Tom.* II. Art. 865.

(87). Let $X = Ay + Bx\dfrac{dy}{dx} + Cx^2\dfrac{d^2y}{dx^2} + Dx^3\dfrac{d^3y}{dx^3}$. ($\alpha$).

Let the factor which renders the equation integrable *per se*, be x^λ: suppose the integration performed, and we get

$$\int x^\lambda X\,dx = A'x^{\lambda+1}y + B'x^{\lambda+2}\frac{dy}{dx} + C'x^{\lambda+3}\frac{d^2y}{dx^2}. \quad (\beta).$$

differentiating this equation, dividing by $x^\lambda dx$, and comparing the result with equation (α), we find

$$A = (\lambda+1)\,A'$$

$$B = (\lambda+2)\,B' + A'$$

$$C = (\lambda+3)\,C' + B'$$

$D = (\lambda+4)\,D' + C' = C'$: consequently

$$(\lambda+1)\,A' = A$$

$$(\lambda+1)(\lambda+2)\,B' = (\lambda+1)\,B - A$$

$$(\lambda+1)(\lambda+2)(\lambda+3)\,C' = (\lambda+1)(\lambda+2)\,C - (\lambda+1)\,B + A$$

$$(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)\,D' = (\lambda+1)(\lambda+2)(\lambda+3)\,D$$
$$- (\lambda+1)(\lambda+2)\,C + (\lambda+1)\,B - A = 0.$$

If $(\lambda+1)(\lambda+2)(\lambda+3)\,D - (\lambda+1)(\lambda+2)\,C + (\lambda+1)B - A$

$= D(\lambda-a)(\lambda-b)(\lambda-c) = P$, then x^a, x^b, x^c,

are severally the factors required.

If we divide equation (β) by x^{a+1}, we get

$$x^{-a-1}\int x^a X\,dx = X'$$

$$= A'y + B'x\frac{dy}{dx} + C'x^2\frac{d^2y}{dx^2}, \quad (\gamma)$$

an equation similar to (α) in form, but whose dimensions are inferior by unity.

If we treat equation (γ) in the same manner as (α) we shall find that the values of λ, in the factor x^{λ}, are determined from the equation

$$Q = (\lambda+1)(\lambda+2)C' - (\lambda+1)B' + A' = 0,$$

which results from the division of P by $\lambda - a$: consequently $Q = D(\lambda - b)(\lambda - c)$: and the factors are x^b or x^c. We thus get the equation

$$x^{-b-1}\int x^b X' dx = X'' = A'' y + B'' \frac{x\,dy}{dx}, \qquad (\delta),$$

which is of the first order: a similar process will give us

$$x^{-c-1}\int x^c X'' dx = A''' y = Dy \qquad (\epsilon),$$
$$= x^{-c-1}\int x^{c-b-1} dx \int X' x^b dx$$
$$= x^{-c-1}\int x^{c-b-1} dx \int x^{b-a-1} X dx;$$

$$\text{or } Dxy = \frac{x^{-a}\int x^a X dx}{(b-a)(c-a)} + \frac{x^{-b}\int x^b X dx}{(a-b)(a-c)}$$
$$+ \frac{x^{-c}\int x^c X dx}{(c-a)(c-b)}.$$

If $X = 0$, the equation becomes

$$Dxy = Cx^{-a} + C'x^{-b} + C''x^{-c},$$

the denominators being included in the arbitrary constants C, C', C''.

It is evident that the same method of investigation may be pursued, whatever be the order of the original equation.

This very elegant process is given by Euler*, who has developed the complete integrals, when any number of the roots of $P = 0$, are equal or imaginary.

The same illustrious analyst by the research of factors of the form $P\,dx + Q\,dy$, $P\,dx^2 + R\,dx\,dy + Q\,dy^2$, &c.

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* *Calc. Integ.* Tom. II. Art. 1226.

where P, Q, R, &c. are functions of x and y, has succeeded in the integration of several equations of the second order, which resist all other methods: we shall subjoin two or three of the most remarkable, omitting all detail of the processes of deducing them, which do not admit of any material abridgement.

(88). Let $\frac{d^2 y}{d x} + 2\, d y \left(\frac{1}{x} + \frac{n x}{a^2 + x^2}\right)$

$$+ \frac{n(n+1)\, y\, d x}{a^2 + x^2} = 0.$$

The multiplier is

$$x^2 (a^2 + x^2)^{n+1} \left\{ \frac{d y}{d x} + \frac{y (a^2 + n x^2)}{x (a^2 + x^2} \right\},$$

and the integral

$$\frac{x^2}{2} (a^2 + x^2)^{n+1} \left\{ \left[\frac{d y}{d x} + \frac{y (a^2 + n x^2)}{x (a^2 + x^2} \right]^2 + \frac{(n-1)^2 a^2 y^2}{(a^2 + x^2)^2} \right\} = C.$$

Euler. *Ib.* Art. 893.

(89). Let $2\, a\, y\, d^2 y - 4\, a\, d y^2 - y^{n+5}\, d x^2 (1 + x^2)^{\frac{n-1}{2}} = 0.$

The factor is $\frac{x\, d x}{y^2} + \frac{(1 + x^2)\, d y}{y^3}$, and the integral

$$\frac{a\, d x^2}{y^2} + \frac{2\, a\, x\, d x\, d y}{y^3} + \frac{a (1 + x^2)\, d y^2}{y^4}$$

$$- \frac{1}{n+3}\, y^{n+1}\, d x^2 (1 + x^2)^{\frac{n+1}{2}} = C\, d x^2.$$

Euler. *Nov. Comm. Petrop.* Tom. VII. 1761. p. 173.

(90). Let $d^2 y + \frac{A\, y\, d x^2}{(a + 2\, b\, x + c\, x^2 + e\, y^2)^2} = 0.$

The factor is $2\,dy\,(a+2bx+cx^2) - 2y\,dx\,(b+cx)$ and the integral

$$\frac{dy^2}{dx^2}(a+2bx+cx^2) - \frac{2y\,dy}{dx}(b+cx)$$
$$-\frac{A(a+2bx+cx^2)}{e(a+2bx+cx^2+ey^2)} + cy^2 = C.$$

If the equation be $\dfrac{d^2y}{dx^2} + \dfrac{a^2y}{(y^2+x^2)^2} = 0$, the first integral is

$$x^2\frac{dy^2}{dx^2} - \frac{2xy\,dy}{dx} + y^2 + \frac{a^2y^2}{x^2+y^2} = c^2;$$

if we make $y=ux$, we shall find

$$\frac{dx}{x^2} = \frac{du\,\sqrt{(1+u^2)}}{\sqrt{[c^2+(c^2-a^2)\,u^2]}},$$

where the variables are separated. Euler. *Opuscula.* Tom. I. p. 82. *Calc. Integ.* Art. 906.

(91). Let $y^2\,d^2y + y\,dy^2 = ax\,dx^2$:

The factor is $3y\,dy^2 - 3ax\,dx^2$, and the integral

$y^3\,dy^3 - 3axy^2\,dx^2\,dy + ay^3\,dx^3 + a^2x^3\,dx^3 = C\,dx^3$.

Euler. *Nov. Comm. Petrop.* Tom. VII. p. 190.

On the Integration of Simultaneous Differential Equations.

(1). LET the simultaneous equations of the first order be

$$\left.\begin{aligned} A\,dx + B\,dy + (Cx + Dy)\,dt &= T\,dt \\ A'\,dx + B'\,dy + (C'x + D'y)\,dt &= T'\,dt \end{aligned}\right\} \quad (a).$$

where A, A', B, B', C, C', D and D' are constant quantities, and T and T' are functions of t.

Eliminate $d\,y$ and $d\,x$ successively from these equations (α), when we get

$$\left.\begin{aligned} d\,x+(a\,x+b\,y)\,d\,t&=\theta\,d\,t \\ d\,y+(a'\,x+b'\,y)\,d\,t&=\theta'\,d\,t \end{aligned}\right\}. \qquad (\beta).$$

Multiply the second of these equations by m, and add the result to the first: assume m_1 and m_2 to represent the two values of m in the equation

$$m=\frac{b+b'\,m}{a+a'\,m};$$

and make $a+a'\,m_1=r_1$, $a+a'\,m_2=r_2$, $\theta+m_1\,\theta'=V_1$ and $\theta+m_2\,\theta'=V_2$: then we have

$$x+m_1\,y=\epsilon^{-r_1 t}\left\{\int\epsilon^{r_1 t}\,V_1\,d\,t+C_1\right\},$$

$$x+m_2\,y=\epsilon^{-r_2 t}\left\{\int\epsilon^{r_2 t}\,V_2\,d\,t+C_2\right\};$$

from which the functions of t which are equal to x and y may be found.

(2). Let

$$\left.\begin{aligned} 4\,d\,x+9\,d\,y+(44\,x+49\,y)\,d\,t&=t\,d\,t \\ 3\,d\,x+7\,d\,y+(34\,x+38\,y)\,d\,t&=\epsilon^t\,d\,t \end{aligned}\right\}. \qquad (\alpha).$$

$$\therefore\ \left.\begin{aligned} d\,x+(2\,x+y)\,d\,t&=(7\,t-9\,\epsilon^t)\,d\,t \\ d\,y+(4\,x+5\,y)\,d\,t&=(4\,\epsilon^t-3\,t)\,d\,t \end{aligned}\right\}. \qquad (\beta).$$

If $m=\dfrac{4+2\,m}{5+m}$, we find $m_1=1$, and $m_2=-4$, and

$$\therefore\ y+x=\epsilon^{-6t}\int\epsilon^{6t}\,(4\,t-5\,\epsilon^t)\,d\,t$$

$$=c_1\,\epsilon^{-6t}-\frac{5\,\epsilon^t}{7}+\frac{2\,t}{3}-\frac{1}{9},$$

$$y-4x=c_2\,\epsilon^{-t}\int\epsilon^t\,(40\,\epsilon^t-31\,t)\,dt$$
$$=c_2\,\epsilon^{-t}+20\,\epsilon^t-31\,t+31.$$

Consequently

$$y=4\,c_1\,\epsilon^{-6t}+c_2\,\epsilon^{-t}+\frac{24}{7}\,\epsilon^t-\frac{17}{3}\,t+\frac{55}{9},\qquad(\gamma).$$

$$x=c_1\,\epsilon^{-6t}-c_2\,\epsilon^{-t}-\frac{29}{7}\,\epsilon^t+\frac{19}{3}\,t-\frac{56}{9}.\qquad(\delta).$$

If the two differential equations (α) for (β) correspond to two curve surfaces whose co-ordinates are x, y and t respectively, the equations (γ), (δ), are those of the curves resulting from projecting orthographically their common intersection upon the planes of $y\,t$ and $x\,t$.

(3). Let $$\left.\begin{aligned}dx+(5\,x+y)\,dt&=\epsilon^t\,dt\\ dy+(3\,y-x)\,dt&=\epsilon^{2t}\,dt\end{aligned}\right\}.$$

In this case, we find $m=\dfrac{1+3\,m}{5-m}$, and therefore

$m_1=m_2=1$: we thus get

$$x+y=c\,\epsilon^{-4t}+\frac{e^t}{5}+\frac{e^{2t}}{6}:$$

Find the value of x in terms of y and t from this equation, and substitute it in the second of the two differential equations, and we find

$$dy+4\,y\,dt=c\,\epsilon^{-4t}\,dt+\frac{e^t\,dt}{5}+\frac{7\,\epsilon^{2t}\,dt}{6},$$

$$\text{and}\ \therefore y=\frac{(c'+c\,t)}{e^{4t}}+\frac{e^t}{25}+\frac{7\,\epsilon^{2t}}{36}.$$

(4). Let us take the linear equations of the second order,

$$\left.\begin{array}{l}\dfrac{2\,d^2 x}{d t^2} + \dfrac{d x}{dt} + x + \dfrac{d^2 y}{d t^2} + \dfrac{d y}{d t} + y = 0, \\ \dfrac{d^2 x}{d t^2} + 96\,\dfrac{d x}{d t} - 9\,x + \dfrac{d^2 y}{d t^2} + 50\,\dfrac{d y}{d t} + 15\,y = 0.\end{array}\right\}.$$

Make $x = \epsilon^{mt}$, and $y = \alpha\,\epsilon^{mt}$: we thus get

$$\left.\begin{array}{l}2\,m^2 + m + 1 + \alpha\,(m^2 + m + 1) = 0 \\ m^2 + 96\,m - 9 + \alpha\,(m^2 + 50\,m + 15) = 0\end{array}\right\}.$$

Eliminating α, we get the equation

$$m^4 + 4\,m^3 - 7\,m^2 - 22\,m + 24 = 0.$$

The values of m are 1, 2, -3, -4, and the corresponding values of α are $\dfrac{-4}{3}$, $\dfrac{-11}{7}$, $\dfrac{-16}{7}$, and $\dfrac{-29}{13}$: consequently

$$x = c_1\,\epsilon^{t} + c_2\,\epsilon^{2t} + c_3\,\epsilon^{-3t} + c_4\,\epsilon^{-4t},$$

$$y = -\frac{4}{3}\,c_1\,\epsilon^{t} - \frac{11}{7}\,c_2\,\epsilon^{2t} - \frac{16}{7}\,c_3\,\epsilon^{-3t} - \frac{27}{13}\,c_4\,\epsilon^{-4t}.$$

(5). Let us take the three linear simultaneous equations of the second order;

$$\frac{2\,d^2 x}{d t^2} - \frac{18\,dx}{dt} + 58\,x + \frac{3\,d^2 y}{dt^2} + \frac{3\,d y}{d t}$$

$$- 48\,y + \frac{d^2 z}{d t^2} - \frac{3\,d z}{d t} + 2\,z = 0. \qquad (1).$$

$$\frac{d^2 x}{d t^2} + \frac{43\,d x}{d t} - 254\,x - \frac{23\,d^2 y}{d t^2} + \frac{d y}{d t}$$

$$+ 250\,y + \frac{d^2 z}{d t^2} + \frac{17\,d z}{d t} - 36\,z = 0. \qquad (2).$$

$$\frac{3\,d^2 x}{d t^2} + \frac{11\,d x}{d t} - 128\,x - \frac{12\,d^2 y}{d t^2} + \frac{2\,d y}{d t}$$

$$+ 130\,y - \frac{d^2 z}{dt^2} + \frac{13\,dz}{dt} - 18\,z = 0. \qquad (3).$$

Make $x = \epsilon^{mt}$, $y = \alpha\,\epsilon^{mt}$, $z = \beta\,\epsilon^{mt}$, which give

$$\left.\begin{array}{l} 2\,m^2 - 18\,m + 58 + \alpha\,(3\,m^2 + 3\,m - 48) + \beta\,(m^2 - 3\,m + 2) = 0, \\ m^2 + 43\,m - 254 - \alpha\,(23\,m^2 - m - 250) + \beta\,(m^2 + 17\,m - 36) = 0, \\ 3\,m^2 + 11\,m - 128 - \alpha(12\,m^2 - 2\,m - 130) - \beta\,(m^2 - 13\,m + 18) = 0 \end{array}\right\}.$$

Eliminate α and β from these equations and we shall find

$$m^6 - 7\,m^5 + 7\,m^4 + 35\,m^3 - 56\,m^2 - 28\,m + 48 = 0.$$

The values of m in this equation are

$$1,\ 2,\ 3,\ 4,\ -1,\ -2,$$

from whence we find the values of α, which are

$$1,\ 1,\ 2,\ -1,\ 2,\ 3,$$

and of β which are

$$1,\ 2,\ 1,\ -1,\ 3,\ 2:$$

consequently

$$x = c_1\,\epsilon^t + c_2\,\epsilon^{2t} + c_3\,\epsilon^{3t} + c_4\,\epsilon^{4t} + c_5\,\epsilon^{-t} + c_6\,\epsilon^{-2t},$$

$$y = c_1\,\epsilon^t + c_2\,\epsilon^{2t} + 2\,c_3\,\epsilon^{3t} - c_4\,\epsilon^{4t} + 2\,c_5\,\epsilon^{5t} + 3\,c_6\,\epsilon^{-2t}$$

$$z = c_1\,\epsilon^t + 2\,c_2\,\epsilon^{2t} + c_3\,\epsilon^{3t} - c_4\,\epsilon^{4t} + 3\,c_5\,\epsilon^{5t} + 2\,c_6\,\epsilon^{-2t}.$$

Paoli. *Elem. d'. Alg*. Tom. II. p. 236.

(6). Let $\left.\begin{array}{l}(A\,x + B\,y + C)\,dt^2 + A_1\,d^2 x + B_1\,d^2 y = 0 \\ (A'x + B'y + C')\,dt^2 + A_1'\,d^2 x + B_1'\,d^2 y = 0\end{array}\right\}.$

Eliminate $d^2 y$ and $d^2 x$ successively from these equations, when we get

$$\left.\begin{array}{l}(a\,x + b\,y + c)\,dt^2 + d^2 x = 0 \\ (a'\,x + b'\,y + c')\,dt^2 + d^2 y = 0\end{array}\right\}.$$

Multiply the second equation by m, add the result to the first, when we find

$$(a+a'm)\left\{x+\frac{b+b'm}{a+a'm}y+\frac{c+c'm}{a+a'm}\right\}dt^2+d^2x+m\,d^2y=0.$$

Make $m=\dfrac{b+b'm}{a+a'm}$, and $u=x+my+\dfrac{c+c'm}{a+a'm}$; consequently

$$(a+a'm)\,u\,dt^2+d^2u=0.$$

If we make $a+a'm=-k^2$, we shall find

$$u=C\epsilon^{kt}+C'\epsilon^{-kt}.$$

If m_1 and m_2 be the two values of m, we have

$$\left.\begin{aligned} x+m_1y+\frac{c+c'm_1}{a+a'm_1}&=c_1\epsilon^{kt}+c_2\epsilon^{-kt}\\ x+m_2y+\frac{c+c'm_2}{a+a'm_2}&=c_3\epsilon^{k't}+c_4\epsilon^{-k't}\end{aligned}\right\}.$$

(7). Let
$$\left.\begin{aligned} d^2x-(3x+4y-3)\,dt^2&=0,\\ d^2y+(x-8y+5)\,dt^2&=0\end{aligned}\right\}.$$

The values of m are -1, and -4: and therefore

$$x=\frac{1}{7}+4c_1\epsilon^{2t}+4c_2\epsilon^{-2t}-3c_3\epsilon^{\sqrt{7}t}-c_4\epsilon^{\sqrt{7}t},$$

$$y=\frac{9}{14}+c_1\epsilon^{2t}+c_2\epsilon^{-2t}-c_3\epsilon^{\sqrt{7}t}-c_4\epsilon^{-\sqrt{7}t}.$$

(8). Let
$$\left.\begin{aligned}(25x+36y-73)\,dt^2-11\,d^2x-8\,d^2y&=0\\ (16x+23y-46)\,dt^2-7\,d^2x-5\,d^2y&=0\end{aligned}\right\}.$$

Eliminating d^2y and d^2x successively, we get

$$\left.\begin{aligned} d^2x-(3x+4y-3)\,dt^2&=0\\ d^2y+(x+y+5)\,dt^2&=0\end{aligned}\right\}.$$

In this case, we find $m_1 = m_2 = 2$: consequently

$$x + 2y = 13 + c_1 \epsilon^t + c_2 \epsilon^{-t}:$$

In the second of the given equations, substitute for x its value derived from this equation, and we get

$$d^2 y - y\,d t^2 + (18 + c_1 \epsilon^t + c_2 \epsilon^{-t}) d t^2 = 0,$$

the integration of which gives

$$y = c_3 \epsilon^t + c_4 \epsilon^{-t} - \frac{c_1 (2t - 1)}{4} \epsilon^t$$

$$+ \frac{c_2 (2t - 1)}{4} \epsilon^{-t} + 18.$$

It would occupy too much space, if we were to attempt to exemplify all the forms and cases of simultaneous equations which admit of integration, which is less necessary as the methods generally resolve themselves into those which have been given before for the integration of linear equations; the subject however is of considerable importance, as such equations frequently occur in the solution of dynamical problems.

On the Integration of Differential Equations by Series and by Approximation.

(1). Let $d y + y d x = a x^m d x$, in which $x = c$, when $y = b$.

Assume

$$y = b + A(x - c)^{\alpha} + B(x - c)^{\alpha+1} + C(x - c)^{\alpha+2} + \&c.$$

$$= b + A t^{\alpha} + B t^{\alpha+1} + C t^{\alpha+2} + \&c.$$

where $t = x - c$, and therefore $x = c + t$.

Substituting for y, dy and x^m in the original equation and equating corresponding coefficients, we find

$$\alpha = 1,$$

$$A = a\,c^m - b,$$

$$B = \frac{a\,m\,c^{m-1} - a\,c^m + b}{2},$$

$$C = \frac{a\,m\,(m-1)\,c^{m-2} - a\,m\,c^{m-1} + a\,c^m - b}{2\,.\,3}, \text{ \&c.}$$

(2). Let $dy + x\,dy - m\,y\,dx = 0$.

Making the same assumptions as in the last Example, we find

$$y = b\left\{1 + \frac{m\,(x-c)}{c+1} + \frac{m\,(m-1)}{1\,.\,2}\,.\,\frac{(x-c)^2}{(c+1)^2} + \text{\&c.}\right\}$$

$$= b\left\{1 + \frac{x-c}{c+1}\right\}^m = \frac{b}{(c+1)^m}\,(1+x)^m$$

$$= C\,(1+x)^m.$$

(3). Let $d^2 y + a\,x^{m-2}\,dx^2 = 0$:

Assume

$$y = A\,x^{\alpha} + B\,x^{\alpha+\delta} + C\,x^{\alpha+2\delta} + \text{\&c.}$$

Substituting the value of y and $d^2 y$, in the equation, we find, by a comparison of the terms of the result, that we must make $\alpha = 0$ or $\alpha = 1$, and also $\delta = m$: from these two hypotheses we deduce two series for y, which combined, furnish the complete integral of the equation: or

$$y = A\left\{1 - \frac{a\,x^m}{(m-1)\,m} + \frac{a^2\,x^{2m}}{1\,.\,2\,.\,(m-1)\,(2\,m-1)\,m^2}\right.$$

$$\left. - \frac{a^3\,x^{3m}}{1\,.\,2\,.\,3\,.\,(m-1)\,(2\,m-1)\,(3\,m-1)\,m^3} + \text{\&c.}\right\}$$

$$+ A'\left\{x - \frac{a\,x^{m+1}}{(m+1)\,m} + \frac{a^2\,x^{2m+1}}{1\,.\,2\;\;(m+1)\,(2\,m+1)\,m^2} - \frac{a^3\,x^{3m+1}}{1\,.\,2\,.\,3\,.\,(m+1)\,(2\,m+1)\,(3\,m+1)\,m^3} + \&c.\right\}$$

Euler, *Calc. Integ.* Tom. II. Art. 931.

Mr. Spence, the author of an Essay on Logarithmic Transcendents, has deduced the same solution of this equation, by means of his *General Series* for the integration of linear differential equations of the second order, of which it affords an easy and satisfactory exemplification: he has applied the same method to several other cases of the general equation, which have been considered by Euler, where one or both of the series for y fail, in consequence of involving terms which are infinite: the Memoir, in which this method is explained, may be seen amongst the Posthumous Works* of this original and able Analyst, which have been arranged and published by Mr. Herschel.

Both the series for y fail, when $m=(0)$; in this case the equation may be integrated by the method given in Ex. 38. p. 384, which gives

$$y = A\,x^{\frac{1}{2}+\sqrt{(\frac{1}{4}-a)}} + A'\,x^{\frac{1}{2}-\sqrt{(\frac{1}{4}-a)}};$$

which becomes, when $a=\frac{1}{4}$,

$$y = (A + A'\log x)\sqrt{x}.$$

If $m=\frac{1}{i}$, where i is a whole number, the first series fails and the second when $m=-\frac{1}{i}$: in both cases, assume

$$y = p + c\,q + q\log x = p + q\log C\,x,$$

where q is the particular integral of the equation, furnished by that series which does not fail: the substitution of this value of y in the original equation gives the equations,

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* Spence, *Mathematical Essays*, p. 315. 1819.

$$d^2q + aqx^{m-2}dx^2 = 0, \qquad (\alpha.)$$

$$d^2p + \frac{2\,dx\,dp}{x} - \frac{q\,dx^2}{x^2} + apx^{m-2}dx^2 = 0. \qquad (\beta)$$

the second of which will determine the value of p by means of a series. Euler. *Ib.* Art. 934.

(4). Let $d^2y + \frac{ay\,dx^2}{x} = 0$, where $m = 1$.

In this case

$$q = A'\left\{x - \frac{ax^2}{1\,.\,2} + \frac{a^2x^3}{1\,.\,2^2\,.\,3} - \frac{a^3x^4}{1\,.\,2^2\,.\,3^2\,.\,4} + \&c.\right\}.$$

The equation to determine p is

$$\frac{d^2p}{dx^2} + \frac{2\,dp}{x\,dx} - \frac{q}{x^2} + \frac{ap}{x} = 0.$$

Assume

$$p = A + Bx + Cx^2 + Dx^3 + \&c.$$

and we shall find

$$A = -\frac{A'}{2}, \quad C = \frac{3a\,A'}{1^2\,.\,2^2}, \quad D = -\frac{14\,a^2A'}{1^3\,.\,2^3\,.\,3^2} \;\&c.$$

making $B = 0$; and

$$y = p + cq + q\log x.$$

Euler, *Ib.* Art. 935.

(5). Let $d^2y + \frac{ay\,dx^2}{x^2\sqrt{x}} = 0$, where $m = -\frac{1}{2}$.

If $y = p + cq + q\log x$, we find

$$q = A\left\{1 - \frac{4\,a}{1\,.\,3}x^{-\frac{1}{2}} + \frac{16\,a^2}{1\,.\,3\,.\,2\,.\,4}x^{-1}\right.$$

$$\left. - \frac{64\,a^3}{1\,.\,3\,.\,2\,.\,4\,.\,3\,.\,5}x^{-\frac{3}{2}} + \&c.\right.$$

The equation for the determination of p is

$$d^2 p + \frac{2\,dxdq}{x} - \frac{q\,d\,x^2}{x^2} + \frac{a\,p\,d\,x^2}{x^2\sqrt{x}} = 0.$$

If $p = K\,x + A'\sqrt{x} + B + C\,x^{-\frac{1}{2}} + D\,x^{-1} + E\,x^{-\frac{3}{2}} + \&c.$,

we shall find

$$A' = \frac{A}{a},\ K = \frac{A}{4\,a^2},\ C = -\frac{8\,.\,4\,.\,a\,A}{1^2\,.\,3^2}$$

$$D = \frac{100 \times 16\,a^2\,A}{1^2\,.\,3^2\,.\,2^2\,.\,4^2},\ \&c.\ \text{if}\ B = 0.$$

Euler. *Ib.* Art. 938.

(6). Let $d^2 y - c^2 x^{2m} y\,dx^2 = 0$, which is the same equation as in Ex. 3.

Make $y = q\,\epsilon^{(\int c\,x^m d\,x)} = q\,\epsilon^{\frac{c\,x^{m+1}}{m+1}}$, when the equation becomes

$$d^2 q + 2\,c\,x^m\,d\,x\,d\,q + m\,c\,x^{m-1}\,q\,d\,x^2 = 0. \qquad (\beta.)$$

If we assume

$$q = A\,x^{\alpha} + B\,x^{\alpha + m + 1} + C\,x^{\alpha + 2m + 2} + \&c.$$

and make $\alpha = 0$ and $\alpha = 1$, successively, we shall find

$$q = A\left\{1 - \frac{m\,c\,x^{m+1}}{m\,(m+1)} + \frac{m\,.\,(3\,m+2)\,c^2\,x^{2m+2}}{1\,.\,2\,.\,m\,(2\,m+1)\,(m+1)^2}\right.$$

$$\left. - \frac{m\,(3\,m+2)\,(5\,m+4)\,c^3\,x^{3m+3}}{1\,.\,2\,.\,3\,.\,m\,(2\,m+1)\,(3\,m+2)\,(m+1)^3} + \&c.\right\}$$

$$+ A'\left\{x - \frac{(m+2)\,c\,x^{m+2}}{(m+2)\,(m+1)} + \frac{(m+2)\,(3\,m+4)\,c^2\,x^{2m+3}}{1\,.\,2\,(m+2)\,(2\,m+3)\,(m+1)^2}\right.$$

$$\left. - \frac{(m+2)\,(3\,m+4)\,(5\,m+6)\,c^3\,x^{3m+4}}{1\,.\,2\,.\,3\,.\,(m+2)\,(2\,m+3)\,(3\,m+4)\,(m+1)^3} + \&c.\right\}$$

If we make $m = -2\,\lambda$, the equation becomes

$$d^2\,y - c^2\,y\,x^{-4\lambda}\,d\,x^2 = 0.$$

and we find $y = q\,\epsilon^{\int p\,d\,x} = q \times \epsilon^{-\frac{c\,x^{-2\,\lambda+1}}{2\,\lambda-1}}$:

if we assume

$$q = A\,x^{\lambda} + B\,x^{3\,\lambda-1} + C\,x^{5\,\lambda-2} + D\,x^{7\,\lambda-3} + \&c.$$

and determine q from the equation

$$d^2\,q + 2\,c\,x^{-2\,\lambda}\,d\,x\,d\,q - 2\,\lambda\,c\,x^{-2\,\lambda-1}\,q\,d\,x^2 = 0,$$

we shall find

$$q = A\left\{x^{\lambda} - \frac{\lambda\,(\lambda-1)\,x^{3\,\lambda-1}}{2\,(2\,\lambda-1)\,c}\right.$$

$$+ \frac{\lambda\,(\lambda-1)\,(3\,\lambda-1)\,(3\,\lambda-2)}{2\,.\,4\,(2\,\lambda-1)^2\,c^2}\,x^{5\,\lambda-2}$$

$$\left. - \frac{\lambda\,(\lambda-1)\,(3\,\lambda-1)\,(3\,\lambda-2)\,(5\,\lambda-2)\,(5\,\lambda-3)\,x^{7\,\lambda-3}}{2\,.\,4\,.\,6\,.\,(2\,\lambda-1)^3\,c^3} + \&c.\right\}.$$

The last series is finite whenever $\lambda = \dfrac{i}{2\,i \pm 1}$, or $m = \dfrac{2\,i}{2\,i \pm 1}$: in all these cases therefore we derive from it a particular integral of the equation, in finite terms.

If we denote those terms of this particular integral, which involve the even powers of c by P and those which involve the odd powers of c by $c\,Q$, we shall find the complete integral or

$$y = A\,\epsilon^{-\frac{c x^{1-2\lambda}}{2\lambda-1}}(P + c\,Q) + A'\,\epsilon^{\frac{c x^{1-2\lambda}}{2\lambda-1}}(P - c\,Q),$$

since c must have a double sign.

The integrable cases of the equation

$$d^2 y - c^2 y\, x^n\, d\,x^2 = 0$$

depend upon those of the equation of Riccati: for if we make $y = \epsilon^{\int u dx}$, we find

$$d\,u + u^2 d\,x = c^2 x^n d\,x,$$

which is integrable whenever $n = -\frac{4\,i}{2\,i \pm 1}$. Euler. *Ib.* Art. 944.

(7). Let $d^2 y - c^2 x^{-4} y\, d\,x^2 = 0$:

In this case $\lambda = 1$, $q = A\,x$, $\int p\,d\,x = -\frac{c}{x}$, $P = x$, and $Q = 0$.

$$\therefore y = \{A\,\epsilon^{-\frac{c}{x}} + A'\,\epsilon^{\frac{c}{x}}\}\,x.$$

Euler. *Ib.* Art. 946.

(8). Let $d^2 y - c^2 y\, x^{-\frac{4}{3}}\, d\,x^2 = 0$:

$$\therefore \lambda = \frac{1}{3},\ q = A\left(x^{\frac{1}{3}} - \frac{1}{3\,c}\right),\ \int p\,d\,x = 3\,c\,x^{\frac{1}{3}},$$

$$P = x^{\frac{1}{3}}, \text{ and } Q = \frac{1}{3\,c^2}\text{: consequently}$$

$$y = (A\,\epsilon^{3cx^{\frac{1}{3}}} + A'\,\epsilon^{-3cx^{\frac{1}{3}}})\,x^{\frac{1}{3}} - \frac{1}{3\,c}(A\epsilon^{3cx^{\frac{1}{3}}} - A'\,\epsilon^{-3cx^{\frac{1}{3}}})$$

If $c^2 = -b^2$, we find

$$y = (A \cos 3bx^{\frac{1}{3}} + A' \sin 3bx^{\frac{1}{3}})\, x^{\frac{1}{3}}$$

$$+ \frac{1}{3b} (A' \cos 3bx^{\frac{1}{3}} - A \sin 3bx^{\frac{1}{3}}).$$

Euler. *Ib.* Art. 947.

(9). Let $d^2 y + b^2 y x^{-\frac{8}{5}} y \, dx^2 = 0$,

$$\therefore y = \left(x^{\frac{2}{5}} - \frac{3}{5^2 b^2}\right) (A \cos 5bx^{\frac{1}{5}} + A' \sin 5bx^{\frac{1}{5}})$$

$$+ \frac{3}{5b} x^{-\frac{1}{5}} (A' \cos 5bx^{\frac{1}{5}} - A \sin 5bx^{\frac{1}{5}}).$$

Euler. *Ib.* Art. 949.

(10). Let $x^2 (a + bx^n) d^2 y + x (c + ex^n) dx\, dy$

$$+ (f + gx^n) y\, dx^2 = 0. \qquad (\beta).$$

If we make $x^n = z$, this equation becomes

$$z^2 (a + bz) d^2 y + z (c' + e' z) dy\, dz + (f' + g' z) y\, dx^2 = 0; \quad (\gamma).$$

we may therefore confine our attention to those cases of equation (β), in which $n = 1$.

Assume

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + \&c.;$$

and substituting for y, dy, $d^2 y$ in the original equation (β), when $n = 1$, we shall find A indeterminate, and that α must be determined from the equation

$$\alpha (\alpha - 1) a + \alpha c + f = 0: \quad (\delta).$$

the two values of α furnish two series for y which combined furnish the complete integral of the equation.

One of these series will terminate, when we find that in addition to equation (δ), we also have

$$(a+i)(a+i-1)b+(a+i)e+g=0. \quad (\epsilon).$$

Euler by different transformations of the equation (β), and the theory of multipliers, has found 9 different cases in which this equation admits of complete integration in finite terms*. The same subject has also been completely discussed by the German analyst Pfaff, who has generalized some of the integrable cases of Euler†.

The integration of differential equations by means of series, of which we have given several examples, will likewise be a method of approximation, when those series are convergent, at least for those values of the independent variable, which come within the limits of the calculation: this however is not always the case, and the series rarely admit of such modification, as may increase their convergency.

Methods of approximation were made use of, even in the very infancy of the Integral Calculus, particularly by the two Bernoullies, and Taylor, which principally depend upon what has been termed the geometrical construction of differential equations: however elegant some of these methods are, they cannot be considered as a substitute for the processes of exact calculation.

Euler has given methods of approximation dependent upon the series

$$y = A + A_1 \frac{(x-a)}{1} + A_2 \frac{(x-a)^2}{1.2} + A_3 \frac{(x-a)^3}{1.2.3} + \&c.$$

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* *Inst. Calc. Integ.* Tom. II. Cap. VIII. *Comm. Petrop.* Tom. XVII. 1773.

† *Disquisitiones Analyticæ. Helmstet.* 1797. p. 135.

where A, A_1, A_2 A_3, &c. are the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. when $x = a$, and where two, three or a greater number of them are considered as arbitrary, according as the differential equation is of the first, second, third or a higher order.*

(11). Let $dy = dx\,(x^n + cy)$, and suppose $y = b$, when $x = a$; then we have

$$A = b$$

$$\frac{dy}{dx} = x^n + cy, \quad \therefore\ A_1 = a^n + cb,$$

$$\frac{d^2y}{dx^2} = nx^{n-1} + c\frac{dy}{dx}, \quad \therefore\ A_2 = na^{n-1} + ca^n + c^2b,$$

$$\frac{d^3y}{dx^3} = n(n-1)\,x^{n-1} + c\frac{d^3y}{dx^2}, \quad \therefore\ A_3 = n(n-1)\,a^{n-2}$$

$$+ nca^{n-1} + c^2a^n + c^3b, \text{ \&c.}$$

Consequently,

$$y = b + (a^n + cb)(x-a) + (na^{n-1} + ca^n + c^2b)\frac{(x-a)^2}{1\,.\,2} + \text{\&c.}$$

If we make $x - a = h$, a very small quantity, we shall find $y = b'$, when $x = a + h$ or a'; these values may be substituted again, and a new value of $y = b''$ corresponding to $x = a' + h$ or a'', may be determined: it is evident that the same process may be continued as long as we please. Euler. *Calc. Integ.* Tom. I. Art. 661.

(12). Let $d^2y + \frac{y\,dx^2}{ex} = 0$, supposing that $y = b$, and $\frac{dy}{dx} = c$, when $x = a$;

$$\therefore \frac{d^2 y}{d x^2} = - \frac{y}{e x}, \text{ and } A_2 = - \frac{b}{e a},$$

$$\frac{d^3 y}{d x^3} = \frac{y - x \frac{d y}{d x}}{e x^2}, \text{ and } A_3 = \frac{b - a c}{e a^2}; \text{ \&c.}$$

Consequently,

$$y = b + c(x - a) - \frac{b}{a e} \frac{(x-a)^2}{1 . 2} + \frac{(b - a c)}{e a^2} \frac{(x-a)^3}{1 . 2 . 3} - \text{\&c.}$$

Making $x - a = h$, a small interval, and $x = a' = a + h$, we find $y = b'$, and $\frac{d y}{d x} = c'$, and so on by successive substitutions, we may find an approximate value of y for any given value of x. Euler. *Ib.* Tom. II. Art. 1098.

This method, however, though of easy application, frequently fails in giving the approximation to sufficient accuracy, in consequence of the accumulation of errors, introduced after each operation: it fails entirely, when any one of the coefficients A_1, A_2, A_3, &c. becomes infinite or extremely large, for any value of x employed in the process of approximation.

Lagrange has attempted* to remedy these inconveniences by expressing y in terms of a continued fraction: this method has the advantage of always terminating when y is expressible in finite and rational terms, and leads in general to a certain and frequently rapid approximation: but the equations from which the quotients as terms of the continued fraction are obtained, become very complicated even in the most simple cases. The integrals of some equations which he has expressed in this manner lead to results of uncommon elegance.

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* *Mémoires de Berlin*, 1776. p. 236.

The method of approximation which is commonly called that of *successive substitutions*, combined with the ordinary processes of integrating linear and other equations, is that which admits of the most extensive and important applications: it is this method that is so much employed in works on Physical Astronomy, in which alone it can properly be studied.

◆

On Partial Differential Equations.

In a subject of such difficulty, as the integration of partial differential equations, it would be useless to multiply examples to any great extent, unless they were accompanied with the principal steps in their solution: we are compelled therefore to omit the exemplification of several important parts of this subject, or at least to notice them in so slight a manner, as to serve rather as a guide to the student in the course of his reading, than as furnishing the materials from which a perfect knowledge of the theory may be obtained.

A considerable number of examples with their solutions have been given by Mr. Herschel in Note *O*, to the Translation of Lacroix.

I. On Equations of Total Differentials, of the form

$$P\,dx + Q\,dy + R\,dz = 0,$$

which satisfy the *criteria* of integrability.

(1). Let $\dfrac{y\,dx}{a-z} + \dfrac{x\,dy}{a-z} + \dfrac{x\,y\,dz}{(a-z)^2} = 0.$

In this case, we have

$$\frac{dP}{dy} = \frac{dQ}{dx} = \frac{1}{a-z}, \quad \frac{dP}{dz} = \frac{dR}{dx} = \frac{y}{(a-z)^2},$$

and

$$\frac{dQ}{dz} = \frac{dR}{dy} = \frac{x}{(a-z)^2};$$

consequently, considering z as constant, we have

$$\int (P\,dx + Q\,dy) = \frac{xy}{a-z} + f(z) = v + Z;$$

and

$$Z = \int \left\{ R - \frac{dv}{dz} \right\} = C,$$

therefore

$$\frac{xy}{a-z} = C.$$

(2). Let $\dfrac{x\,dx + y\,dy + z\,dz}{\sqrt{\{x^2 + y^2 + z^2\}}} + \dfrac{z\,dx - x\,dz}{x^2 + z^2} + z\,dz = 0.$

Considering z in the first place as constant, and proceeding as before, we find

$$\sqrt{\{x^2 + y^2 + z^2\}} + \tan^{-1}\frac{x}{z} + \frac{z^2}{2} = C.$$

II. When the equation

$$P\,dx + Q\,dy + K\,dz = 0, \qquad (\alpha).$$

does not satisfy the *criteria* of integrability, but may be made to satisfy them by means of a multiplier, which can only be the case, when the equation of condition

$$P\left\{\frac{dQ}{dz} - \frac{dR}{dy}\right\} + Q\left\{\frac{dR}{dx} - \frac{dP}{dz}\right\}$$
$$+ R\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\} = 0. \qquad (\beta).$$

obtains.

Or, if the equation (α), is reduced to the form

$$dz = p\,dx + q\,dy, \qquad (\gamma.)$$

the equation of condition becomes

$$\frac{dp}{dy} + q\frac{dp}{dz} - \frac{dq}{dx} - p\frac{dq}{dz} = 0. \qquad (\delta.)$$

from which the equation (β) may be readily deduced.

For it is evident, if z be a function of x and y considered as independent variables, that the second member of equation (γ) must be a complete differential: and the equation (δ) is the expression of that condition, p and q being considered as functions of z, x and y.

The process of integration is to consider one of the three variables as constant, and to apply the rules of integration which are given for equations of two variables.

(3). Let $dx(y + z) + dy(x + z) + dz(x + y) = 0$:

The equation (β) is satisfied: make therefore z constant, and we have

$$dx(y + z) + dy(x + z) = 0,$$

or

$$\frac{dx}{x + z} + \frac{dy}{y + z} = 0;$$

therefore

$$\log(x + z) + \log(y + z) = f(z),$$

or

$$(x + z)(y + z) = Z.$$

Differentiating this equation, we find

$$dx(y + z) + dy(x + z) + dz(x + y + 2z) = dZ;$$

consequently

$$dZ = 2z\,dz, \text{ and } Z = z^2 + c:$$

we thus get

$$(x + y)(y + z) = z^2 + c, \text{ or } xy + xz + yz = c.$$

Euler, *Calc. Integ.* Tom. III. Cap. 1. Art. 12.

(4). Let $dx + dy + dz + (x + y + z)\,dz = 0$, therefore, $x + y = f(z) = Z$: consequently,

$$dZ + Z\,dz = -dz - z\,dz.$$

therefore

$$Z = c\,\varepsilon^{-z} - z = x + y,$$

or

$$(x + y + z)\,\varepsilon^{z} = c.$$

(5). Let $2dx(y + z) + dy(x + 3y + 2z) + dz(x + y) = 0$: therefore

$$(x + y)^2 (y + z) = c. \quad \text{Euler. } \textit{Ib.} \text{ Art. 11.}$$

(6). Let $dx(ay - bz) + dy(cz - ax) + dz(bx - cy) = 0$, therefore

$$\frac{ay - bz}{cz - ax} = c. \quad \text{Euler. } \textit{Ib.} \text{ Art. 13.}$$

Other examples are given by Euler, which are remarkable for the artifices of analysis which they exhibit.

III. When the equation (β or δ) is not satisfied.

In this case, the second member of the equation

$$dz = p\,dx + q\,dy,$$

is no longer a complete differential, unless we cease to consider x and y as independent variables, and assume one of them equal to an arbitrary function of the other; it then becomes an equation between two variables, and may be integrated by the ordinary processes for such equations.

The integral will thus be exhibited by a system of two equations, which are necessarily indeterminate, as they both involve an arbitrary function.

As the integral in the former case might be considered as the equation of a determinate curve surface, so in this case it may be considered as the equation of an infinite number of curves of double curvature.

(7). Let $d z = a y d x + b d y$.

The equation of condition is not satisfied; assume therefore $y = \frac{d X}{d x} = X'$, where X is an arbitrary function of x; therefore

$$d z = a X' d x + b d X',$$

and

$$z = a X + b X'.$$

The system of equations

$$y = X' \text{ and } z = a X + b X',$$

constitute the integral of the proposed equation.

(8). Let $z d x + x d y + y d z = 0$.

Make $y = X$, and we shall find

$$z = \varepsilon^{-\int \frac{d x}{X}} \left\{ c - \int \varepsilon^{\int \frac{d x}{X}} \frac{x d X}{X} \right\}.$$

(9). Let $d z^2 = a^2 (d x^2 + d y^2)$. (a.)

Make y constant; consequently we have

$d z = a d y$, $\therefore z = a y + X$, where $X = f(x)$,

and $d z = a d y + d X = a d y + X' d x$;

substitute this value of $d z$, in the given equation (a), and we get

$$2\,dy = \frac{a\,dx}{X'} - \frac{X'\,dx}{a},$$

and $$2\,y = a\int\frac{dx}{X'} - \frac{X}{a}.$$

The condition expressed by the equation (α) is satisfied by any curve of double curvature, whose tangents make a given angle with the plane of $x\,y$; thus the straight line determined by the system of equations

$$x = b\,z + c$$

$$y = \frac{z\sqrt{(1 - a^2 b^2)}}{a} + d,$$

will satisfy the equation.

Monge, in the *Mémoires de l'Académie des Sciences* for 1784, has given general solutions of this equation and of others of the same class, which do not involve an integral sign, and which therefore furnish any number of algebraical solutions: the reader who wishes thoroughly to understand the theory and geometrical character of those equations, would do well to study this *Memoir.*

IV. Partial differential equations in which the differential coefficient $\frac{dz}{dx}$ is given in terms of x, y, and z.

(10). Let $\frac{dz}{dx} = a$,

$$\therefore\ z = a\,x + \phi\,y.$$

Euler, *Calc. Integ.* Tom. III. Art. 33.

(11). Let $\frac{dz}{dx} = \frac{y}{\sqrt{(y^2 - x^2)}}$,

$\therefore\ z = y\,.\sin^{-1}\frac{x}{y} + \phi\,y.$ Euler, *Ib.* Art. 48.

(12). Let $\frac{dz}{dy} = \frac{x^2}{x^2 + y^2}$,

$$\therefore\ z = x \tan^{-1} \frac{y}{x} + \phi x.$$

(13). Let $\frac{dz}{dx} = \frac{y}{z}$,

$\therefore\ z = \sqrt{\{2xy + \phi y\}}$. Euler, *Ib.* Art. 56.

(14). Let $\frac{dz}{dx} = \frac{\sqrt{(y^2 - z^2)}}{z}$,

$\therefore\ dx = \frac{z\,dz}{\sqrt{\{y^2 - z^2\}}}$, $\therefore\ x = \phi y - \sqrt{\{y^2 - z^2\}}$. Euler, *Ib.* Art. 57.

(15). Let $\frac{dz}{dx} = \frac{nz}{x}$,

$\therefore\ z = x^n \phi y$. Euler, *Ib.* Art. 61.

(16). Let $\frac{dz}{dx} = nx - z$. See Ex. p. 329.

$\therefore\ z = n(x - 1) + \varepsilon^{-x} \phi y$. Euler, *Ib.* Art. 62.

(17). Let $\frac{dz}{dx} = \frac{xz}{x^2 + z^2}$. See Ex. p. 325.

$\therefore\ \log z = \frac{x^2}{2z^2} + \phi y$. Euler, *Ib.* Art. 63.

(18). Let $\frac{dz}{dx} = \frac{xz}{ay}$,

$\therefore\ \log z = \frac{x^2}{2ay} + \phi y$; or if $z = c$, when $x = a$, then we have

$$\log \frac{z}{c} = \frac{x^2 - a^2}{2ay} + \phi y.$$

Euler, *Ib.* Art. 69.

(19). Let $\frac{dz}{dx} = \frac{y}{x + z}$, or

$$dx - \frac{x\,dz}{y} = \frac{z\,dz}{y},$$

$\therefore\ \epsilon^{-\frac{z}{y}}(x + y + z) = \phi y$, or if $z = b$, when $x = a$, the integral may be put under the form

$$\epsilon^{-\frac{z}{y}}(x+y+z) - \epsilon^{-\frac{b}{y}}(a+b+y) = \phi y.$$

Euler, *Ib.* Art. 70.

(20). Let $\frac{dz}{dx} = \frac{y^2 + z^2}{y^2 + x^2}$,

$$\therefore\ \tan^{-1}\frac{z}{y} - \tan^{-1}\frac{x}{y} = \tan^{-1}\frac{yz - yx}{y^2 + xz} = \phi y,$$

or if $z = b$, when $x = a$,

$$\tan^{-1}\frac{y(z - x)}{y^2 + x^2} - \tan^{-1}\frac{(b - a)y}{y^2 + x^2} = \phi y,$$

Euler, *Ib.* Art. 71.

V. Equations of the form

$$Pp + Qq = R$$

where $p = \frac{dz}{dx}$, $q = \frac{dz}{dy}$ and P, Q, R are quantities any how involving the variables x, y, z.

The following theorem of Lagrange will enable us to integrate a great number of equations which are included under this very general formula; "if we integrate the equations

$$P\,dx - Q\,dy = 0. \qquad (\alpha).$$

$$\left.\begin{aligned} P\,dz - R\,dx &= 0 \\ \text{or } Q\,dz - R\,dy &= 0 \end{aligned}\right\}. \qquad (\beta).$$

and make the integral of the first $(\alpha) = a$, and of the second $(\beta) = b$, then the complete integral of the proposed equation is

$$b = \phi\,(a),$$

where ϕ denotes an arbitrary function." See *Lacroix*, Art. 314.

The success of this method will depend upon one at least of the equations (α) or (β), involving two variables only: the integral of this will enable us to exterminate, if necessary, one of the three variables from one of the remaining equations, after which it may be integrated in the ordinary manner.

(21). Let $a\,p + b\,q = 1$;

The equations to integrate are

$$\left.\begin{aligned} a\,dy - b\,dx &= 0 \\ a\,dz - dx &= 0 \end{aligned}\right\},$$

and therefore

$$a\,y - b\,x = \alpha$$

$$a\,z - x = \beta.$$

Consequently, since $\beta = \phi\,(\alpha)$, we have

$$z = \frac{x}{a} + \phi\,(a\,y - b\,x).$$

The solution is the equation of all cylindric surfaces whatever be the nature of the base. Monge, *Application de l'Analyse à la Géométrie*. p. 5.

(22). Let $p = \dfrac{q\,x}{a}$;

$$\therefore \; z = \phi\left(\frac{y}{a} + \log x\right).$$

Euler, *Calc. Integ.* Tom. III. Art. 97.

(23). Let $p\,x + q\,y = 0$.

$$\therefore \; z = \phi\left(\frac{x}{y}\right). \qquad \text{Euler, } \textit{Ib.} \text{ Art. 139.}$$

This equation expresses the condition which is fulfilled by all curve surfaces, which are generated by the motion of a straight line, which is always parallel to a given plane, and which always passes through a given line at right angles to that plane. Monge, *Ib.* p. 22.

(24). Let $X\,p + Y\,q = Z$, where X, Y and Z are functions of x, y and z respectively,

$$\therefore \int \frac{dz}{Z} - \int \frac{dx}{X} = \phi\left\{\int \frac{dx}{X} - \int \frac{dy}{Y}\right\}.$$

(25). Let $x^2 p + y^2 q = \dfrac{\sqrt{(1+z^2)}}{z}$;

$$\therefore \; \sqrt{(1+z^2)} + \frac{1}{x} = \phi\left(\frac{x-y}{x\,y}\right).$$

(26). Let $p + q = \dfrac{z}{a}$;

$$\therefore \; z = \varepsilon^{\frac{y}{a}}\phi(x - y). \qquad \text{Euler, } \textit{Ib.} \text{ Art. 129.}$$

(27). Let $m\,p\,x + n\,q\,y = a$,

$$\therefore \; z = \frac{a}{n}.\log y + \phi\left(\frac{x^n}{y^m}\right). \qquad \text{Euler, } \textit{Ib.} \text{ Art. 141.}$$

(28). Let $q = \dfrac{p\,x}{y} + \dfrac{y}{x}$;

The equations to integrate are

$$y\,d\,x + x\,d\,y = 0, \text{ and } d\,z - \frac{y}{x}.\,d\,y = 0.$$

Consequently $xy = \alpha$, and the second equation becomes

$$dz - \frac{y^2\,dy}{\alpha} = 0, \text{ which gives;}$$

$$z - \frac{y^3}{3\,\alpha} = z - \frac{y^2}{3\,x} = \beta.$$

$$\therefore \quad z = \frac{y^2}{3\,x} + \phi\,(x\,y).$$ Euler, *Ib.* Art. 152.

The same process is followed in most of the examples which follow.

(29). Let $p\,x + q\,z + y = 0$;

$$\therefore \quad x\,\varepsilon^{\sin^{-1}\frac{z}{\sqrt{(y^2+z^2)}}} = \phi\,(y^2 + z^2).$$

(30). Let $p + \frac{y}{x}q = \frac{x}{z}$;

$$\therefore \quad z^2 = x^2 + \phi\left(\frac{x}{y}\right).$$

(31). Let $y^2q - x\,y\,p = \frac{a\,x\,z}{y}$;

$$\therefore \quad \log z + \frac{a\,x}{3\,y^2} = \phi\,(x\,y).$$

(32). Let $p\,y + q\,x = z$;

$$\therefore \quad z = (x + y)\,\phi\,(x^2 - y^2).$$

Euler, *Ib.* Art. 195.

(33). Let $p\,(x + y) + q\,(y - x) = z$.

The equations to integrate are

$$(x + y)\,dy - (y - x)\,dx = 0.$$

$$(x + y)\,dz - z\,dx = 0,$$

$$(y - x)\,dz - z\,dy = 0.$$

The first gives

$$\tan^{-1}\frac{x}{y} - \log\sqrt{\{x^2 + y^2\}} = \alpha.$$

the form of which prevents our exterminating, without extreme difficulty, y or x from the second or third equations; make therefore, since the equations are homogeneous, $x = t z$ and $y = u z$; the first equation gives $\frac{dz}{z} = \frac{dt}{u}$, and the second $\frac{dz}{z} = \frac{-du}{t}$:

consequently, we have

$\frac{dt}{u} = -\frac{du}{t}$, or $t^2 + u^2 = \frac{x^2 + y^2}{z^2} = \beta$; therefore,

$$z = \sqrt{(x^2 + y^2)}\,\phi\left\{\tan^{-1}\frac{x}{y} - \log\sqrt{(x^2 + y^2)}\right\}.$$

Euler, *Ib.* Art. 196.

(34). Let $p(x + y) - q(x + y) = z$.

By a similar process, we find

$$z = \varepsilon^{\frac{x}{x+y}}\,\phi(x + y).\quad \text{Euler, } \textit{Ib.} \text{ Art. 206.}$$

(35). Let $p(x - 2y) + q(2x - 3y) = z$;

$$\therefore\ z = \frac{\varepsilon^{\frac{x}{x-y}}}{x - y}\,\phi(x - y)^2.\quad \text{Euler, } \textit{Ib.} \text{ Art. 207.}$$

(36). Let $p x z + q y z = x y$;

$$\therefore\ z = x y + \phi\left(\frac{x}{y}\right).$$

(37). Let $c - z = (a - x)p + (b - y)q$;

$$\therefore\ \frac{y - b}{z - c} = \phi\left\{\frac{x - a}{z - c}\right\}.$$

The solution is the general equation of all conical surfaces. Monge, *Application*, . . . p. 10.

(38). Let $(y - bz)p - (x - az)q = bx - ay$; we hence get,

$$(y - bz)\,dy + (x - az)\,dx = 0, \qquad (1).$$

$$(y - bz)\,dz - (bx - ay)\,dx = 0, \qquad (2).$$

$$(x - az)\,dz + (bx - ay)\,dy = 0, \qquad (3).$$

Multiply (2) by a, and (3) by b, subtract the products, divide by $bx - ay$, and we get

$$dz + a\,dx + b\,dy = 0, \text{ or } z + ax + by = \alpha.$$

Again, multiply (2) by x, and (3) by y, subtract the products, divide by $bx - ay$, and we have

$$z\,dz + x\,dx + y\,dy = 0, \text{ or } x^2 + y^2 + z^2 = \beta.$$

consequently

$$x^2 + y^2 + z^2 = \phi(z + ax + by)$$

which is the general equation of all surfaces of revolution. Monge, *Ib*. p. 21.

V. Partial differential equations containing four or a greater number of variables.

The principle and method of integration employed for equations of three variables, may be readily extended to those which involve four, or a greater number of variables.

(39). Let $nu + px + qy = az$, where $n = \dfrac{dz}{du}$, and $z = f(x, y, u)$.

In this case, we have

$$u\,dx - x\,du = 0, \qquad (1).$$

$$u\,dy - y\,du = 0, \qquad (2).$$

$$u\,dz - az\,du = 0, \qquad (3).$$

from which we get

$$\frac{x}{u} = \alpha, \ \frac{y}{u} = \beta, \ \frac{z}{u^a} = \gamma,$$

consequently, since $\gamma = \phi(\alpha, \beta)$, we have

$$z = u^a \phi\left(\frac{x}{u}, \frac{y}{u}\right).$$

(40). Let $ap + bq + cn = 0$;

$$\therefore \ z = \phi\{(cx - az), (cy - bz)\}.$$

(41). Let $px + qy + nu = az + \frac{xy}{u}$;

$$\therefore \ (a - 1)z + \frac{xy}{u} = x^a \phi\left(\frac{y}{x}, \frac{u}{x}\right).$$

(42). Let $(y + u)p + (x + u)q + (x + y)n = 0$;

$$\therefore \ dx = \frac{y + u}{x + y} du, \ dy = \frac{x + u}{x + y} du;$$

$$\therefore \ dx - dy = \frac{(y - x)}{x + y} du, \text{ or } \frac{du}{x + y} = \frac{dx - dy}{y - x}.$$

Also $dx + dy + du = \frac{2(x + y + u)du}{x + y}$, and therefore

$$\frac{du}{x + y} = \frac{dx + dy + du}{2(x + y + z)} = \frac{dx - dy}{y - x};$$

$\therefore \ (x - y)^2(x + y + u) = \alpha$, and in the same manner,

$(x - u)^2(x + y + u) = \beta$, and since $dz = 0$, and $z = \gamma$,

we have

$$z = \phi\{(x - y)^2(x + y + u), (x - u)^2(x + y + u)\}.$$

Euler, *Ib.* Art. 483.

(43). Let $(z + y)n + xp + (z + u)q = u + y$;

$$\therefore z + y + u = y^2 \phi \{ (xz - xy), (xy - xu) \}.$$

Paoli, *Elementi d' Alg*. Vol. II. p. 274.

VI. Equations in which the differential coefficients p and q exceed the first degree.

Consider q as a function of p, x, y, z, and substitute for $\frac{dq}{dx}$ and $\frac{dq}{dz}$ in the equation of condition,

$$\frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz} = 0; \qquad (\delta).$$

we thus get a partial differential equation of four variables p, x, y, z, which being integrated will furnish us with a value of p, and therefore of q in terms of x, y, z, and an arbitrary constant α; substituting these values of p and q in the equation $dz = p\,dx + q\,dy$, and integrating, we shall find

$$f(x, y, z, \alpha) = \beta = \phi\alpha;$$

from which α may be eliminated by the ordinary processes of the Differential Calculus, when a particular value of ϕ is assigned. Lagrange, *Mém. de Berlin*. 1774.

Other methods of solution have been given, particularly by Euler, which though less systematic and general than the preceding, yet frequently lead to the results in a more simple manner.

(44). Let $pq = 1$.

Making $\frac{dp}{dx} = p'$, and $\frac{dp}{dy} = q'$, the equation (δ) becomes

$$q' + \frac{p'}{p^2} = 0,$$

consequently, $dp = 0$, and $p = \alpha$.

Again, since $dz - p\,dx - q\,dy = 0$,

we have $$dz - a\,dx - \frac{dy}{a} = 0,$$

and $$z - a\,x - \frac{y}{a} = \beta = \phi\,(a),$$

and $x - \frac{y}{a^2} + \phi' a = 0$, where $\phi' a = \frac{d\,\phi\,a}{d\,a}$.

Thus, if $\phi\,a = a$, we find

$$z = 2\sqrt{\{y(1+x)\}}.$$

Euler, *Ib.* Art. 83.

(45). Let $p^2 + q^2 = 1$.

By making $\frac{dp}{dz} = n'$, we get

$$q' + \frac{p}{\sqrt{(1-p^2)}}\,p' + \frac{1}{\sqrt{(1-p^2)}}\,n' = 0;$$

consequently $dp = 0$, $p = a$, and the integral is found by eliminating a from the equations

$$\left.\begin{array}{l} z - a\,x - y\sqrt{(1-a^2)} - \phi\,a = 0, \\ x - \frac{a\,y}{\sqrt{(1-a^2)}} + \phi' a = 0, \end{array}\right\}$$

Euler, *Ib.* Art. 90.

(46). Let $q = x\,p + p^2$;

$$\therefore\ q' - (x + 2p)p' - p^2 n' = p,$$

we hence get $$y - \log p = a,$$

$$\left.\begin{array}{l} z - x\,p - \frac{p^2}{2} = \phi\,a = \phi\,(y - \log p) \\ x + p + \frac{1}{p}\cdot\phi'(y - \log p) = 0, \end{array}\right\}$$

and

(47). Let $q = p^2 z$; eliminate α from the equations

$$\left.\begin{aligned} \frac{z}{2\alpha^2} - \frac{x}{\alpha} - y - \phi\alpha = 0, \\ \frac{z}{\alpha^3} - \frac{x}{\alpha^2} + \phi'\alpha = 0. \end{aligned}\right\}$$

(48). Let $q = p^2 x y^2 z^3$,

consequently

$$q' - 2xy^2z^3pp' - p^2xy^2z^3n' = p^2y^2z^3 + 3p^2xy^2z^2,$$

from which we get the equations

$$\left.\begin{aligned} 2xz\frac{dp}{p} + (z + 3px)\,dx = 0, \\ 2\,dz - p\,dx = 0; \end{aligned}\right\}$$

putting $\dfrac{2\,dz}{dx}$ for p in the second member of the first equation, and dividing by $2xz$, we get

$$\frac{dp}{p} + \frac{dx}{2x} + \frac{3\,dz}{z} = 0;$$

$\therefore\ p x^{\frac{1}{2}} z^3 = \alpha$, and $q = \dfrac{\alpha^2 y^2}{z^3}$; substituting for p and q in the equation

$$dz - p\,dx - q\,dy = 0,$$

and integrating, we get

$$\left.\begin{aligned} \frac{z^4}{4} - 2\alpha\sqrt{x} - \frac{\alpha^2 y^3}{3} - \phi\alpha = 0, \\ 2\sqrt{x} + \frac{2\alpha y^3}{3} + \phi'\alpha = 0. \end{aligned}\right\}$$

The same method of solution will apply to any equation of the form

$$q = p^n X\ Y\ Z, \text{ where } X,\ Y,\ Z,$$

are functions of x, y, z, respectively.

(49). Let $z = a p q$,

we hence get $$a q' + \frac{z}{p^2} p' + \frac{2 z}{p} n' = 1,$$

which gives $$\frac{z}{p^2} = \alpha,$$

and $$\left.\begin{aligned} 2\sqrt{z} - \frac{x}{\sqrt{\alpha}} - \frac{y\sqrt{\alpha}}{a} - \phi\,\alpha &= 0, \\ \frac{x}{2\,\alpha^{\frac{3}{2}}} - \frac{y}{2\,a\sqrt{\alpha}} - \phi'\,\alpha &= 0. \end{aligned}\right\}$$

Thus, if $\phi\,\alpha = \sqrt{\alpha}$, we get

$$z = x + \frac{x y}{a}.$$ Euler, *Ib.* Art. 136.

(50). Let $y q^2 = x^2 p^2$,

by the common process, we find

$$\left.\begin{aligned} z - \alpha \log x - 2\,\alpha\sqrt{y} - \phi\,\alpha &= 0, \\ \log x + 2\sqrt{y} + \phi'\alpha &= 0, \end{aligned}\right\}$$

or $$z = f(\log x + 2\sqrt{y}).$$

This example properly belongs to those given in Sect. IV. p. 438.

(51). Let $q = \dfrac{x^2 y^2}{a^4 p}$;

$$\left.\begin{aligned} \therefore\ \varepsilon &= \frac{1}{3\,a^2}\left\{\frac{x^3}{3\,\alpha} + \alpha\,y^3 + \phi\,\alpha\right\}, \\ y\,\frac{x^3}{\alpha^2} - y^3 &= \phi'\,\alpha. \end{aligned}\right\}$$

Euler, *Ib.* Art. 166.

(52). Let $q = \frac{px}{y} + \frac{1}{p}$;

$$\left.\begin{array}{l} \therefore\ z - \alpha xy - \frac{1}{\alpha}\log y - \phi\alpha = 0, \\ xy - \frac{1}{a^2}\log y + \phi' a = 0. \end{array}\right\}$$

(53.) Let $px + qy = apq$;

$$\therefore\ (ap - y)q' + \frac{xyp'}{ap - y} + \frac{axp^2}{ap - y}n' = p;$$

consequently $$\frac{ap^2}{2} - yp = \alpha,$$

also $\int\{dz - p\,dx - q\,dy\} = z - xp + \int(x\,dp - q\,dy)$

$= z - xp$, since $x\,dp - q\,dy = 0$;

$$\left.\begin{array}{l} \therefore\ z - xp - \phi\left(\frac{ap^2}{2} - yp\right) = 0, \\ x + (ap - y)\,\phi'\left(\frac{ap^2}{2} - yp\right) = 0. \end{array}\right\}$$

Euler, *Ib.* Art. 139.

(54). Let $pqxy = az$;

$$\therefore\ dz = p\,dx + \frac{az}{pxy}dy;\ \text{make}\ \frac{px}{z} = t,$$

and $t = \frac{u}{\sqrt{z}}$;

we thus get $$\frac{a\,dy}{y} = \frac{u\,dz}{\sqrt{z}} - \frac{u^2\,dx}{x};$$

$\therefore\ a\log y = 2u\sqrt{z} - u^2\log x - \int du\,\{2\sqrt{z} - 2u\log x\}$.

Make $\sqrt{z} - u\log x = \phi' u$;

$$\therefore\ a \log y = 2u\sqrt{z} - u^2 . \log x - 2\phi u,$$

$$= u^2 \log x + 2u\phi^1 u - 2\phi u.$$

Euler, *Ib.* Art. 218.

On the Integration of Equations of Partial Differentials of the Second and Higher Orders.

I. Equations of the form,

$$\frac{d^2 z}{dx\,dy} + P\frac{dz}{dx} = Q, \text{ or } \frac{d^2 z}{dx^2} + P\frac{dz}{dx} = Q.$$

where P and Q are functions of x and y.

(1). Let $\dfrac{d^2 z}{dx\,dy} = x + y$;

$$\therefore\ \frac{dz}{dx} = \int dy\,(x + y) = xy + \frac{y^2}{2} + \phi' x,$$

$$z = \int dx \int dy\,(x+y) = \frac{x^2 y}{2} + \frac{xy^2}{2} + \phi x + \psi y,$$

where ϕx and ψy are arbitrary functions of x and y respectively.

(2). **Problem.** To find the volume of a cone whose altitude is h and the radius of whose base is a.

Let $z = m\sqrt{(x^2 + y^2)} = \dfrac{h}{a}\sqrt{(x^2 + y^2)}$, be the equation of the cone; then if u be the volume, we have (Lacroix, Art. 246.)

$$\frac{d^2 u}{dx\,dy} = z = \frac{h}{a}\sqrt{(x^2 + y^2)};$$

$$\therefore\ \frac{du}{dx} = h\sqrt{(a^2-x^2)} + \frac{hx^2}{2a}\log\left\{\frac{a+\sqrt{(a^2-x^2)}}{x}\right\},$$

integrating between the limits $y = +\sqrt{(a^2-x^2)}$ and $y = -\sqrt{(a^2-x^2)}$; and

$u = \frac{\pi a^2 h}{3}$, the integral being taken between the limits $x = +a$ and $x = -a$.

(3). **Prob.** To find the volume of an ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

$$\therefore\ \frac{d^2u}{dx\,dy} = c\sqrt{\left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\}},$$

$$\frac{du}{dx} = \frac{\pi cb}{2}\left(1 - \frac{x^2}{a^2}\right),$$

$$\left(\text{from } \frac{y}{b} = +\sqrt{\left(1-\frac{x^2}{a^2}\right)} \text{ to } \frac{y}{b} = -\sqrt{\left(1-\frac{x^2}{a^2}\right)}\right);$$

$$u = \frac{2\pi abc}{3}\left(\text{from } \frac{x}{a} = +1 \text{ to } \frac{x}{a} = -1\right),$$

and for the whole ellipsoid, on both sides of the plane of xy, we have

$$2u = \frac{4\pi abc}{3}.$$

(4). **Prob.** The axes of two equal cylinders intersect at right angles; to find the volume of the portion which is common to both.

Let $z^2 + y^2 = a^2$, and $x^2 + y^2 = a^2$, be the equations of the cylinders;

$$\therefore\ \frac{d^2u}{dx\,dy} = z = \sqrt{\{a^2 - y^2\}},$$

$$\frac{du}{dy} = 2(a^2 - y^2), \left\{ x = \begin{matrix} + \sqrt{(a^2 - y^2)} \\ - \sqrt{(a^2 - y^2)} \end{matrix} \right\};$$

$$u = \frac{8a^3}{3} \ldots\ldots \left\{ y = \begin{matrix} +a \\ -a \end{matrix} \right\};$$

and the whole volume

$$= \frac{16a^3}{3}.$$

(5). Prob. With the data as above, to find the surface of one cylinder which is intercepted by the other.

If u be the surface required, we have

$$\frac{d^2u}{dx\,dy} = \sqrt{\left\{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right\}} = \frac{a}{\sqrt{(a^2 - y^2)}},$$

$$\frac{du}{dy} = 2a, \ldots\ldots\ldots\ldots \left\{ x = \begin{matrix} + \sqrt{(a^2 - y^2)} \\ - \sqrt{(a^2 - y^2)} \end{matrix} \right\};$$

$$2u = 8a^2, \ldots\ldots\ldots \left\{ y = \begin{matrix} +a \\ -a \end{matrix} \right\}.$$

(6). Prob. Two paraboloids of revolution have their axes coincident, but the vertex of one of them is upon the base of the other; to find the volume of the portion which is common to both.

Let $x^2 + y^2 = az$, be the equation of the first paraboloid, then $x^2 + y^2 = a'(b - z)$, is the equation of the second;

$$\therefore \frac{d^2u}{dx\,dy} = \frac{x^2 + y^2}{a}$$

$$\frac{du}{dx} = \frac{(4x^2 + 2c^2)}{3a}\sqrt{(c^2 - x^2)}, \left\{ y = \begin{matrix} + \sqrt{(c^2 - x^2)} \\ - \sqrt{(c^2 - x^2)} \end{matrix} \right\}$$

$$\text{where } c^2 = \left\{\frac{aa'b}{a + a'}\right\}$$

$$u = \frac{\pi a a'^2 b^2}{2(a + a')^2} \ldots\ldots\ldots\ldots \left\{ x = \begin{matrix} +c \\ -c \end{matrix} \right\},$$

(7). **Prob.** The axis of a given cylinder passes through the centre of a given sphere; to find the volume of the portion common to both.

Let $x^2+y^2+z^2=a^2$ and $x^2+y^2=b^2$, be the equations of the sphere and cylinder respectively;

$$\therefore \frac{d^2 u}{dx\,dy} = \sqrt{(a^2-x^2-y^2)},$$

$$\frac{du}{dx} = \sqrt{(a^2-b^2)}\sqrt{(b^2-x^2)} + (a^2-x^2)\times$$

$$\sin^{-1}\sqrt{\left\{\frac{b^2-x^2}{a^2-x^2}\right\}},\ \left\{y = \begin{matrix}+\sqrt{(b^2-x^2)}\\ -\sqrt{(b^2-x^2)}\end{matrix}\right\},$$

$$u = \frac{\sqrt{(a^2-b^2)}}{3}x\sqrt{(b^2-x^2)} + \left(a^2x - \frac{x^3}{3}\right)$$

$$\sin^{-1}\sqrt{\left\{\frac{b^2-x^2}{a^2-x^2}\right\}} + \frac{a^3}{3}\tan^{-1}\frac{x\sqrt{(a^2-b^2)}}{a\sqrt{(b^2-x^2)}}$$

$$-\frac{(a^2-b^2)^{\frac{3}{2}}}{3}\sin^{-1}\frac{x}{b},$$

$$2u = \frac{4\pi}{3}\{a^3-(a^2-b^2)^{\frac{3}{2}}\},\ \left\{x = \begin{matrix}+b\\ -b\end{matrix}\right\}.$$

(8). **Prob.** The axis of a cylinder bisects the radius of a sphere at right angles, its radius being one-half that of the sphere; to find the surface of the sphere included within the cylinder.

If $x^2+y^2+z^2=a^2$, is the equation of the sphere, then $x^2+y^2=cx$ is the equation of the cylinder;

$$\therefore \frac{d^2 u}{dx\,dy} = \frac{a}{\sqrt{(a^2-x^2-y^2)}},$$

$$\frac{du}{dx} = 2a\sin^{-1}\sqrt{\frac{x}{a+x}}\quad \left\{y = \begin{matrix}+\sqrt{(ax-x^2)}\\ -\sqrt{(ax-x^2)}\end{matrix}\right\}$$

$$u = \pi a^2 - 2a^2 \ldots\ldots\ldots \left\{x = \begin{matrix}+a\\ -a\end{matrix}\right\}$$

If similar portions be cut out from all the four quadrants of the sphere, the whole remainder will be equal to twice the square of the diameter of the sphere.

(9). Prob. With the same data as above, to find the volume common to the sphere and the cylinder.

$$\therefore \frac{d^2 u}{d x\, d y} = \sqrt{(a^2 - x^2 - y^2)}.$$

$$\frac{d u}{d x} = (a - x)\sqrt{(a x)} + (a^2 - x^2) \sin^{-1} \sqrt{\frac{x}{a + x}},$$

$$\left\{ y = \begin{matrix} +\sqrt{(a x - x^2} \\ -\sqrt{(a x - x^2} \end{matrix} \right\},$$

$$u = \frac{4 a^3}{3} \cdot \frac{\pi}{4} - \frac{4 a^3}{9}, \quad \left\{ x = \begin{matrix} + a \\ - a \end{matrix} \right\};$$

consequently the portion of the hemisphere which is not included in the cylinder is an algebraical quantity and equal to $\frac{1}{9}$ of the cube of the diameter of the sphere.

(10). Prob. With the same data as above, to find the surface of the cylinder, which is included within the sphere.

Let s = arc of the base of the cylinder

$$\therefore d s = \frac{\frac{a}{2} d x}{\sqrt{(a x - x^2)}},$$

and
$$u = \int z\, d s = \frac{a}{2} \int \frac{\sqrt{(a^2 - a x)}\, d x}{\sqrt{(a x - x^2)}}$$

$$= 2 a^2 \ldots\ldots \left\{ x = \begin{matrix} + a \\ - a \end{matrix} \right\},$$

or the whole cylindrical surface enveloped by the hemisphere is equal to the square of the diameter of the sphere.

(11.) Let $\frac{d^2 z}{d x\, d y} = \frac{n}{y} \frac{d z}{d x} + \frac{m}{x}$; make $\frac{d z}{d x} = v$, and integrate considering x as constant;

$$\therefore\ v = \frac{dz}{dx} = \frac{-my}{(n-1)x} + y^n \phi' x;$$

integrate again, considering y as constant

$$\therefore\ z = \frac{-m}{n-1} \cdot y \log x + y^n \phi x + \psi y.$$

Euler, *Calc. Integ.* Tom. III. Art. 299.

(12). Let $\dfrac{d^2 z}{dx\,dy} = \dfrac{y}{x^2+y^2} \cdot \dfrac{dz}{dx} + \dfrac{a}{x^2+y^2}$;

$$\therefore\ z = \frac{-ay}{x} + \psi y + \int dx \sqrt{(x^2+y^2)}\, \phi x.$$

Euler, *Ib.* Art. 280.

(13). Let $\dfrac{d^2 z}{dx^2} - \dfrac{ax}{\sqrt{(x^2+y^2)}}$;

$$\therefore\ \frac{dz}{dx} = a\sqrt{(x^2+y^2)} + \phi y,$$

$$z = \frac{1}{2} a x \sqrt{(x^2+y^2)} + \frac{1}{2} a y^2 \log . \{ x + \sqrt{(x^2+y^2)} \}$$

$$+ x \phi y + \psi y.$$

Euler, *Ib.* Art. 231.

(14). Let $\dfrac{d^2 z}{dx^2} = \dfrac{n}{x} \cdot \dfrac{dz}{dx}$, make $\dfrac{dz}{dx} = v$,

$$\therefore\ \frac{dv}{dx} = \frac{n}{x} v;$$

$$\therefore\ v = x^n \phi y$$

$$z = \frac{x^{n+1}}{n+1} \phi y + \psi y.$$

Euler, *Ib.* Art 258.

(15). Let $\frac{d^2 z}{d x^2} = \frac{n}{x} \cdot \frac{d z}{d x} + \frac{a}{x y}$;

$$z = -\frac{a x}{n y} + \frac{x^{n+1}}{n+1} \phi y + \psi y.$$

Euler, *Ib.* Art. 259.

(16). Let $\frac{d^2 z}{d x^2} = \frac{2 x}{x^2 + y^2} \cdot \frac{d z}{d x} + \frac{x}{a y}$;

$$\therefore z = \frac{x(x^2+3y^2)}{6ay} \log(x^2+y^2) - \frac{1}{9ay}\left(x^3+6xy^2-6y^3 \tan^{-1}\frac{x}{y}\right)$$

$$+ \frac{1}{3} x (x^2+3y^2) \phi y + \psi y.$$

Euler. *Ib.* Art. 260.

(11). Partial differential equations of the form

$$\frac{d^2 z}{d x^2} + P \frac{d^2 z}{d x \, d y} + Q \frac{d^2 z}{d y^2} = R,$$

$$\text{or } r + P s + Q t = R,$$

where $r = \frac{d^2 z}{d x^2}$, $s = \frac{d^2 z}{d x \, d y}$, $t = \frac{d^2 z}{d y^2}$, and where P, Q, R are functions of x, y, z, p, q.

The following process of solution, first given by Monge, is of very general application: Form the two systems of equations,

$$\left.\begin{array}{l} d y - k \, d x = 0 \\ k \, d p + Q \, d q - R k \, d x = 0, \end{array}\right\} \quad (\alpha).$$

$$\left.\begin{array}{l} d y - k' \, d x = 0 \\ k' \, d p + Q \, d q - R k' \, d x = 0, \end{array}\right\} \quad (\beta)$$

where k and k' are the roots of the equation

$$k^2 - P k + Q = 0 : \qquad (\gamma).$$

if from either of the two systems (α) or (β) we can deduce either separately or by adding any multiple of one to any multiple of the other, two primitive equations $M = a$ and $N = b$, then $N = \phi(M)$, will be a first integral of the proposed equation; the integral of this equation of the first order, determined by the ordinary processes, will be the second or complete integral required.

This determination, however, of the second integral from the first, frequently presents great difficulties, in consequence of its involving an arbitrary function of the integral of the first equation in the systems (α) or (β); it is generally therefore more convenient, to determine, *when possible*, another first integral $N' = \phi_1(M')$, from the second of the two systems (α) or (β); we shall thus be enabled to eliminate p or q from the two equations $N = \phi(M')$, and $N' = \phi_1(M')$ and the resulting equation involving only one differential coefficient, may be integrated in the same manner as the Examples given in the last Section.

(17). Let $\dfrac{d^2 z}{d x^2} = c^2 \dfrac{d^2 z}{d y^2}$, or $r - c^2 t = 0$.

The system of equations (α) and (β), become

$$\left.\begin{aligned} dy - c\,dx &= 0 \\ dp - c\,dq &= 0 \end{aligned}\right\} \qquad (\alpha).$$

$$\left.\begin{aligned} dy + c\,dx &= 0 \\ dp + c\,dq &= 0 \end{aligned}\right\} \qquad (\beta).$$

From system (α), we get $y - cx = a$ and $p - cq = b$;

$$\therefore p - cq = \phi(y - cx), \qquad (1).$$

From system (β), we get $y + cx = a'$, and $p + cq = b'$,

$$\therefore\ p + cq = \psi(y + cx) \qquad (2).$$

Eliminating q from (1) and (2), we get

$$p = \frac{dz}{dx} = \frac{1}{2}\phi(y - cx) + \frac{1}{2}\psi(y + cx);$$

$$\therefore\ z = \phi_1(y - cx) + \psi_1(y + cx).$$

This differential equation is that which occurs in the solution of the problem of vibrating chords, and was first integrated by D'Alembert in the Berlin Acts for 1747, by a process extremely different from the preceding, but very simple and elegant. The same equation was afterwards integrated in a different manner by Euler, who maintained that the arbitrary functions which it involves, might be discontinuous; contrary to the opinion of D'Alembert: this circumstance gave rise to a controversy between these two illustrious analysts, which was finally decided by Lagrange in favour of the opinion of Euler, in a masterly discussion of the problem in the *Mémoires de Turin* for 1759.

(18). Let $r - c^2 t = xy$.

From system (α) $p - cq - \frac{1}{2}x^2 y + \frac{1}{6}cx^3 = \phi(y - cx)$, (1).

$$\text{for}\quad \int R\,dx = \int xy\,dx = \frac{1}{2}x^2 y - \frac{1}{2}\int x^2\,dy$$

$$= \frac{1}{2}x^2 y - \frac{1}{2}\int cx^2\,dx = \frac{1}{2}x^2 y - \frac{1}{6}cx^3.$$

Also, by system (β),

$$p + cq - \frac{1}{2}x^2 y - \frac{1}{6}cx^3 = \psi(y + cx); \qquad (2).$$

$$\therefore\ p = \frac{1}{2} x^2 y + \frac{1}{2} \phi(y - cx) + \frac{1}{2} \psi(y + cx),$$

and $$z = \frac{1}{6} x^3 y + \phi_1(y - cx) + \psi_1(y + cx).$$

(19). Let $r + (a + b)s + abt = \dfrac{x}{x + y}$,

$$p - bq - \frac{1}{1+a}\left\{x - \frac{(y - ax)}{1+a} \log\left(\frac{x+y}{1+a}\right)\right\} = \phi(y - ax),$$

$$p - aq - \frac{1}{1+b}\left\{x - \frac{(y - bx)}{1+b} \log\left(\frac{x+y}{1+b}\right)\right\} = \psi(y - bx),$$

$$\therefore\ (a - b)z = \frac{(a - b)}{(1+a)(1+b)}\left\{\frac{x^2}{2} - xy + y^2 . \log(x + y) - \frac{(a + b + 2ab)}{(1+a)(1+b)} \cdot \frac{x^2}{4}\right\} + \frac{a}{(1+a)^2} \cdot \left(xy - \frac{ax^2}{2}\right) \log\left(\frac{x+y}{1+a}\right)$$
$$- \frac{b}{(1+b)^2}\left(xy - \frac{bx^2}{2}\right) \log\left(\frac{x+y}{1+b}\right)$$
$$+ \phi_1(y - ax) + \psi_1(y - bx).$$

(20). Let $r - \dfrac{2a}{b} s + \dfrac{a^2}{b^2} t = 0$.

In this case $k = -\dfrac{a}{b} = k'$ and the two systems (α) and (β) become identical: they are

$$\left.\begin{aligned} dy + \frac{a}{b} dx &= 0 \\ dp - \frac{a}{b} dq &= 0, \end{aligned}\right\}$$

$$\therefore\ bp - aq = \phi(ax + by).$$

This equation must be solved by the process in Sect. V.

p. 438. as we have no means of deducing a second equation of the first order, as in the last examples; consequently

$$a\,d\,x + b\,d\,y = 0, \text{ and } d\,x\,\phi(a\,x + b\,y) - b\,d\,z = 0;$$

$$\therefore\ a\,x + b\,y = \alpha \text{ and } d\,x\,\phi\,\alpha - b\,d\,z = 0,$$

$$\text{or } x\,\phi\,\alpha - b\,z = \psi\,\alpha,$$

$$\text{or } z = x\,\phi_1(a\,x + b\,y) + \psi_1(a\,x + b\,y).$$

Monge, *Application*, . . . p. 64.

(21). Let $r + \frac{2\,y}{x}s + \frac{y^2}{x^2}t = 0$;

$$\left.\begin{aligned} \therefore\ d\,y - \frac{y}{x}\,d\,x &= 0 \\ d\,p + \frac{y}{x}\,d\,q &= 0 \end{aligned}\right\} \quad (\alpha).$$

and

$$\therefore\ \frac{y}{x} = a, \text{ and } p + a\,q = p + \frac{y}{x}\,q = \phi\left(\frac{y}{x}\right);$$

and in the same manner as in Example 20, we find

$$z = x\,\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right).$$

Monge, *Application*, p. 75. Lagrange, *Mem. de Berlin.* 1774. p. 273.

(22). Let $r - \frac{2\,p}{q}s + \frac{p^2}{q^2}t = 0$.

$$\therefore\ d\,y + \frac{p}{q}\,d\,x = 0, \text{ or } d\,z = 0, \text{ and } \therefore z = a,$$

$$d\,p - \frac{p}{q}\,d\,q = 0, \text{ or } \frac{p}{q} = \phi(z),$$

and as in Example 21, we find

$$z = \psi\{x\phi(z) + y\} \quad \text{or} \quad y = \psi_1(z) + x\phi_1(z).$$

(23). Let $r - t + \dfrac{4p}{x+y} = 0$,

$$\left.\begin{aligned} \therefore\ dy - dx &= 0 \\ dp - dq + \frac{4p}{x+y}\,dx &= 0, \end{aligned}\right\} \quad (\alpha).$$

$$\left.\begin{aligned} dy + dx &= 0 \\ dp + dq - \frac{4p\,dx}{x+y} &= 0, \end{aligned}\right\} \quad (\beta).$$

consequently, $y - x = a$ and the second equation of system (α) becomes

$$dp - dq + \frac{4p\,dx}{2y - a} = 0,$$

from which subtracting

$$\frac{2p}{2y-a}(dy - dx) = \frac{2p\,dy}{2y-a} + \frac{2q\,dy}{2y-a} - \frac{2\,dz}{2y-a},$$

we get

$$(2y - a)(dp - dq) + (2p - 2q)\,dy + 2\,dz = 0,$$

$$\text{and} \quad \therefore\ p - q + \frac{2z}{x+y} = \frac{\phi(y-x)}{x+y}, \qquad (1).$$

From this equation we get

$$dy + dx = 0, \text{ and } dz - \left(\frac{2z - \phi(y-x)}{x+y}\right) dy = 0,$$

or

$$y + x = b,$$

and therefore $dz - \dfrac{2z\,dy}{b} = \dfrac{dy}{b}\phi(2y - b),$

and $$z = \varepsilon^{\frac{2y}{b}} \left\{ \psi(b) - \frac{1}{b} \int \varepsilon^{-\frac{2y}{b}} dy\, \phi(2y - b) \right\}$$

where $b = x + y$.

If we had proceeded with the system (β), we should have got $x + y = b$,

and $$dp + dq - \frac{4p\,dq}{b} = 0,$$

which cannot be integrated.

The differential equation in this Example admits but of one first integral, a circumstance of frequent occurrence and which may have been remarked in Example 21.; it arises from the form of the primitive equation which admits of the elimination of one of the arbitrary functions only, by combining it with the derivative equations involving differential coefficients of the first order; thus the process in question will enable us to eliminate $\psi(x+y)$ from the complete integral of the preceding equation, but not the other arbitrary function $\phi(y - x)$.

In some cases, indeed, we can eliminate neither of the arbitrary functions, without proceeding to derivative equations involving differential coefficients of higher orders than the first and therefore the partial differential equation of the second order, will admit of no first integral whatever; an example of this may be seen in the equation

$$r - t - \frac{2p}{x} = 0,$$

whose complete integral is

$$z = \phi(y + x) + \psi(y - x) - x \{ \phi'(y + x) - \psi'(y - x) \}.$$

See Paoli, *Elementi d'Alg*. Tom. II. p. 293.

(24). Let $s + ap + bq + abz = V$, where $V = f(x, y)$.

Make $\frac{dz}{dy} + az = v$, which gives

$$\frac{dv}{dx} = \frac{d^2 z}{dx\,dy} + \frac{a\,dz}{dx} = s + ap;$$

and $$\frac{dv}{dx} + bv = V;$$

by the method in p. 329, we get

$$v = \varepsilon^{-bx} \int \varepsilon^{bx} V\,dx = \frac{dz}{dy} + az;$$

and repeating the same process, we get

$$z = \varepsilon^{-ay} \phi x + \varepsilon^{-bx} \psi y + \varepsilon^{-ay-bx} \int \varepsilon^{ay} dy \int \varepsilon^{bx} V\,dx.$$

Euler, *Ib.* Art. 292.

(25). Let $s + \frac{a}{y} p + \frac{b}{x} q + \frac{ab}{xy} z = V$,

where $V = f(x, y)$.

Make $\frac{dz}{dy} + \frac{a}{y} z = v$,

which gives $$\frac{dv}{dx} + \frac{b}{x} v = V;$$

consequently

$$z = x^{-b} y^{-a} \{ \phi x + \psi y + \int y^a dy \int x^b V\,dx \}.$$

Euler. *Ib.* Art. 293.

The same method is obviously applicable to the equations

$$s + Pp + Qq + \left\{ PQ + \frac{dP}{dx} \right\} z = V,$$

and $$s + Pp + Qq + \left(PQ + \frac{dQ}{dy} \right) z = V,$$

where P, Q and V are functions of x and y.

Laplace in the *Mémoires de l'Académie* for 1773, has shewn how to reduce other cases of the equation

$$s + Pp + Qq + Rz = V, \qquad (2).$$

in which R is not $= PQ + \frac{dP}{dy}$, or $PQ + \frac{dQ}{dy}$, to similar equations, in which that condition is satisfied; the method consists in assuming $\frac{dz}{dx} + Pz = v$, when the equation (2) becomes

$$\frac{dv}{dx} + Qv + \alpha z = V, \qquad (3).$$

From this we get values of z and of $\frac{dz}{dy}$, which substituted in the equation $\frac{dz}{dy} + Pz = v$, give us the equation

$$\frac{d^2 v}{dx\,dy} + P_1 \frac{dv}{dx} + Q \frac{dv}{dy} + R_1 V = V_1,$$

which if $R_1 = P_1 Q + \frac{dP_1}{dx}$, may be integrated in the same manner as the two last examples; if not, the same process must be repeated, and if the equation be integrable, we shall finally get an equation in which this essential condition is satisfied. We shall thus get a value of v involving two arbitrary functions: the value of z is easily derived from equation (3), at least when the condition is satisfied after the second operation; the method of determining z will easily suggest itself when a greater number of operations are required.

(26). Let $s + \frac{a}{x+y} p + \frac{b}{x+y} q + \frac{(ab + b - 2a - 2)}{(x+y)^2} z = 0.$

Make $\frac{dz}{dy} + \frac{az}{x+y} = v$, and we get

$$\frac{dv}{dx} + \frac{bv}{x+y} + \frac{(b-a-2)}{(x+y)^2} z = 0;$$

$$\therefore \; z = \frac{(x+y)^2}{A} \cdot \frac{dv}{dx} + \frac{b(x+y)}{A} v, \text{ where } A = a + 2 - b,$$

$$\frac{dz}{dy} = \frac{(x+y)^2}{A} \cdot \frac{d^2 v}{dx\,dy} + \frac{2(x+y)}{A} \cdot \frac{dv}{dx} + \frac{b(x+y)}{A} \cdot \frac{dv}{dy} - \frac{bv}{A};$$

consequently substituting these values of z and $\frac{dz}{dy}$ in $\frac{dz}{dy} + \frac{az}{x+y} = 0$, and dividing by $\frac{(x+y)^2}{A}$, we find

$$\frac{d^2 v}{dx\,dy} + \frac{a+2}{x+y} \cdot \frac{dv}{dx} + \frac{b}{x+y} \cdot \frac{dv}{dy} + \frac{(ab + 2b - a - 2)}{(x+y)^2} v = 0.$$

Make $\frac{dv}{dy} + \frac{(a+2)}{x+y} v = u$,

and we get $$\frac{du}{dx} + \frac{bu}{x+y} = 0;$$

$$\therefore \; u = (x+y)^{-b} \psi y = \frac{dv}{dy} + \frac{a+2)}{(x+y)} v,$$

$$\therefore \; v = (x+y)^{-a-2} \left\{ \int (x+y)^{a-b+2} \psi(y)\, dy + f(x) \right\},$$

and $z = \frac{(x+y)^2}{a+2-b} \cdot \frac{dv}{dx} + \frac{b(x+y)}{a+2-b} v$

$$= (x+y)^{-a} \int (x+y)^{a-b+1} \psi(y)\, dy$$

$$- (x+y)^{-a-1} \int (x+y)^{a-b+2} \psi(y)\, dy$$

$$- (x+y)^{-a-1} f(x) + \frac{(x+y)^{-a}}{a+2-b} f'(x),$$

which becomes, when $a = b$

$$z = \frac{(x+y)^{-a}}{2} \{ F'(x) + \phi' y \} - (x+y)^{-a-1} (Fx + \phi y).$$

The equation

$$Rr + Ss + Tt + Pp + Qq + Nz = V,$$

where R, S, T, P, Q, N, and V, are functions of x and y, may always be reduced to the form

$$s' + P'p' + Q'q' + N'z = V',$$

by considering x and y as functions of two new variables, u and v, to be determined from general or particular integrals of the equations,

$$du = \frac{du}{dy}(dy + k\,dx), \text{ and } dv = \frac{dv}{dy}(dy + k'\,dx),$$

where k and k' are roots of the equation

$$Rk^2 + Sk + T = 0.$$

In determining particular values of u and v, we must consider $\frac{du}{dy}$ and $\frac{dv}{dy}$ as the most simple multipliers necessary to render $dy + k\,dx$ and $dy + k'\,dx$ complete differentials; thus, if k or k' be constant quantities or functions of x alone, $\frac{du}{dy}$ and $\frac{dv}{dy}$ are each equal to unity; if $k = \frac{y}{x}$ and $k' = -\frac{y}{x}$, then $\frac{du}{dy} = x$ and $\frac{dv}{dy} = \frac{1}{x}$, and so on for other cases.

As this transformation is of great and frequent use, it may be convenient to observe, that if

$$\frac{d^2z}{du^2} = r', \quad \frac{d^2z}{du\,dv} = s', \quad \frac{d^2z}{dv^2} = t', \quad \frac{dz}{du} = p', \quad \frac{dz}{dv} = q',$$

$$\frac{du}{dy} = n, \quad \frac{dv}{dy} = n', \quad \frac{du}{dx} = m, \text{ and } \frac{dv}{dx} = m',$$

we have

$$p = m p' + m' q',$$

$$q = n p' + n' q',$$

$$r = m^2 r' + 2 m m' s' + m'^2 t' + \frac{d m}{d x} p' + \frac{d m'}{d x} q',$$

$$s = m n r' + (m n' + m' n) s' + m' n' t' + \frac{d m}{d y} p' + \frac{d m'}{d y} q',$$

$$t = n^2 r' + 2 n n' s' + n'^2 t' + \frac{d n}{d y} p' + \frac{d n'}{d y} q'.$$

(27). Let $r - a^2 t + 2 a b p + 2 a b^2 q = 0$.

In this case, $k = a$, and $k' = - a$; making $\frac{d u}{d y} = 1 = \frac{d v}{d y}$, we have

$$d u = d y + a d x, \text{ and } d v = d y - a d x,$$

particular integrals of which are

$$u = y + a x, \text{ and } v = y - a x;$$

$$\therefore r = a^2 r' - 2 a^2 s' + a^2 t'$$

$$t = r' + 2 s' + t'$$

$$p = a p' - a q', \text{ and } q = p' + q';$$

consequently the equation becomes

$$- 4 a^2 s' + 4 a^2 b p' = 0,$$

or

$$\frac{d^2 z}{d u d v} - \frac{b d z}{d u} = 0,$$

$$\therefore \frac{d z}{d u} = \epsilon^{b v} \psi' u,$$

and

$$z = \epsilon^{b v} \{ \psi u + \phi v \}$$

$$= \epsilon^{b(y - a x)} \{ \psi (y + a x) + \phi (y - a x) \}.$$

Euler, *Ib.* Art. 316.

(28). Let $r - a^2 t + \frac{a}{y} . p + \frac{a^2}{y} . q = 0$.

By the same process we find

$$\frac{d^2 z}{d u\, d v} - \frac{1}{u + v} \cdot \frac{d z}{d u} = 0,$$

$$\therefore\ z = (u + v)\, \psi' u - \psi u + \phi v$$

$$= 2 y\, \psi'(y + a x) - \psi (y + a x) + \phi (y - a x).$$

Euler, *Ib.* Art. 317.

(29). Let $r - \frac{y^2}{x^2} . t = 0.$

In this case $k = \frac{y}{x}$, $k' = - \frac{y}{x}$,

$$d u = \frac{d u}{d y}\left(d y + \frac{y}{x}\, d x\right),\quad d v = \frac{d v}{d y}\left(d y - \frac{y}{x}\, d x\right);$$

$$\therefore\ u = x y, \text{ and } v = \frac{y}{x};$$

$$\therefore\ m = y,\ m' = - \frac{y}{x^2},\ n = x,\ n' = \frac{1}{x},$$

$$r = y^2 r' - \frac{2 y^2}{x^2}\, s' + \frac{y^2}{x^4} . t' + \frac{2 y}{x^3} \cdot q',$$

$$t = x^2 r' + 2 s' + \frac{1}{x^2} . t',$$

consequently

$$\frac{d^2 z}{d u\, d v} - \frac{1}{2 u} \cdot \frac{d z}{d v} = 0,$$

$$\therefore\ z = \sqrt{(x y)}\, \phi \left(\frac{y}{x}\right) + \psi (x y).$$

Euler, *Ib.* Art. 307.

(30). Let $r - \frac{y^2}{x^2} t + \frac{1}{y} . p - \frac{1}{x} . q = 0.$

In the same manner as in the last Example, we find

$$z = (xy)^{\frac{1}{2}} \int (xy)^{-\frac{x}{2y}} \psi\left(\frac{x}{y}\right) d \cdot \left(\frac{x}{y}\right) + \phi(xy).$$

Euler, *Ib.* Art. 318.

(31). To integrate the equation

$$\frac{d^2 z}{dx\,dy} + \frac{a}{x+y}\frac{dz}{dx} + \frac{a}{x+y}\frac{dz}{dy} + \frac{bz}{(x+y)^2} = 0,$$

by a series, and to assign the values of b which make that series terminate.

It is evident, that x and y must be similarly involved in the integral, the form of which may be conjectured from the result given in Ex. 26: we may assume, therefore,

$$z = A(x+y)^{\lambda}(fx + Fy) + B(x+y)^{\lambda+1}(f'x + F'y)$$
$$+ C(x+y)^{\lambda+2}(f''x + F''y) + \&c.$$

or, for greater simplicity, omitting the functions of y in this assumption and replacing them after the determination of λ, B, C, D, &c. we have

$$z = A(x+y)^{\lambda} fx + B(x+y)^{\lambda+1} f'x + C(x+y)^{\lambda+2} f''x + \&c.;$$

and substituting for z, $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{d^2 z}{dx\,dy}$ in the equation, we get

$$\begin{array}{llll}
0 = bA(x+y)^{\lambda} fx & + bB(x+y)^{\lambda+1} f'x & + bC(x+y)^{\lambda+2} f''x & + \&c. \\
+ 2a\lambda A & + aA & + aB & \\
+ \lambda(\lambda-1)A & + 2a(\lambda+1)B & + 2a(\lambda+2)C & \\
 & + \lambda A & + (\lambda+1)B & \\
 & + (\lambda+1)\lambda B & + (\lambda+2)(\lambda+1)C, &
\end{array}$$

we from hence get

$$\lambda = \frac{1}{2} - a \pm \sqrt{\left\{\frac{1}{4} - a - b + a^2\right\}}$$

$$B = \frac{-(a+\lambda)\,A}{2\,(a+\lambda)} = \frac{-A}{2}, \text{ since } b + 2\,a\,\lambda + \lambda^2 - \lambda = 0,$$

$$C = \frac{-\,(a+\lambda+1)\,B}{2\,(2\,a+2\,\lambda+1)} = \frac{+\,(i-1)\,A}{2\,.\,2\,.\,(2\,i-1)}, \text{ if } a+\lambda = -i,$$

$$D = \frac{-\,(a+\lambda+2)\,C}{3\,(2\,a+2\,\lambda+2)} = \frac{-\,(i-2)\,A}{2^2\,.\,2\,.\,3\,(2\,i-1)},$$

$$E = \frac{-\,(a+\lambda+3)\,D}{4\,(2\,a+2\,\lambda+3)} = \frac{+\,(i-2)\,(i-3)\,A}{2^3\,.\,2\,.\,3\,.\,4\,.(2\,i-1)\,(2\,i-3)},$$

$$F = \frac{-\,(a+\lambda+4)\,D}{5\,(2\,a+2\,\lambda+4)} = \frac{-\,(i-3)\,(i-4)\,A}{2^4\,.\,3\,.\,4\,.\,5\,.(2\,i-1)\,(2\,i-3)} \;\&c.$$

The series must terminate whenever $a + \lambda + n = n - i$, is a whole number, n being the denomination of the last term but one in the series; the constant A may be included in the arbitrary functions of x and y.

Thus let $\lambda = -\,a$; it is evident from the formation of the series, that $B = 0$, $C = 0$, &c.

$$\therefore\; z = (x+y)^{-a}\,(f\,x + F\,y)$$

Let $\lambda = -\,a - 2$, and therefore $b = (a+2)\,(a-3)$; then

$$z = (x+y)^{-a-2}\,(f\,x + F\,x) - \frac{1}{2}\,(x+y)^{-a-1}\,(f'\,x + F'\,x)$$

$$+ \frac{1}{12}\,(x+y)^{-a}\,(f''\,x + F''\,x).$$

The series will terminate, whenever

$$b = (a+i)\,(a-i-1).$$

If $b = 0$, the equation becomes

$$s + \frac{a}{x+y}\,p + \frac{a}{x+y}\,q = 0,$$

and we must have $a = -i$, or $a = i + 1$, or any positive or negative whole number, and therefore, $\lambda = 0$, or $\lambda = -2i - 1$: in the first case the integral is

$$z = f(x) + F(y) - \frac{1}{2}(x + y)(f'x + F'y)$$

$$+ \frac{1}{2} \cdot \frac{(i-1)}{2 \cdot (2i-1)} \cdot (x + y)^2 (f''x + F''y) - \&c:$$

and in the second we have $(x + y)^{2i+1} z$ equal to the same series. Euler, *Ib.* Art. 322.

(32). Let $\frac{d^2 z}{dx^2} - \frac{a^2}{b^2} \cdot \frac{d^2 z}{dy^2} + \frac{2m}{x} \cdot \frac{dz}{dx} = 0.$

If we transform this equation by the method given in page 466, we shall find

$u = \frac{1}{2} x + \frac{by}{2a}$, and $v = \frac{1}{2} x - \frac{by}{2a}$, so that $u + v = x$;

the transformed equation is

$$\frac{d^2 z}{du\,dv} + \frac{m}{u + v} \cdot \frac{dz}{du} + \frac{m}{u + v} \cdot \frac{dz}{dv} = 0,$$

which may be integrated whenever m is an integer.

Thus, if $m = -1$, we have

$$z = f\left(\frac{ax + by}{2a}\right) + F\left(\frac{ax - by}{2a}\right)$$

$$- \frac{x}{2}\left\{f'\left(\frac{ax + by}{2a}\right) + F'\left(\frac{ax - by}{2a}\right)\right\}.$$

If $m = 2$, we have

$$x^3 z = f\left(\frac{ax + by}{2a}\right) + F\left(\frac{ax - by}{2a}\right)$$

$$- \frac{x}{2}\left\{f'\left(\frac{ax + by}{2a}\right) + F'\left(\frac{ax - by}{2a}\right)\right\}.$$

Euler, *Ib.* Art. 343.

(33). Let $\dfrac{d^2 z}{d x^2} = \dfrac{a^2}{b^2} x^{-\frac{4m}{2m-1}} \dfrac{d^2 z}{d y^2}$.

Transforming the equation, as in the last Example, we shall find

$$u = \frac{1}{2} x^{-\frac{1}{2m-1}} + \frac{b y}{2(2m-1)a},$$

and
$$v = \frac{1}{2} x^{-\frac{1}{2m-1}} - \frac{b y}{2(2m-1)a},$$

so that $u + v = x^{-\frac{1}{2m-1}}$; the original equation becomes

$$\frac{d^2 z}{d u\, d v} + \frac{m}{u+v} \cdot \frac{d z}{d u} + \frac{m}{u+v} \cdot \frac{d z}{d v} = 0.$$

If $m=0$, we have $u = \dfrac{1}{2} x - \dfrac{b y}{2a}$, and $v = \dfrac{1}{2} x + \dfrac{b y}{2a}$,

and
$$z = f(ax + by) + F(ax - by).$$

If $m = -1$, we have $u = \dfrac{1}{2} x^{\frac{1}{3}} - \dfrac{b y}{6a}$,

and
$$v = \frac{1}{2} x^{\frac{1}{3}} + \frac{b y}{6a},$$

$$\therefore\ z = fu + Fv - \frac{1}{2} x^{\frac{1}{3}} (f'u + F'u).$$

If $m = 3$, we have $u = \dfrac{1}{2} x^{-\frac{1}{5}} + \dfrac{b y}{10a}$,

$$v = \frac{1}{2} x^{-\frac{1}{5}} - \frac{b y}{10a}, \text{ and}$$

$$\therefore\ z = x(fu + Fv) - \frac{2}{4} x^{\frac{4}{5}} (f'u + F'v)$$

$$+ \frac{1}{4 \cdot 3} \cdot x^{\frac{3}{5}} (f''u + F''v).$$

This equation is evidently analogous to that of Riccati, and integrable under the same circumstances. Euler, *Ib.* Art. 345.

(34). Let $\frac{d^2 z}{d x^2} = a^2 \frac{d^2 z}{d y^2} + \frac{a^2 b z}{y^2}$.

Make $u = y + a x$, $v = y - a x$, and the equation becomes

$$\frac{d^2 z}{d u\, d v} - \frac{b z}{(u + v)^2} = 0,$$

this becomes a case of the equation in Ex. 31, when $a = 0$, and is integrable in finite terms, whenever

$$b = - i\,(i + 1).$$

Thus let $i = 1$, and therefore $b = - 2$; then

$$z = \frac{1}{2 y} \{ f(y + a x) + F(y - a x) \}$$

$$- \{ f'(y + a x) + F'(y - a x) \}.$$

Euler, *Ib.* Art. 333.

(35). Let $\frac{d^2 z}{d x\, d y} = a z$; a particular integral of this equation may be found, by making

$$z = \varepsilon^{a x}\, Y,$$

$$\therefore \quad \frac{d^2 z}{d x\, d y} = a\, \varepsilon^{a x} \frac{d Y}{d y} = a\, \varepsilon^{a x}\, Y,$$

$$\therefore \quad \frac{a\, d Y}{Y} = a\, d y, \quad Y = \varepsilon^{\frac{a y}{\alpha}}, \text{ and } z = \varepsilon^{\alpha x + \frac{a y}{\alpha}},$$

or more generally

$$z = A\, \varepsilon^{\alpha x + \frac{a y}{\alpha}} + B\, \varepsilon^{\beta x + \frac{a y}{\beta}} + C\, \varepsilon^{\gamma x + \frac{a y}{\gamma}} + \&c.$$

where $A, \alpha, B, \beta, C, \gamma$, &c. are arbitrary constants. Or if $m n = a$, $m' n' = a$, &c, we have also

$$z = A \sin (m x - n y) + B \sin (m' x - n' y)$$

$$+ C \sin (m'' x - n'' y) + \&c.$$

$$z = A \cos (m x - n y) + B \cos (m' x - n' y)$$

$$+ C \cos (m'' x - n'' y) + \&c.$$

Euler, *Ib.* Art. 282.

(36). Let $\frac{d^2 z}{d x^2} = a^2 z$. Make $\frac{d z}{d x} = k z$;

$$\therefore \frac{d^2 z}{d x^2} = k . \frac{d z}{d x} = k^2 z; \qquad \therefore k = a, \text{ or } -a,$$

$$\text{and } z = \epsilon^{ax} \phi y + \epsilon^{-ax} \psi y.$$

(37). Let $A \frac{d^3 z}{d x^3} + B \frac{d^3 z}{d x^2 d y} + C \frac{d^3 z}{d x d y^2} + D \frac{d^3 z}{d y^3} = 0$;

where A, B, C, D are constant quantities.

If k, k_1, k_2, are roots of the equation

$$A k^3 + C k^2 + C k + D = 0,$$

then

$$z = \phi (y + k x) + \phi_1 (y + k_1 x) + \phi_2 (y + k_2 x).$$

Euler, *Ib.* Art. 421.

(38). Let $\frac{d^3 z}{d x^3} - (2 a + b) \frac{d^3 z}{d x^2 d y}$

$$+ (2 a b + a^2) \frac{d^3 z}{d x d y^2} - a^2 b \frac{d^3 z}{d y^3}.$$

In this case, $k_2 = b$, $k = k_1 = a$,

$$\therefore z = \psi (y + a x) + x \psi_1 (y + a x) + \phi (y + b x).$$

(39). Let $\frac{d^3 z}{d x^3} - 3 a \frac{d^3 z}{d x^2 d y} + 3 a^2 \frac{d^3 z}{d x d y^2} - a^3 \frac{d^3 z}{d y^3} = 0.$

Therefore $k = k_1 = k_2 = a$,

and $z = \psi (y + a x) + x \psi_1 (y + x) + x^2 \psi_2 (y + a x).$

(40). Let $\frac{d^3 z}{d x^3} = a^3 z.$ Make $\frac{d z}{d x} = k z$;

$$\therefore \frac{d^3 z}{d x^3} = k^3 z = a^3 z;$$

consequently $k = a, \ \frac{-1 + \sqrt{(-3)}}{2} a, \ \frac{-1 - \sqrt{(-3)}}{2} a,$

and

$$z = \epsilon^{ax} \phi y + \epsilon^{-\frac{1}{2} ax} \cos \frac{a x \sqrt{3}}{2} \phi_1 y + \epsilon^{-\frac{1}{2} ax} \sin \frac{a x \sqrt{3}}{2} \phi_2 y.$$

Euler. *Ib.* Art. 395.

(41). Let $\frac{d^2 z}{d x^2} + b \frac{d^2 z}{d x d y} - 2 a \frac{d z}{d x} - a b \frac{d z}{d y} + a^2 z = 0.$

Make $z = \epsilon^{ax} v$; after the proper substitutions and dividing by ϵ^{ax}, we get

$$\frac{d^2 v}{d x^2} + b \frac{d^2 v}{d x d y} = 0;$$

$$\therefore z = \epsilon^{ax} \{ \phi (y - a x) + \psi y \}.$$

Euler, *Ib.* Art. 403.

(42). Let $(a + 2 b) z - (2 a + 3 b) \frac{d z}{d x} + c . \frac{d z}{d y} + a \frac{d^2 z}{d x^2}$

$$- 2 c . \frac{d^2 z}{d x d y} + b \frac{d^3 z}{d x^3} + c \frac{d^3 z}{d x^2 d y}.$$

Make $z = \epsilon^x v$, and proceeding as in the last Example, we get

$$0 = (a + 3 b) \frac{d^2 v}{d x^2} + b \frac{d^3 v}{d x^3} + c \frac{d^3 v}{d x^2 d y}.$$

Making $\frac{d^2 v}{d x^2} = u$, and integrating, we get

$$u = \epsilon^{-\frac{(a+3b)}{c} y} \phi''(c x - b y),$$

and $$z = \epsilon^{x} \{ \epsilon^{-\frac{(a+3b)}{c} y} \phi(c x - b y) + x \phi_1 y + \phi_2 y \}.$$

Euler, *Ib.* Art. 404.

(43). Let $\frac{d^2 z}{d x^2} \cdot \frac{d^2 z}{d y^2} - \left(\frac{d^2 z}{d x d y}\right)^2 = 0$, or $r t - s^2 = 0$;

$$\text{let } \frac{r}{s} = \frac{s}{t} = v, \text{ or } r = v s, \; s = v t;$$

$$\therefore \; d p = r d x + s d y = v (s d x + t d y) = v d q;$$

$$\therefore \; p = \phi(q);$$

and the complete integral is found by eliminating a from the equations,

$$\left.\begin{aligned} y - \phi a &= (z - a) \phi' a \\ x - \psi a &= (z - a) \psi' a \end{aligned}\right\}$$

This is the equation of all curve surfaces, which admit of developement, or which can be spread flat upon a plane without tearing or doubling; of this kind are all cylindrical and conical surfaces. Monge, *Application*....p. 82.

(44). Let $r^2 - t^2 = r t - s^2$.

Make $q = \phi(p)$, $s = r \phi'(p)$, $t = s \phi'(p) = r \{ \phi'(p) \}^2$;

$$\therefore \; r^2 - t^2 = r^2 \{ 1 - (\phi' p)^2 \} = 0;$$

by last Example

$$\therefore \; \phi' p = 1, \text{ and } q = p + c,$$

where c is an arbitrary constant; the integral of this equation gives

$$z - c x = \psi(x + y),$$

a *particular* integral of the original equation, involving only one arbitrary function.

This method of solution applies to all equations of the form

$$P = (rt - s^2)^n Q,$$

where P is a function of p, q, r, s, t, homogeneous with respect to r, s, and t; n any positive number, whole or fractional, and Q any function whatever of x, y, z, p, q, r, s, t, &c. which does not become infinite when $rt - s^2 = 0$; it was given by Poisson in the *Correspondance sur l'Ecole Polytechnique*. Vol. II. p. 410.

(45). Let $t + 2ps + (p^2 - a^2) r = 0$; in this case $Q = 0$.

$$\therefore \{ (\phi' p)^2 + 2p\phi' p + (p^2 - a^2) \} r = 0,$$

$$\text{or } dq^2 + 2p\,dp\,dq + (p^2 - a^2)\,dp^2 = 0,$$

$$\therefore dq = -(p \pm a)\,dp,$$

and therefore $$q + \frac{1}{2}p^2 \pm ap = c,$$

consequently by the method given in p. 445, we find

$$z = px + (c \pm ap - \frac{1}{2}p^2)y + \psi p$$

$$0 = x \pm ay - py + \psi' p,$$

which is a particular integral of the equation. This equation expresses accurately the law of the propagation of sound in an horizontal cylindrical canal, when the vibrations of the air are not considered as infinitely small and when the temperature is constant. Poisson, *Ib*.

(46). Let $(1 + q^2) r - 2pqs + (1 + p^2) t = 0$;

in the same manner, we get

$$(1 + q^2)\,dp^2 - 2pq\,dp\,dq + (1 + p^2)\,dq^2 = 0. \quad (1).$$

by differentiating this equation, considering $d\,p$ as constant, we get

$$d^2 q = 0, \text{ or } q = ap + b,$$

where b is determined in terms of a, by substitution in equation (1), which gives

$$1 + a^2 + b^2 = 0\,;$$

consequently $q = ap + \surd(-1 - a^2)$,

and $z = \phi(x + ay) + y\surd(-1 - a^2)$ (2).

a particular integral.

The original equation expresses the condition that the two radii of curvature of a curve surface are equal, and upon opposite sides of the surface; the second equation (2) is that of a cylindrical surface; the only surface which answers both these conditions is a *plane* which is obviously included in the equation (2), and which indeed is the only surface it can represent.

The complete integral of this equation has been given both by Monge and Legendre; under its most simple form, it is the result of the elimination of a and b from the equations

$$x = a + b$$

$$y = \phi\, a + \psi\, b$$

$$z = \int d\,a \surd\{1 - (\phi'\, a)^2\} + \int d\,b \surd\{1 - (\psi'\, b)^2\}.$$

The extent to which these examples have already proceeded, compel us to omit all notice and illustration of many other transformations and methods of solution of partial differential equations of the second and higher orders, which have been given by Laplace, Legendre, Poisson, Monge, and other analysts; the same reason induces us omit the discussion of their geometrical characters, and what is of most importance, the exemplification of the methods of determining the arbitrary functions which enter into their complete integrals; a subject which has been noticed by Mr. Herschel in Note *P*, to the Translation of Lacroix, but which in most cases belongs to a different branch of analysis.

Partial differential equations occur in many of the most interesting and difficult researches of Natural Philosophy, and thus the most important results are connected with the improvement of methods for their solution; amongst these may be mentioned the vibrations of a cord, whether of uniform or variable thickness, in media both resisting and non-resisting, the motion of waves and the propagation of sound upon all hypotheses, as alone sufficient to convince the reader of the importance of a subject which is hardly noticed by the mathematical writers of this country.

On the Particular Solutions of Differential Equations.

I. GIVEN the complete integral of a differential equation, to find it particular solutions, if any exist.

If $U=0$, be the complete integral cleared of radicals, and if c be the arbitrary constant which it involves, we must eliminate c from $U=0$, by means of the *variable* values of c which are given by the equation $\frac{dU}{dc}=0$.

(1). Take $\sqrt{(x^2+y^2-a^2)}=y+c$,

the complete integral of

$$x\,dx+y\,dy=dy\sqrt{(x^2+y^2-a^2)}.$$

Exterminating the radical, we get

$$U=x^2-2cy-c^2-a^2=0,$$

$$\frac{dU}{dc}=-2y-2c=0, \text{ or } c=-y,$$

and therefore

$$x^2 + y^2 - a^2 = 0,$$

which is a *particular* solution. Lagrange, *Mémoires de Berlin*. 1774. p. 199. *Calcul des Fonctions. Lèçon* 14.

(2). Take $x^2 = \sin(2y + c)$, the complete integral of $dy = \frac{x\,dx}{\sqrt{(1-x)^4}}$;

$$\therefore \frac{dU}{dc} = \cos(2y + c) = 0, \text{ or } 2y + c = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}:$$

the first gives $1 - x^2 = 0$, and the second $1 + x^2 = 0$, both of which are *particular* solutions.

(3). Take Ex. 67. p. 345, whose complete integral is

$$\sqrt{X} \pm \sqrt{Y} = (x - y)\sqrt{(C + u)}, \text{ where}$$

$$u = e(x + y) + f(x + y)^2.$$

Exterminating the radicals, we find

$$\frac{(X - Y)^2}{(x - y)^4} - C^2 - 2Cu - u^2 = U = 0,$$

and therefore $\frac{dU}{dC} = -2C - 2u = 0$, or $C = -u$,

consequently $X - Y = 0$, is a *particular* solution. We must except the case, when $e = 0$, and $f = 0$, as in Ex. 68. p. 347, when the integral appears under a different form, but of which $X - Y = 0$ is a *particular* solution, where $C = -2b(x+y) - c(x+y)^2$: and also that in which two pairs of roots, or the four roots of $X = 0$, and $Y = 0$, are equal to each other as in Ex. 71. of which the integral involves no radical, and only the first power of C, and consequently admits not of a *particular* solution. Lagrange, *Memoires de Berlin*. 1773. p. 206.

(4). Let $y - a\,x - \frac{1}{2}\,x^2\,(c-a)^2 + \frac{b^2}{4}\,(c-a)^4 = 0$,

be the complete integral of

$$y - a\,x - \frac{1}{2}\,x\,(p-a) + \frac{b^2}{4\,x^2}\,.\,(p-a)^2 = 0,$$

$$\frac{d\,U}{d\,c} = -\,x^2\,(c-a) + b^2\,(c-a)^3 = 0,$$

an equation which is satisfied by making $c - a = 0$, or

$$(c-a)^2 = \frac{x^2}{b^2};$$

the first gives a *particular integral,*

$$y - a\,x = 0\,;$$

the second gives a *particular solution,* which is

$$y - a\,x - \frac{x^4}{4\,b^2} = 0.$$

(5). Let $a^2 + b^2 + b\,x + \frac{1}{2}\,a\,x^2 - y = 0 = U$, be the complete integral of the equation of the second order and degree

$$y - x\,\frac{d\,y}{d\,x} + \frac{1}{2}\,x^2\,\frac{d^2\,y}{d\,x^2} - \left(\frac{d\,y}{d\,x} - x\,\frac{d^2\,y}{d\,x^2}\right)^2 - \frac{d^2\,y^2}{d\,x^4} = 0,$$

or $y - x\,p + \frac{1}{2}\,x^2\,q - (p - x\,q)^2 - q^2 = 0.$

Let $d'\,U$ denote the differential of U, considering a and b as variable, and b as a function of a; then we have

$$\frac{d\,U}{d\,x} = (a\,x + b) - p = 0,$$

$$\frac{d'\,U}{d\,a} = \frac{1}{2}\,x^2 + 2\,a + (x + 2\,b)\,\frac{d\,b}{d\,a} = 0,$$

$$\frac{d\,d'\,U}{d\,x\,d\,a} = x + \frac{d\,b}{d\,a} = 0, \text{ or } \frac{d\,b}{d\,a} = -\,x,$$

a variable quantity; eliminating a, b, and $\frac{db}{da}$ from equations

$$U = 0,\quad \frac{dU}{dx} = 0,\quad \frac{d'U}{da} = 0, \text{ and } \frac{dd'U}{dx\,da} = 0,$$

we get

$$\frac{dy^2}{dx^2} + \left(\frac{1}{2}x^3 + x\right)\frac{dy}{dx} - \frac{x^4}{16} - y(1 + x^2) = 0; \qquad (\alpha).$$

by solving this equation (α), with respect to $\frac{dy}{dx}$, we shall readily reduce it to the form,

$$\frac{4\,dy + 2x\,dx + x^3\,dx}{\sqrt{(16y + 4x^2 + x^4)}} = dx\sqrt{(1 + x^2)}$$

and

$$\therefore\ \sqrt{(16y + 4x^2 + x^4)} = x\sqrt{(1 + x^2)} + \log\{x + \sqrt{(1 + x^2)}\} + C,$$

a *particular* solution of the differential equation of the second order, involving only one arbitrary constant. Lagrange, *Calcul des Fonctions, Leçon* 14.

(6). Let $\{p - (x-1)b\}^2 + px - \frac{bx^2}{2} - y = 0 = U \ (1)$

and

$$\frac{p^2}{x^2} + \frac{px}{2} + \frac{ax}{2} + \frac{2ap(x-1)}{x^2} + \frac{a^2(x-1)^2}{x^2} - y = 0 = U_1 \quad (2).$$

be the two first integrals of the equation

$$y - px + \frac{x^2 q}{2} - \{p - (x-1)q\}^2 = 0.$$

In the first place, taking equation (1), we find

$$\frac{dU}{db} = -2(x-1)\{p - (x-1)b\} - \frac{x^2}{2} = 0,$$

and eliminating b from $U = 0$, and $\frac{dU}{db} = 0$, we get

$$(x-1)^2 y + \frac{x^4}{16} - \frac{px}{2}(x-1)(x-2) = 0:$$

if this be reduced to the form

$$dy - \frac{2y(x-1)\,dx}{x(x-2)} = \frac{x^3\,dx}{8(x-1)(x-2)},$$

and integrated (see p. 320.) we get

$$y = \frac{x(x-2)}{8}\left\{c + \log(x-1) - \frac{4}{x-2}\right\}$$

a *particular* solution of the given equation.

If we proceed with the second equation (2), we find

$$\frac{dU_1}{da} = \frac{x}{2} + 2p\frac{(x-1)}{x^2} + 2a\frac{(x-1)^2}{x^2} = 0,$$

and substituting the value of a deduced from hence in equation (2) or $U_1 = 0$, we get

$$(x-1)^2 y + \frac{x^4}{16} - \frac{px}{2}(x-1)(x-2) = 0,$$

the same result, as deduced from the first equation; this confirms the remark of Lagrange*, that it is indifferent from which of the two first integrals of an equation of the second order, the *particular* solution is derived, as it is in both cases the same.

II. Given the differential equation, to find its *particular* solutions, if any exist, without the intervention of the complete integral.

If $p = \frac{dy}{dx}$ and $\frac{dp}{dy} = \frac{M}{N}$, where M and N are in their lowest terms, then all the factors of N made equal to nothing, which *satisfy* the differential equation, are *particular* solutions of it.

There may however exist *particular* solutions which this process alone will not ascertain; as, for instance, in the case where $p = f(x)$: it is only necessary to make $p_1 = \frac{dx}{dy}$ and $\frac{dp_1}{dx}$

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* *Calcul des Fonctions, Leçon* 14.

$= \frac{M_1}{N_1}$ and proceed as before, in order to derive all those solutions which are not determined by the other trial.

If $U = f(x, y, p) = 0$, where the equation is not solved with respect to p, then we have

$$\frac{dp}{dy} = -\frac{\frac{dU}{dy}}{\frac{dU}{dp}} \text{ or } \frac{dp_1}{dx} = -\frac{\frac{dU}{dx}}{\frac{dU}{dp_1}}:$$

the *particular* solutions are determined by eliminating p from $\frac{dU}{dp} = 0$ and $U = 0$, provided always that these solutions do not make $\frac{dU}{dy} = 0$; they may likewise be similarly derived by eliminating p_1 from $\frac{dU}{dp_1} = 0$, and $U = 0$.

Lagrange remarked that *particular* solutions render $\frac{dp}{dx}$ or $\frac{d^2 y}{dx^2} = \frac{0}{0}$, a test which is a necessary consequence of the preceding, since

$$dp = \frac{d^2 y}{dx} = -\frac{\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy\right)}{\frac{dU}{dp}};$$

of which the numerator and denominator must be equal to nothing respectively, in virtue of the equations

$$\frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dU}{dp}dp = 0, \text{ and } \frac{dU}{dp} = 0.$$

It is amongst equations of a degree exceeding the first, that we must look for *particular* solutions; for if the first power of p alone enters into $U = 0$, it will not appear in $\frac{dU}{dp} = 0$, which will not therefore enable us to eliminate p from $U=0$, upon which the determination of the particular solution depends; the same remark extends to equations which are linear with respect to p, but which involve radicals, for we can only consider such equations, as factors made equal to zero of equations of higher degrees, depending upon the number and nature of the radicals involved. Corresponding processes for determining the particular solutions of equations of higher orders, will enable us to extend these remarks to such equations; and we may assert generally, that no *linear* equation, properly so called, whatever be its order, admits of a *particular* solution.

(7). Let $dy = dx\sqrt{(y-a)}$; in this case we have

$$p = \sqrt{(y-a)} \text{ and } \frac{dp}{dy} = \frac{1}{2\sqrt{(y-a)}} = \infty,$$

when $y = a$, which is therefore a *particular* solution.

The complete integral is $x = 2\sqrt{(y-a)} + c$, which gives the particular solution, by making $c = x$, a variable quantity.

(8). Let $dy = \frac{x\,dx}{\sqrt{(1-x^4)}}$; $\therefore \frac{dx}{dy} = p_1 = \frac{\sqrt{(1-x^4)}}{x}$;

$$\therefore U = x^2 p_1^2 - 1 + x^4 = 0;$$

$$\therefore \frac{dU}{dp_1} = 2x^2 p_1 = 0, \text{ and } \frac{dU}{dx} = 2p_1^2 x + 4x^3.$$

If $p_1 = 0$, we have $1 - x^4 = 0$, which does not make $\frac{dU}{dx} = 0$; it is therefore a *particular* solution, since we have $\frac{dy}{dx} = \frac{x}{0}$, and therefore y must be variable, otherwise $\frac{dy}{dx}$ would in this case $= \frac{0}{0}$. If $x = 0$, we have also $\frac{dU}{dx} = 0$; also in this case $dy = 0$, or $y = c$ a constant quantity; $x = 0$, is not therefore a solution of the equation.

(9). Let $y\,dx - x\sqrt{(dx^2 + dy^2)} = 0$,

or
$$U = y^2 - x^2 - x^2 p^2 = 0,$$
$$\frac{dU}{ap} = -\,2\,x^2 p = 0.$$

If $p = 0$, we have $y^2 - x^2 = 0$; but since $p = 0$, we must have y and therefore in this case x, a constant quantity: it is not therefore a solution of the equation.

If $x = 0$, we have also $y = 0$, to which the same remark applies.

The complete integral of this equation is
$$\frac{\{y + \sqrt{(y^2 - x^2)}\}^2}{4\,x^2} - \frac{1}{2}\log\{y + \sqrt{(y^2 - x^2)}\} - \frac{3}{2}\log x = C.$$
Euler, *Calc. Integ.* Tom. I. Art. 681.

(10). Let $y^2 p^2 - 2\,x\,y\,p + a\,x + by = 0 = U$;
$$\therefore\ \frac{dU}{dp} = 2\,y^2 p - 2\,x\,y = 0,$$
the elimination of p from these equations, gives us $a\,x + b\,y - x^2 = 0$, which does *not satisfy* the differential equation: it is therefore absolutely necessary to examine in all cases, whether the equation which is the result of the elimination of p, be a solution of the original equation or not.

(11). Let $y - xp - a\,(1 + p^2) = 0 = U$;
$$\therefore\ \frac{dU}{dp} = -\,x - 2\,a\,p = 0,$$
and
$$4\,a\,y - 4\,a^2 + x^2 = 0,$$
is a *particular* solution.

Or thus, $p = -\,\dfrac{x}{2\,a} \pm \dfrac{\sqrt{(x^2 + 4\,a\,y - 4\,a^2)}}{2\,a}$, and $$\frac{dp}{dy} = \pm \frac{1}{\sqrt{(x^2 + 4ay - 4a^2)}} = \infty,$$

when $$x^2 + 4ay - 4a^2 = 0.$$

Euler, *Ib.* Art. 701.

(12). Let $(x - p)^2 (x^2 - 2xyp + a^2) = 0 = U$;

$$\therefore \frac{dU}{dp} = 2(x - p)(x^2 + a^2 - xyp - x^2 y) = 0.$$

If $p = x$, we get

$$(x - x)^2 x^2 + a^2 - 2x^2 y) = 0;$$

from the first factor $(x - x)^2 = 0$, nothing can be inferred: the second factor $x^2 + a^2 - 2x^2 y = 0$, is no solution of the equation.

If we wish to extend this method to equations of the second order, we must consider two cases; 1st, when it is proposed to determine a particular solution of the equation which involves p, x, and y only, or whose integral involves one arbitrary constant, or 2dly, when the particular solution is a function of x and y only, and involves no arbitrary constant whatever.

In the first case, if $q = \frac{d^2 y}{dx^2}$, we must eliminate q from $U = 0$, and $\frac{dU}{dq} = 0$; in the second case, we must eliminate p and q from $U = 0$, $\frac{dU}{dp} = 0$ and $\frac{dU}{dq} = 0$, and in both cases, we must try whether the equation which is the result of elimination, satisfies the original equation or not.

(13). Let $y - xp + \frac{x^2}{2} q - pq + xq^2 = 0 = U$;

$$\therefore\ \frac{dU}{dq} = \frac{1}{2}x^2 - p + 2xq = 0;$$

eliminating from these equations, we get

$$4xy - 3px^2 - p^2 - \frac{x^4}{4} = 0. \qquad (\alpha).$$

a *particular* solution, since it satisfies the original equation.

This equation admits of no *particular* solution of the second kind, or which is a function of x and y only, without an arbitrary constant, since the equation $\frac{dU}{dp} = 0$, does not involve p, and therefore does not enable us to eliminate it; but $y + \frac{x^3}{2} = 0$, is a *particular* solution of the first *particular* solution (α), but which does not satisfy the original equation $U = 0$; an important remark the truth of which has been demonstrated generally by Lagrange, *Calcul des Fonctions Leçon* 14.

The complete integral of the equation $U=0$, is

$$ab + bx + \frac{1}{2}ax^2 - y = 0.$$

(14). Let $y - xp + \frac{x^2}{2}q - p^2 + (2x - 1)pq$

$$- (x^2 - x + 1)q^2 = 0 = U;$$

$$\frac{dU}{dq} = \frac{1}{2}x^2 + (2x - 1)p - 2(x^2 - x + 1)q = 0.$$

Eliminating q we get

$$(x^2 - x + 1)y + \frac{x^4}{16} - \frac{x^3p}{2} + \frac{3x^2p}{4} - xp - \frac{3p^2}{4} = 0, \qquad (\alpha).$$

a *particular* solution of the first order,

By eliminating p and q between the equations $U = 0$,

$$\frac{dU}{dp} = 0,\ \frac{dU}{dq} = 0,$$

we get $$y + \frac{x^2}{12}\cdot(x^2 - 2x + 4) = 0,$$

which is no solution of the original equation.

The complete integral of the equation is

$$a^2 + ab + b^2 + bx + \frac{1}{2}ax^2 - y = 0.$$

(15). Let $y - xp + \frac{x^2 q}{2} - \frac{x^3 r}{6} - p^2 + 2pqx - q^2x^2 -$

$$prx^2 + qrx^3 - \frac{r^2x^4}{4} = 0 = U,$$

$$\text{where } r = \frac{d^3y}{dx^3}.$$

Eliminating r from the equations $U = 0$, and $\frac{dU}{dr} = 0$, we get

$$y - \frac{2px}{3} + \frac{x^2 q}{6} + \frac{x^2}{36} = 0,$$

a *particular* solution, being a linear equation of the second order only, whose integral is

$$y = cx^3 + c_1 x^2 - \frac{x^2}{6}(1 + \log x).$$

The differential equation admits not of a particular solution, whose order is lower than the second; for the equations $\frac{dU}{dq} = 0$, and $\frac{dU}{dp} = 0$, are inconsistent with each other, and with $\frac{dU}{dr} = 0$, and therefore cannot exist simultaneously; the reason of this is evident from the complete integral of the equation, which is

$$y - ax + \frac{bx^2}{2} - \frac{cx^3}{6} - a^2 = 0.$$

III. Given a solution of a differential equation, to find whether it is included in the complete integral or not.

Let $y=X$, be the given solution; in the equation reduced to the form $dy=p\,dx$, substitute $X+h$ for y; if the expansion involve a power of h, whose exponent is a proper fraction, then the solution in question is a *particular* solution; otherwise not.

If the given solution, involve x alone, we must reduce the equation to the form $dx=p_1\,dy$, and substitute $a \pm h$ for x, where a is the value of x deduced from the solution $X=0$; the test is the same as before.

From the principles explained in Note F, it is obvious that when the solution $y=X$, answers the preceding conditions, it must also make $\frac{dp}{dy}=\infty$ in one case, or $\frac{dp_1}{dx}=\infty$ in the other; or if $U=0$, be the given equation and a rational function of x, y, and p, then if the solution $y=X$, makes $\frac{dU}{dp}=0$, and not $\frac{dU}{dy}=0$, it is a particular solution; otherwise not.

In equations of the second order; if $p=X$, be a solution, which involves x or y or both: for q substitute $\frac{dp}{dx}+h$, or $X'+h$, in the equation: if the result involves no power of h whose exponent is a proper fraction, $p=X$ is a particular solution, otherwise not: or more simply thus; $p=X$, is a particular solution when it makes $U=0$ and $\frac{dU}{dp}=0$.

If the solution does not involve p, it is a particular solution, when it makes $U=0$, $\frac{dU}{dp}=0$, and $\frac{dU}{dq}=0$.

It would perhaps have been more natural to have made the problems considered in this section precede those given in the last, such at least being the order which they occupy in the works in which they are investigated, and the propositions are obviously the converse of each other; in the exemplification of them, however, it does not seem to be a matter of great consequence to adhere to the precise order in which they are deduced, as the student must be in possession of the theory, of both the propositions before he can proceed to apply either one or the other.

(16). Let $a^2 dy + y^2 dx = a^2 dx + xy dx$, which is satisfied by making $y=x$.

In this case $p = 1 + \frac{xy - y^2}{a^2}$; for y put $x + h$;

$\therefore\ p' = 1 - \frac{xh}{a^2} - \frac{h^2}{a^2}$, in which no power of h is involved, whose exponent is a proper fraction; $y = x$ is therefore a *particular integral*, as is evident from other considerations.

In order to find the complete integral, make

$$y = x + \frac{a^2}{v};$$

this gives $$v = \epsilon^{\frac{x^2}{2a^2}} \left\{ c + \int \epsilon^{-\frac{x^2}{2a^2}} dx \right\},$$

and $$y = x + \frac{a^2}{v} = x + \frac{a^2 \epsilon^{-\frac{x^2}{2a^2}}}{\left\{ c + \int \epsilon^{-\frac{x^2}{2a^2}} dx \right\}} = x,$$

when $c = \infty$. Euler, *Calc. Integ.* Tom. I. Art. 544.

(17). Let $y = x\frac{dy}{dx} - \frac{1}{4} \cdot \frac{dy^2}{dx^2}$, which is satisfied by making $y = x^2$;

$$\therefore\ U = y - x p + \frac{1}{4} p^2 = 0,$$

$$\frac{d U}{d p} = - x + \frac{1}{2} p = 0, \text{ and } \frac{d U}{d y} = 1;$$

by making $y = x^2$, we satisfy the equation $\frac{d U}{d p} = 0$; it is therefore a *particular* solution.

(18). Let $d y = \frac{x\, d x}{\sqrt{(1 - x^4)}}$, which is satisfied by making $1 - x^4 = 0$.

$\therefore\ p_1 = \frac{\sqrt{(1 - x^4)}}{x}$; make $x = a - h$, where a is one of the roots of $1 - x^4 = 0$: we hence get

$$p_1' = \frac{\sqrt{\{1 - 1 + 4 a^3 h - 6 a^2 h^2 + 4 a h^2 - h^4\}}}{a - h}$$

$$= 2 a^{\frac{1}{2}} \sqrt{h} + \&c.;$$

consequently $1 - x^4 = 0$, is a *particular* solution.

(19). Let $a\, d y - a\, d x = d x \sqrt{(y^2 - x^2)}$, which is satisfied by making $y = x$.

$$\therefore\ U = (a p - a)^2 - y^2 + x^2 = 0,$$

$\frac{d U}{d p} = a^2 (p - 1) = 0$; which is satisfied by making $y = x$, which is therefore a *particular* solution.

(20). Let $(x y - 1)(2 x y q - y p + x p^2)^2 - x y (y + x p)^2 = 0$, which is satisfied by $x y - 1 = 0$,

$$\frac{d U}{d p} = 2 (x y - 1)(2 x p - y)(2 x y q - y p + x p^2)$$

$$- 2 x^2 y (y + x p) = 0$$

$$\frac{dU}{dq} = 4(xy - 1)\,xy\,(2xyq - yp + xp^2) = 0$$

and both these equations are satisfied by making $xy - 1 = 0$, which is therefore a *particular* solution.

IV. Every particular solution must render *infinite* a *multiplier* of a differential equation; but the converse of this proposition, that every solution of a differential equation which renders a *multiplier* infinite, is a *particular* solution, is not generally true*.

(21). Take Ex. p. 479, of which $\frac{1}{\sqrt{(x^2 + y^2 - a^2)}}$ is a multiplier, which the particular solution $x^2 + y^2 - a^2 = 0$, renders infinite.

(22). Take the equation

$$\frac{2xy\,dx}{\sqrt{(1 + x^2)}} + x\,dx - 3y\,dy = 0,$$

whose integral is

$$\log\{y + \sqrt{(1 + x^2)}\} + \frac{\sqrt{(1 + x^2)}}{2y + \sqrt{(1 + x^2)}} = c;$$

in this case $y = -\sqrt{(1 + x^2)}$, which renders the multiplier $\frac{1}{\{y + \sqrt{(1 + x^2)}\}^2}$ infinite, is a *particular integral* corresponding to $c = -\infty$.

(23). Take the equation in Ex. 62. p. 343, whose multiplier

$$\frac{1}{(x - a^2)\sqrt{(y^2 + 2ay + x)}}$$

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[*] Laplace, *Mémoires de l'Acad. des Sciences.* 1772.

becomes infinite by making $x - a^2 = 0$, or $y^2 + 2ay + x = 0$; both of these solutions are particular integrals, one corresponding to $c = \infty$, and the other to $c = 0$.

(24). Take Ex. 17. p. 357, of which the multiplier

$$\frac{1}{(x^2 - a^2)\sqrt{(x^2 + y^2 - a^2)}}$$

becomes infinite by making $x^2 + y^2 - a^2 = 0$, or $x^2 - a^2 = 0$; the first is a *particular* solution; the second does not satisfy the original equation, and is therefore no solution whatever.

Legendre* has shewn that every differential equation admitting a particular solution, may be so modified that the particular solution *may become a factor of it*; and Poisson† has shewn that any given particular solution may be introduced into a differential equation, by multiplying it by a factor containing that solution, and subsequently effecting a change of the variables; though these problems are extremely interesting in the theory of particular solutions, yet as they are of little use in the practical parts of analysis, we shall not exemplify them; we shall conclude with a few remarks on the particular solutions of partial differential equations.

V. Given a solution of a partial differential equation, to find whether it is included in the general solution or not.

If $U = f(x, y, z, p, q) = 0$, be an equation of the first order, then the solution $z = F(x, y)$, is a particular solution, if it makes $\frac{dU}{dp} = 0$, and $\frac{dU}{dq} = 0$; otherwise not.

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* *Mémoires de l'Académie des Sciences*, 1790.

† *Journal de l'Ecole Polytechnique*, Cah. XIII. p. 73.

If $U = f(x, y, z, p, q, r, s, t) = 0$, be an equation of the second order, then the solution $z = F(x, y)$, is a particular solution, if it makes

$$\frac{dU}{dp} = 0, \frac{dU}{dq} = 0, \frac{dU}{dr} = 0, \frac{dU}{ds} = 0, \frac{dU}{dt} = 0;$$

or if $z = F(z, y, p, q)$, be an equation of the first order and a solution of $U = 0$, then it is a particular solution, if it also makes

$$\frac{dU}{dr} = 0, \quad \frac{dU}{ds} = 0, \quad \frac{dU}{dt} = 0.$$

The mode of extending this rule to partial differential equations of all orders is obvious.

The solution of this problem, immediately suggests the solution of its converse; namely, to find the particular solutions which a partial differential equation admits, without knowing its general solution; thus, in the case of equations of the first order, if $V = 0$, be the result of the elimination of p and q from the equations $U = 0$, $\frac{dU}{dp} = 0$, and $\frac{dU}{dq} = 0$; the factors of $V = 0$, which satisfy these equations, are particular solutions; a similar method is applicable to equations of higher orders.

(25). Let $z = px + qy + pq$, which is satisfied by making $x = -xy$.

$$\therefore \; U = z - px - qy - pq = 0,$$

$$\left.\begin{array}{l} \dfrac{dU}{dp} = -x - q = 0, \text{ if } z = -xy \\[2ex] \dfrac{dU}{dq} = -y - p = 0, \text{ if } z = -xy; \end{array}\right\}$$

the requisite equations are therefore satisfied, and $z = -xy$

is a *particular* solution, not included in the general solution, which is the result of the elimination of p from the equations

$$\left.\begin{aligned} z &= p x + y \phi p + p \phi p \\ 0 &= x + y \phi' p + \phi p + p \phi' p. \end{aligned}\right\}$$

(26). **Prob.** To find the nature of the curve surface, so that the perpendiculars drawn from a given point upon its tangent planes may be always of the same length.

Let the given point be the origin of the co-ordinates, x', y', z', the co-ordinates of the tangent plane, and x, y, z, the co-ordinates of the surface; the equation of the plane is

$$z' = a + b x' + c y',$$

and at the point of contact, we have

$$z' = z, \; x' = x, \; y' = y,$$

$$\frac{d z'}{d x'} = b = \frac{d z}{d x} = p, \quad \frac{d z'}{d y'} = c = \frac{d z}{d y} = q;$$

and therefore its equation for that point becomes

$$z = a + p x + q y,$$

$$\text{or } a = z - p x - q y.$$

Again, if u be the distance of the given point and the point of contact, we have

$$u^2 = x'^2 + y'^2 + z'^2,$$

and since this line is perpendicular to the given plane, it is a *minimum*, and $d u = 0$; consequently

$$u \, d u = x' d x' + y' d y' + z' d z' = 0,$$

but $\qquad d z' = b \, d x' + c \, d y';$

therefore $\qquad (x' + b z') d x' + (y' + c z') d y' = 0,$

or $$x' + b z' = 0, \text{ and } y' + c z' = 0,$$

or $$x = -b z, \quad \text{and } y = -c z$$

at the point of contact:

$$\therefore z = a - b^2 z - c^2 z, \quad \text{or } z = \frac{a}{1 + b^2 + c^2},$$

and $$u = h = \frac{a}{\sqrt{(1 + b^2 + c^2)}} = \frac{z - x p - y q}{\sqrt{(1 + p^2 + q^2)}},$$

or $$z - x p - y q - h \sqrt{(1 + p^2 + q^2)} = 0 = U,$$

$$\frac{d U}{d p} = - x - \frac{h p}{\sqrt{(1 + p^2 + q^2)}} = 0,$$

$$\frac{d U}{d q} = - y - \frac{h q}{\sqrt{(1 + p^2 + q^2)}} = 0;$$

and the elimination of p and q, gives us

$$x^2 + y^2 + z^2 = h^2$$

which is the equation of a sphere, and a *particular* solution of the differential equation.

The general solution is derived from the equations

$$z = b x + y \phi b + h \sqrt{\{ 1 + b^2 + (\phi b)^2 \}} \qquad (1).$$

$$x + y \phi' b + \frac{h (b + \phi b \phi' b)}{\sqrt{\{ 1 + b^2 + (\phi b)^2 \}}} = 0. \qquad (2).$$

If in equation (1.) we make $\phi b = c$, we get

$$z = b x + c y + h \sqrt{(1 + b^2 + c^2)}, \qquad (3).$$

the equation of a plane, involving two arbitrary constants and which is evidently a solution of the differential equation, as it gives

$$\frac{d z}{d x} = p = b, \text{ and } \frac{d z}{d x} = q = c.$$

The surface given by the particular solution, always touches this plane, whatever be the value of the constants b and c. See the remarks in p. 358, 359.

Lagrange has denominated the equation (3), the *complete solution* of the differential equation, to distinguish it from the *general solution*, which involves an arbitrary function; under this form alone the theory exemplified in p. 479, by which the *particular solutions* are deduced from the complete integrals, admits of application in the case of partial differential equations.

The connection between the *complete solution* and the *general solution* is very simple in the case of partial differential equations of the first order, and the second may be always deduced from the first, by making $c = \phi b$; thus

$$z = a x + b y + a^2 + b^2$$

is the complete solution of

$$z = p x + q y + p^2 + q^2.$$

If we make $b = \phi a$, we get

$$\left.\begin{aligned} &z = a x + y \phi a + a^2 + (\phi a)^2 \\ \text{and} \quad &x + y \phi' a + 2 a + 2 \phi a \phi' a = 0, \end{aligned}\right\}$$

a system of equations from which the *general solution* is determined.

For equations of the second order, the *complete solution* $U = 0$, will involve *five arbitrary constants*, the elimination of which from the equations

$$U = 0, \ \frac{d U}{d x} = 0, \ \frac{d U}{d y} = 0, \ \frac{d^2 U}{d x^2} = 0, \ \frac{d^2 U}{d x d y} = 0, \ \frac{d^2 U}{d y^2} = 0,$$

will give the original equation; but in this case the passage from the *complete solution* to the *general solution* involves considerable difficulties, even in the most simple cases; it is on this account, that we can derive very little advantage in the determination of the general solutions of equations of the second and higher orders, from a knowledge of their *complete solutions*, which in some cases can be determined very readily. For further information on this subject, the reader may consult a *Memoir* of Lagrange in the Berlin Acts for 1774, p. 266.

(27). Let $r^2 - 2rq\left(p - \frac{z}{1+x}\right)$

$$+ y\left(p - \frac{z}{1+x}\right) = 0 = U.$$

$$\frac{dU}{dr} = 2r - 2q\left(p - \frac{z}{1+x}\right) = 0;$$

eliminating r from $U = 0$, and $\frac{dU}{dr} = 0$, we find

$$q^2\left(p - \frac{z}{1+x}\right)^2 - y\left(p - \frac{z}{1+x}\right) = 0;$$

and since the factor $p - \frac{z}{1+x} = 0$,

makes $r = \frac{1}{1+x}\left(p - \frac{z}{1+x}\right) = 0$,

it satisfies both the equations, and is therefore a *particular solution*, its integral being

$$z = (1 + x)\phi y.$$

Poisson, *Journal de l'Ecole Polytechnique*, Cah. XIII. p. 115.

(28). Let $r^2 - t^2 + \left(\frac{p}{x} - \frac{q}{y}\right)(z - x - y) = 0 = U;$

$$\frac{dU}{dr} = 2r, \quad \frac{dU}{dt} = -2t, \quad \frac{dU}{dp} = \frac{z - x - y}{x}$$

and $\frac{dU}{dq} = \frac{z - x - y}{z};$

and all these quantities vanish by making $z = x + y$; it is therefore a *particular* solution involving no arbitrary function.

(29). To find whether $q = -\frac{1}{2}p^2 + cp + a$, which is a solution of

$$t + 2ps + (p^2 - c^2)r = 0 = U,$$

is a particular solution, or a particular integral:

$$\therefore \frac{dU}{dr} = p^2 - c^2, \quad \frac{dU}{ds} = 2p, \quad \frac{dU}{dt} = 1,$$

which the solution in question does not render severally equal to nothing; it is therefore a *particular integral.* See Ex. 45. p. 477.

There are many circumstances in which the analogy between the arbitrary functions which enter into the integrals of partial differential equations, and the arbitrary constants which enter into the integrals of differential equations of two variables, is not maintained, some of which have been already noticed (p. 462.): one of the most remarkable of these anomalies, occurs in a class of equations, whose integrals exhibited in a series by means of the theorem of Taylor, involve a number of arbitrary functions less than the order of the equation; thus in the equation $\frac{d^2z}{dy^2} = \frac{dz}{dx}$, not integrable under a finite form by the ordinary methods, if we assume

$$z = Z_0 + xZ_1 + \frac{x^2}{1.2}.Z_2 + \frac{x^3}{1.2.3}Z_3 + \&c.$$

where Z_0, Z_1, Z_2 &c, are the values of z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$ &c. when $x = 0$, we shall find

$$z = Fy + x^2.\frac{d^2Fy}{dy^2} + \frac{x^2}{2}.\frac{d^4Fy}{dy^4} + \&c.: \qquad (a.)$$

which involves only one arbitrary function.

But if Z_0, Z_1, Z_2, &c. are the values of z, $\frac{dz}{dy}$, $\frac{d^2 z}{dy^2}$ &c. when $y = 0$, by proceeding as before we shall find

$$z = fx + y\,\phi x + \frac{y^2}{1\,.\,2} \cdot \frac{d\,.\,fx}{dx} + \frac{y^3}{1\,.\,2\,.\,3} \cdot \frac{d^2\,.\,fx}{dx^2} + \&c. \quad (b).$$

which involves two arbitrary functions.

But this solution (b) is not more complete than the first (a), as the arbitrary functions may be reduced into one, and the solutions made to coincide with each other; thus, assume

$$fx = A + Bx + \frac{Cx^2}{1\,.\,2} + \frac{Dx^2}{1\,.\,2\,.\,3} + \&c.$$

$$\phi x = A' + B'x + \frac{C'x^2}{1\,.\,2} + \frac{D'x^3}{1\,.\,2\,.\,3} + \&c.$$

and $$z = A + A'y + \frac{By^2}{1\,.\,2} + \frac{B'\,.\,y^3}{1\,.\,2\,.\,3} + \frac{Cy^4}{1\,.\,2\,.\,3\,.\,4}$$

$$+ \frac{C'y^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} + \&c.$$

$$+ x\left(B + B'y + \frac{Cy^2}{1\,.\,2} + \frac{C'\,.\,y^3}{1\,.\,2\,.\,3}\right)$$

$$+ \frac{x^2}{1\,.\,2}(C + C'y + \&c.) + \&c.$$

$$= Fy + x\,.\,\frac{d^2\,.\,Fy}{dy^2} + \frac{x^2}{1\,.\,2} \cdot \frac{d^4\,.\,Fy}{dy^4} + \&c.$$

which is identical with the first solution.

And whenever the differential coefficient relative to one of the variables, is of a higher order than it is relative to the other, the number of arbitrary functions which enter into the integral of the equation, exhibited in a series, is equal to the exponent of the lowest order, or they may be always reduced to that number, even when the integral in the first instance involves a number of arbitrary functions equal to the exponent of the highest order *.

A question may naturally be asked, whether such integrals are particular solutions, or not; in one sense they might be considered as such, though in reality the most general solutions which the equations admit of; they more properly, therefore, constitute an exception to a proposition which is generally true, that the number of arbitrary functions, which enter into the integral of a partial differential equation, is equal to the exponent of the order of the equation.

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* Poisson, *Journal de l'Ecole Polytechnique,* Cah. XIII. p. 109.

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ADDITIONS AND CORRECTIONS.

PART I.

Page line

2. 3 *from bottom*, for $du =$ read $\frac{du}{u} =$

21. 13. This theorem, though usually attributed to Maclaurin, is certainly due to Stirling, who gave it in page 32 of his *Lineæ Tertii Ordinis Newtonianæ*, published in 1717, two years after the appearance of the *Methodus Incrementorum* of Brook Taylor: he has given several applications of it, including the Binomial theorem and the series for an arc in terms of its cosine.

67. 8. This theorem is deduced as follows: if

$\tan a = \frac{x}{y}$, $\tan b = \frac{1}{e}$, $\tan b_1 = \frac{1}{e_1}$, $\tan b_2 = \frac{1}{e_2}$, &c.

then since

$$a = (a - b) + b,$$
$$b = (b - b_1) + b_1, \text{ \&c.}$$

it is obvious that

$$\tan^{-1}\frac{x}{y} = \tan^{-1}\frac{\frac{x}{y} - \frac{1}{e}}{1 + \frac{x}{ey}} + \tan^{-1}\frac{1}{e}$$

$$= \tan^{-1}\frac{ex - y}{ey + x} + \tan^{-1}\frac{1}{e},$$

$$\tan^{-1}\frac{1}{e} = \tan^{-1}\frac{\frac{1}{e} - \frac{1}{e_1}}{1 + \frac{1}{ee_1}} + \tan^{-1}\frac{1}{e_1}$$

$$= \tan^{-1}\frac{e_1 - e}{ee_1 + 1} + \tan^{-1}\frac{1}{e_1},$$

Page line

$$\tan^{-1}\frac{1}{e_1} = \tan^{-1}\frac{\frac{1}{e_1} - \frac{1}{e_2}}{1 + \frac{1}{e_1 e_2}} + \tan^{-1}\frac{1}{e_2}$$

$$= \tan^{-1}\frac{e_2 - e_1}{e_1 e_2 + 1} + \tan^{-1}\frac{1}{e_2},$$

&c. &c.

and substituting successively for $\tan^{-1}\frac{1}{e}$, $\tan^{-1}\frac{1}{e_1}$, $\tan^{-1}\frac{1}{e_2}$, &c. their values given in each succeeding equation, we shall get the expression in the text.

90. 5 & 3 *from botttom.* We cannot consider these equations as *arithmetically* true in all cases; this is obvious, if we make $x = \frac{\pi}{2}$, in the first of the series to which we refer: the reader will find an explanation of this *paradox*, which has long been a source of embarrassment to Analysts, in some series of this kind, in a Paper by Mr. Babbage in the Philosophical Transactions for 1819.

108. 4. for $u = -27$, read $u = -26$.

123. 18. for $\frac{bc}{e^2 - bx + x^2}$, read $\frac{ec}{e^2 - cx + x^2}$.

— 19. for $x = \frac{b}{c}$, read $x = \frac{c}{2}$.

— last line. for $u =$, read $3u =$.

127. 5 *from bottom. for* Prop. 16. *read Theorem* 16.

129. 10. for $u = \frac{s}{3}\sqrt{\left(\frac{2s}{3\pi}\right)}$, read $u = \frac{u}{3}\sqrt{\left(\frac{2s}{3\pi\sqrt{3}}\right)}$.

132. 21. for CM, read CM'.

142. 11. for $y' = -\frac{ab}{x^2}x'$, read $y' = -\frac{ab}{x^2}x' + \frac{2ab}{x}$.

Page line

148. 17, 18. Instead of the expressions in the text, *read*

$$y = \pm (1 + \sqrt{2})\, x \pm \frac{a\,(3\sqrt{2} - 7)}{8\,(1 + \sqrt{2})} + \frac{A}{x} + \&c.$$

152. 19. for $y = \pm \sqrt{\left(\frac{-c}{5}\right)}$, read $y = \pm \sqrt{\left(\frac{-c}{3}\right)}$.

168. 15. for $u = a \cos 2\theta$, read $u^2 = a^2 \cos 2\theta$.

175. 17. for TM, read Tm.

176. 20, 21. for MH read MT, and for AT read AY.

183. 2. *for* in both these cases, *read* in the two first cases.

186. 20. for AEM, read AEQ.

188. 5. for $\frac{a\,\phi}{2 \tan \phi}$, read $\frac{a\,\phi}{2 \tan \frac{\phi}{4}}$.

190. 5. Where a is the radius of the generating circle, and b the distance of the describing point.

210. 14. for $-\frac{a\,d^2 z}{d y^2}$, read $-\frac{a^2\,d^2 z}{d y^2}$.

262. 20. *for* equal to the square of the diameter, *read* equal to twice the square upon the diameter.

PART II.

Page 267. Ex. 3. It ought to be remarked of this integral and of many others, which are contained in this collection, that its form is modified by the addition or subtraction of some constant quantity, in order to render it more simple: thus the ordinary process of integration, would give us

$$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{-x^2}{b(a+bx)} + \frac{2x}{b^2} - \frac{2a}{b^3}\log(a+bx),$$

and by subtracting $\frac{2a}{b^3}$ from this result, we get the expression in the text; a similar process is followed in Examples 4, 7, &c.

In Ex. 56. p. 276, the constant quantity added is

$$\frac{2}{3}\sin\frac{\pi}{3}\,.\,\tan^{-1}.\,\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}:$$

similar additions are made in Ex. 57, 58, 59, 62, 64, and 65.

We have been prevented by the short time which has elapsed, between the printing and the publication of the latter part of this Work, and other occupations, from giving a sufficiently minute examination of it to form a list of errors, of which it is very probable that the number is considerable; we hope, however, that they are not so numerous, or of such a kind, as to cause much embarrassment to the student.

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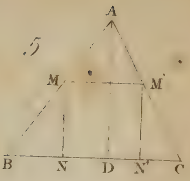

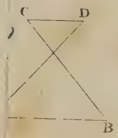

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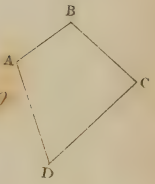

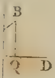

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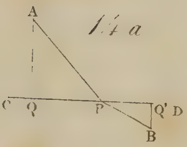

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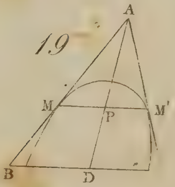

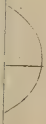

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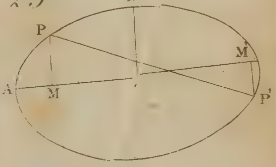

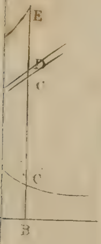

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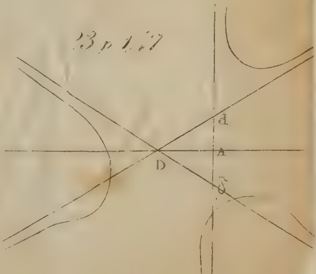

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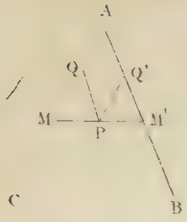

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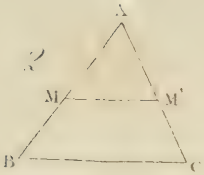

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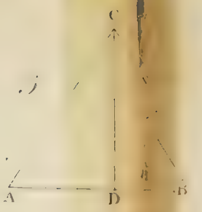

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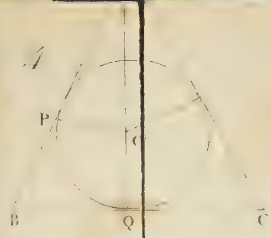

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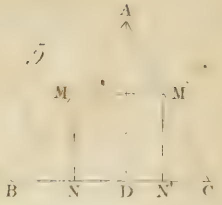

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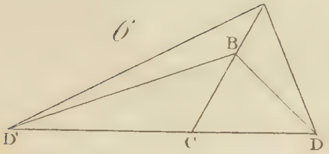

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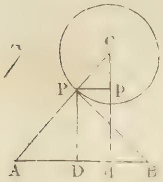

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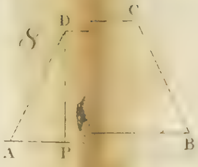

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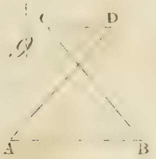

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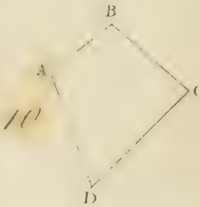

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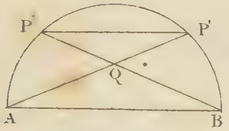

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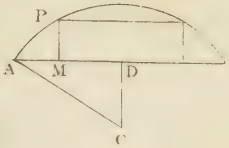

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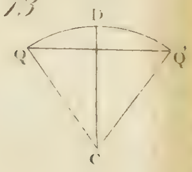

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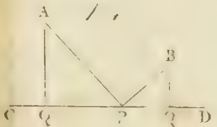

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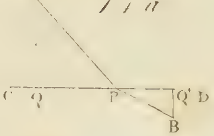

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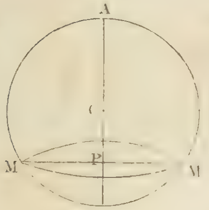

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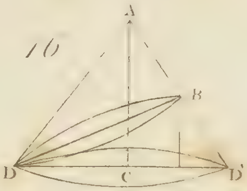

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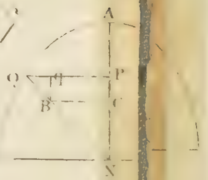

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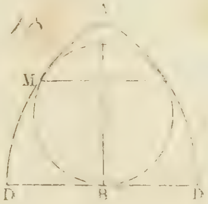

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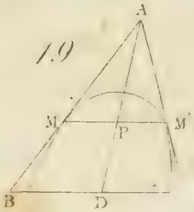

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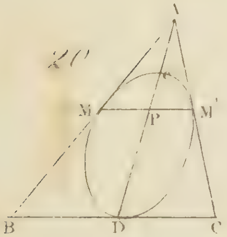

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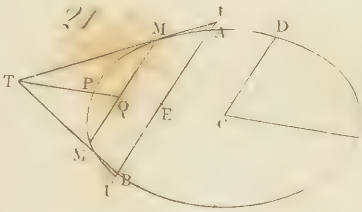

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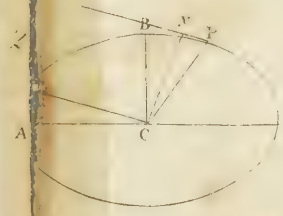

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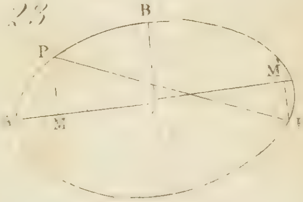

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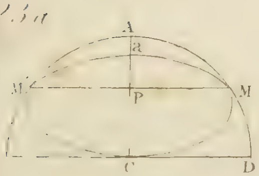

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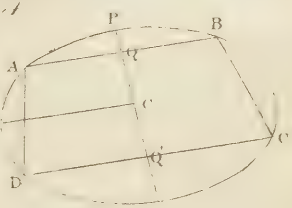

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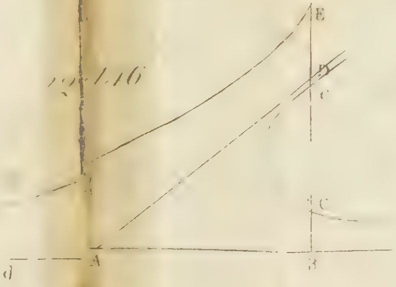

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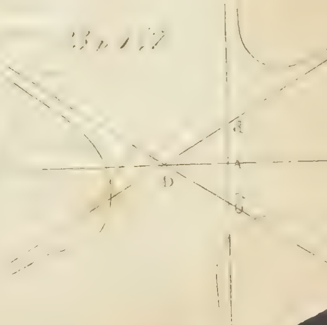

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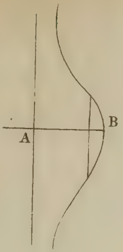

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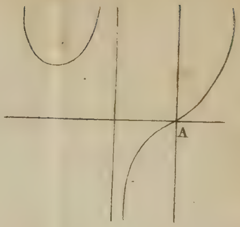

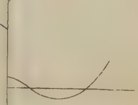

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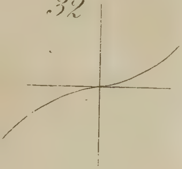

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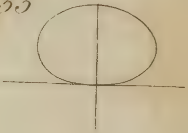

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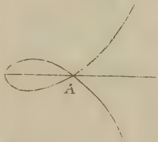

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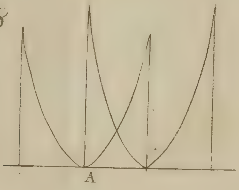

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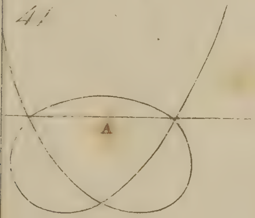

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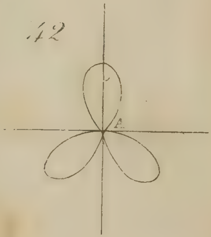

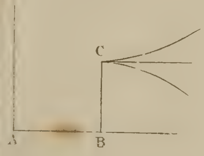

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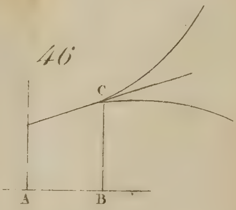

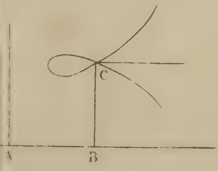

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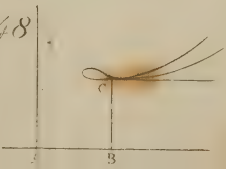

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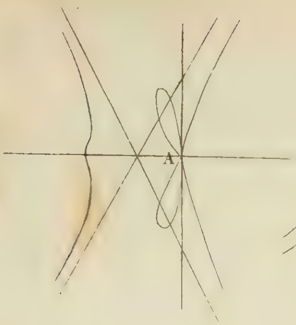

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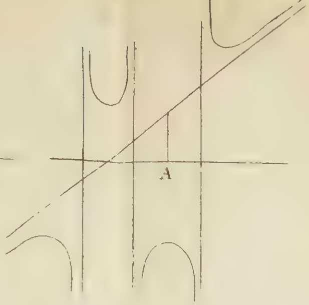

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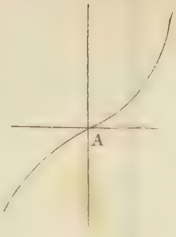

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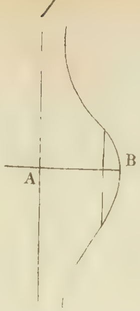

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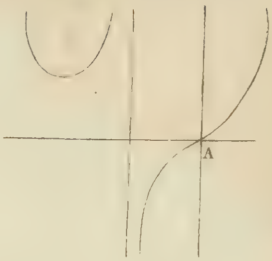

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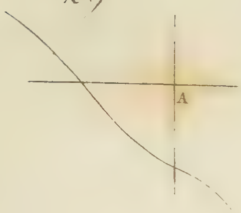

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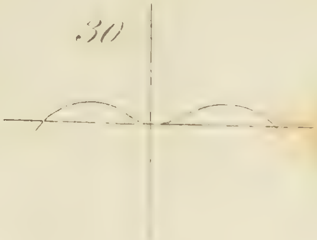

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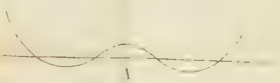

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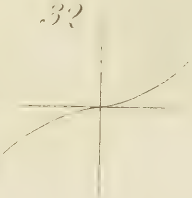

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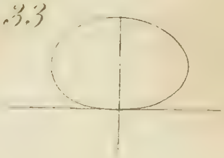

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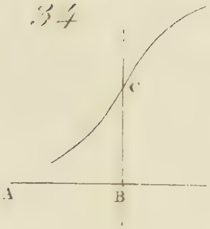

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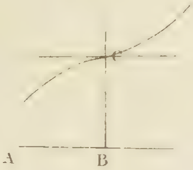

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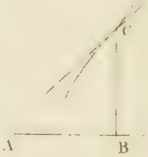

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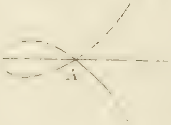

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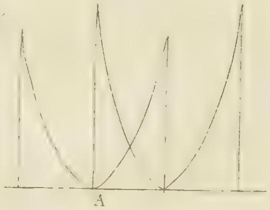

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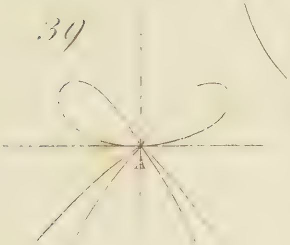

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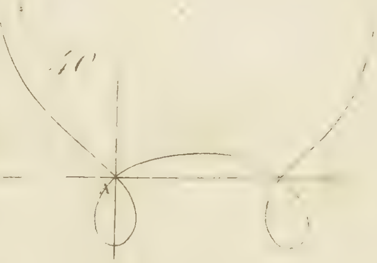

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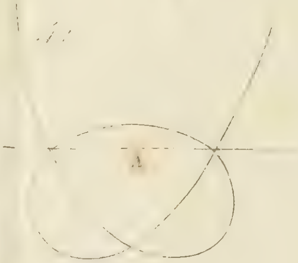

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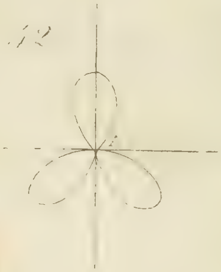

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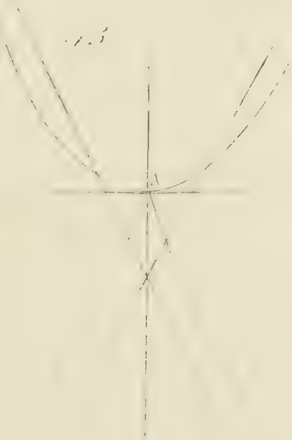

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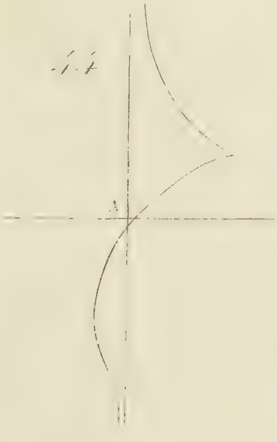

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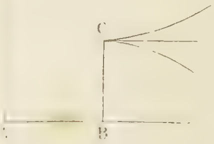

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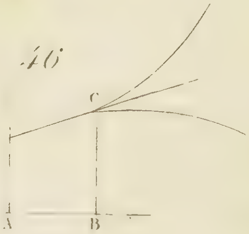

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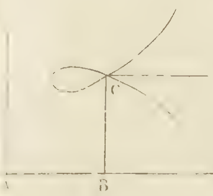

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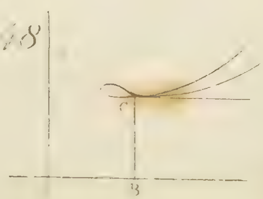

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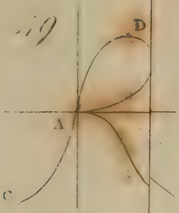

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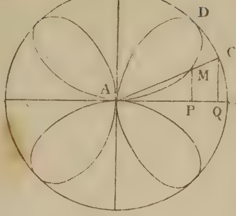

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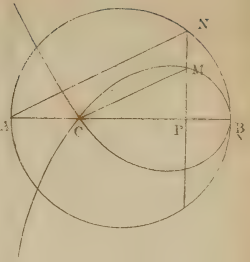

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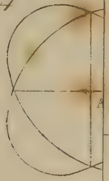

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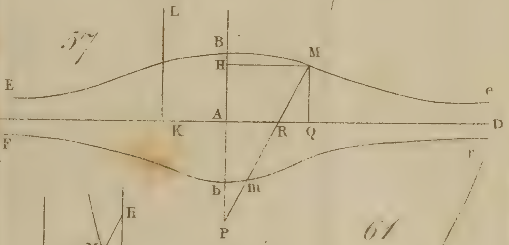

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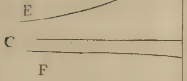

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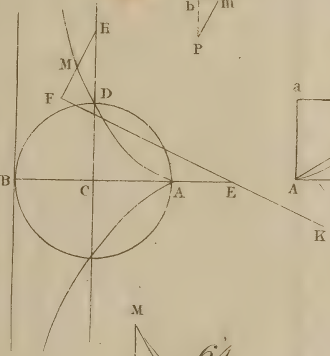

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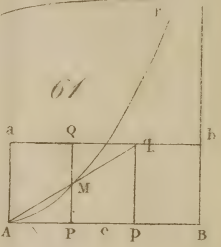

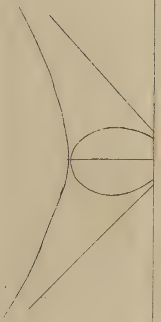

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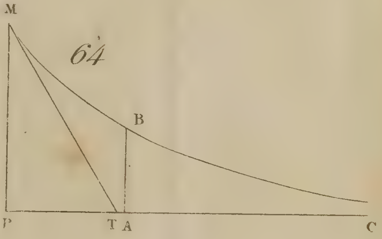

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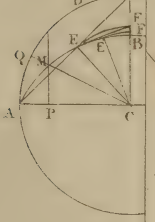

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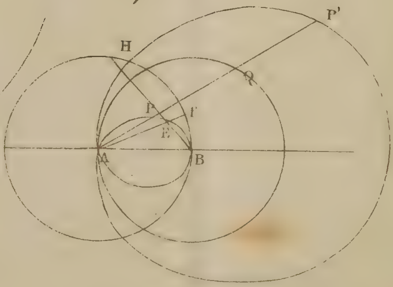

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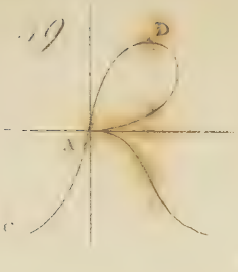

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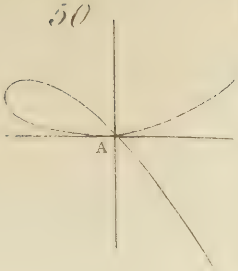

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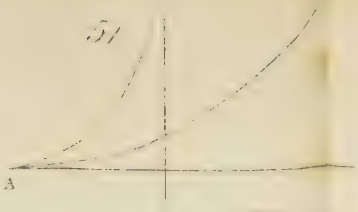

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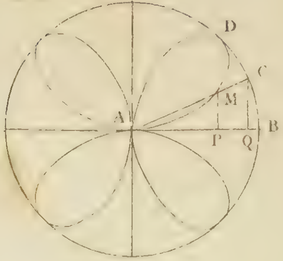

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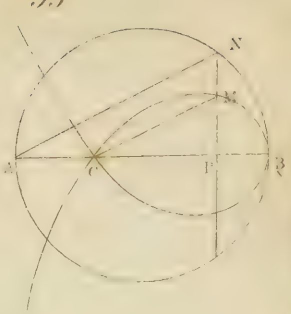

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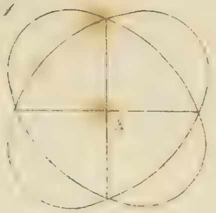

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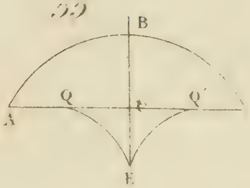

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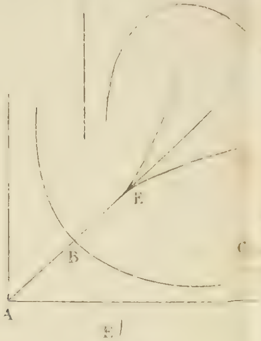

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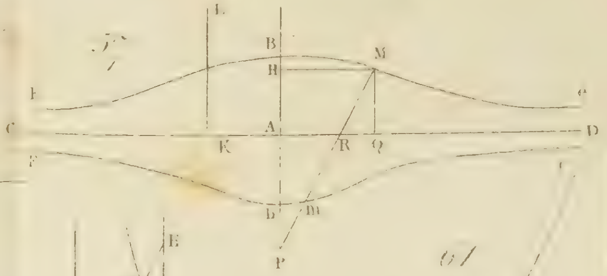

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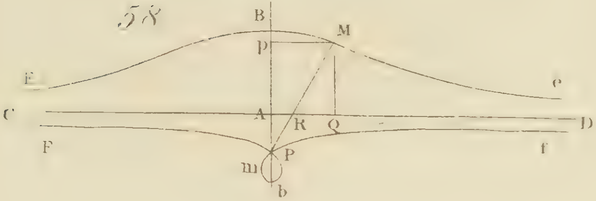

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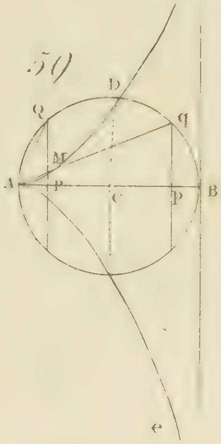

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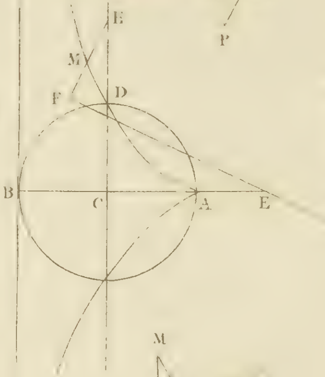

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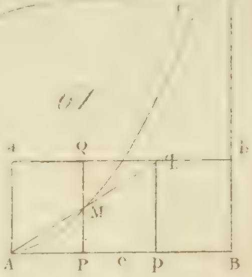

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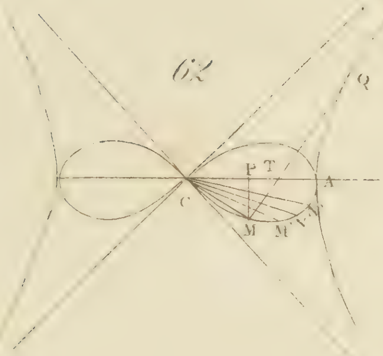

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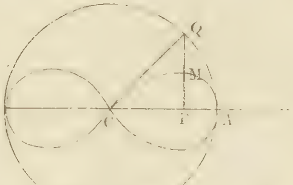

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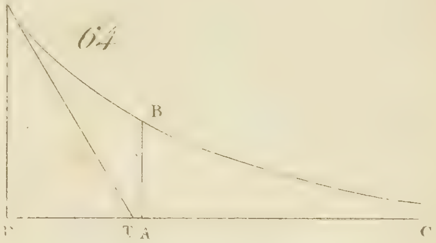

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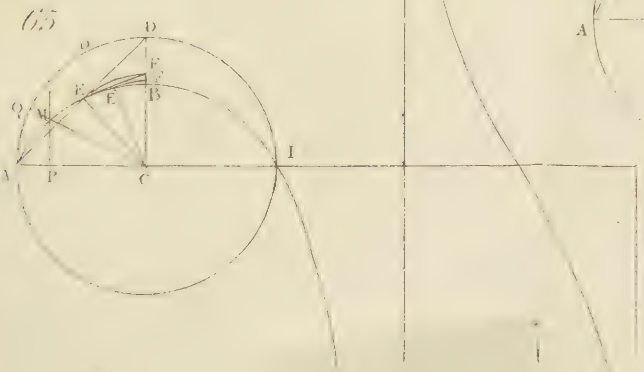

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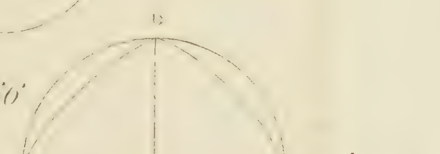

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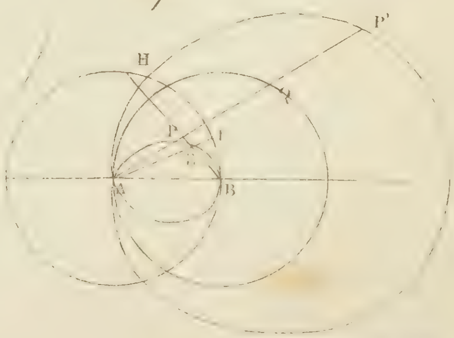

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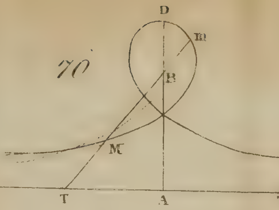

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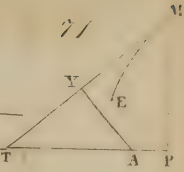

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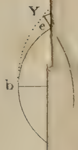

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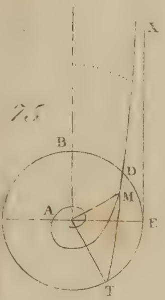

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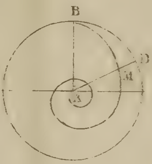

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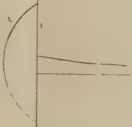

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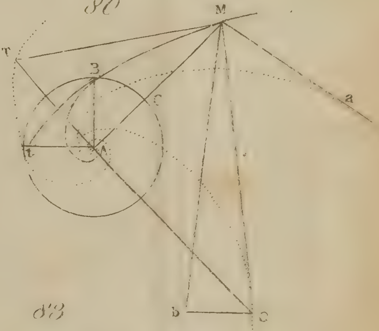

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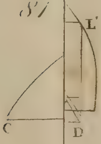

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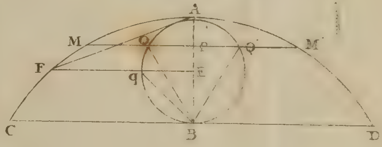

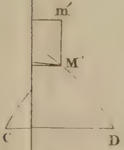

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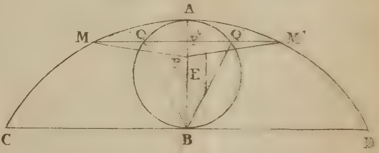

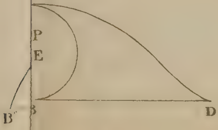

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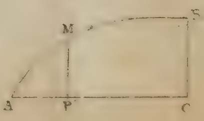

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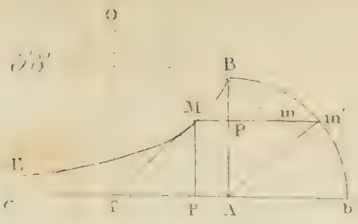

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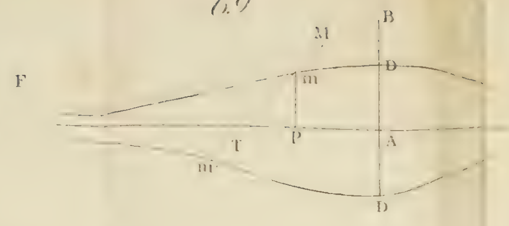

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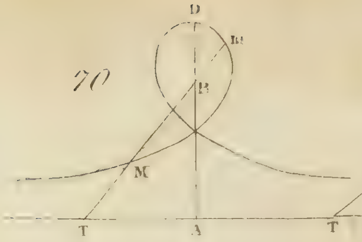

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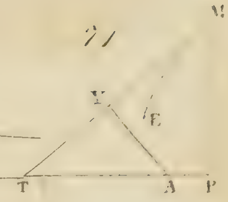

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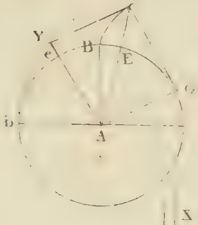

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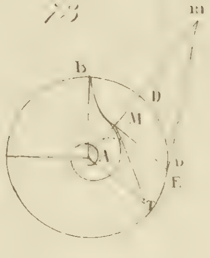

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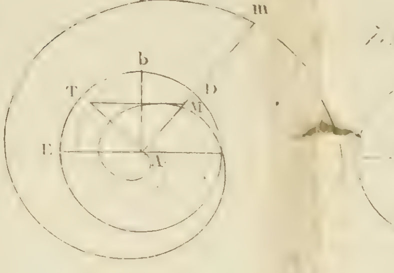

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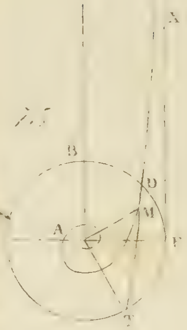

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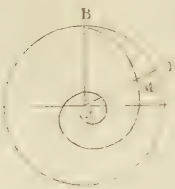

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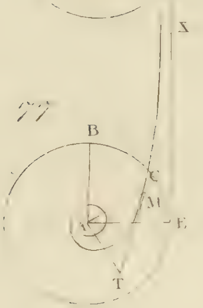

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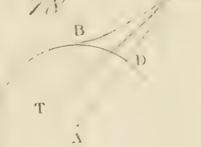

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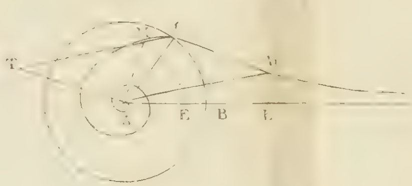

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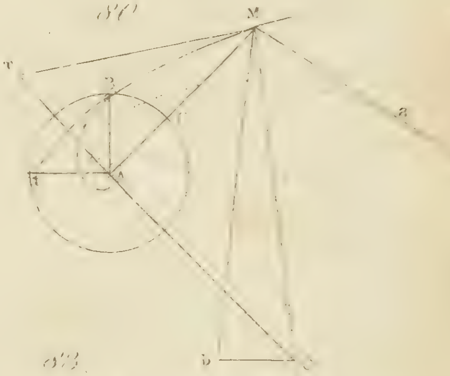

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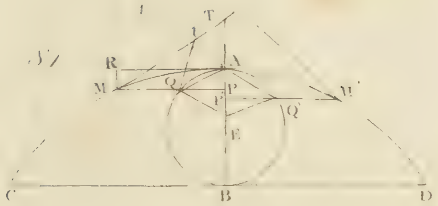

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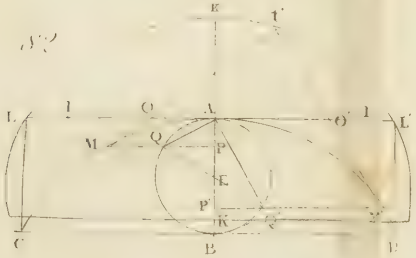

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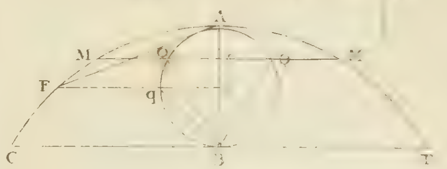

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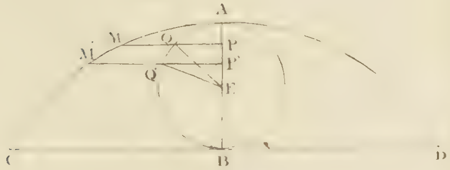

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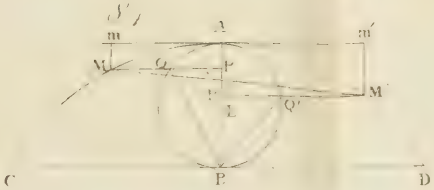

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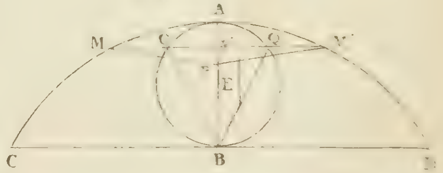

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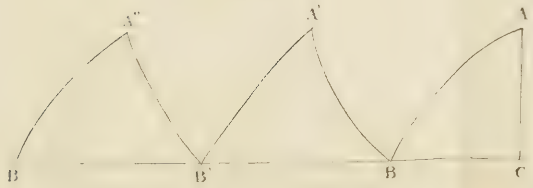

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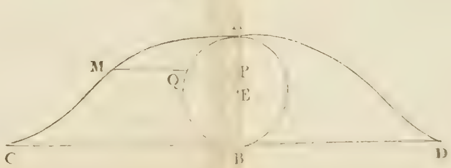

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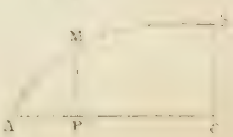

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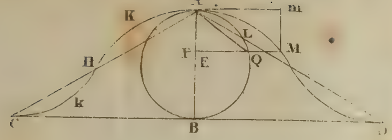

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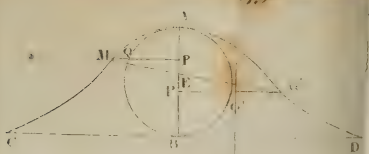

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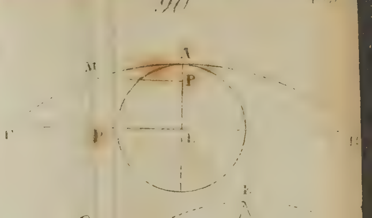

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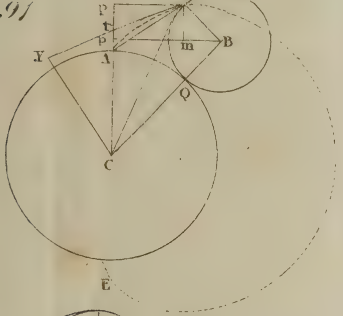

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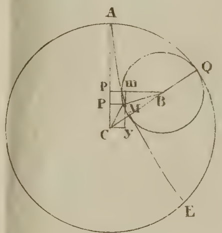

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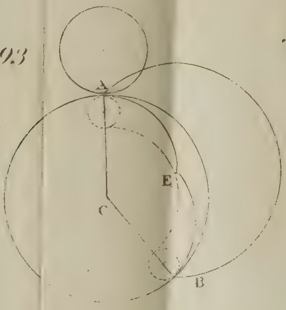

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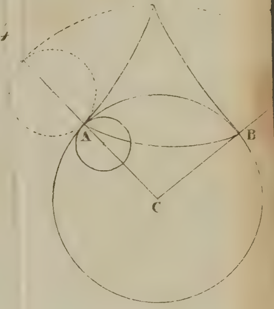

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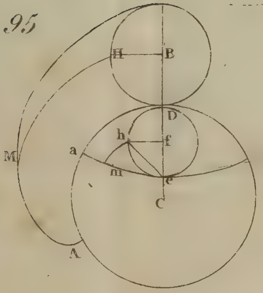

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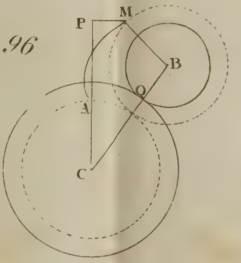

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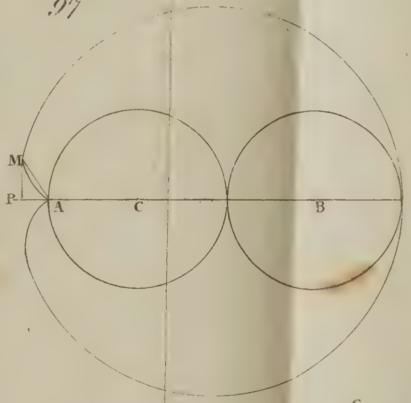

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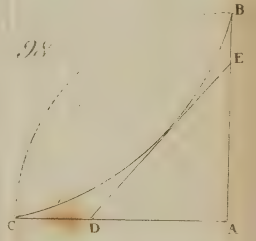

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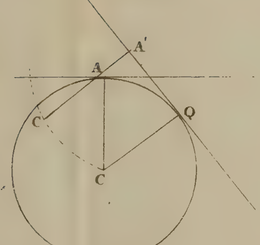

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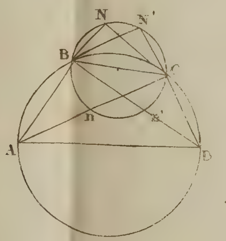

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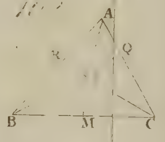

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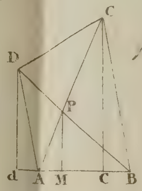

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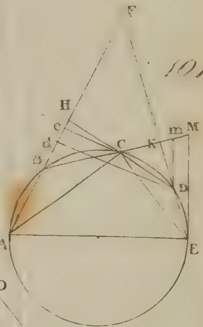

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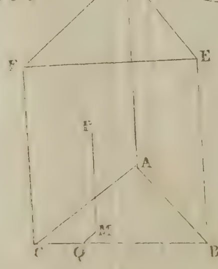

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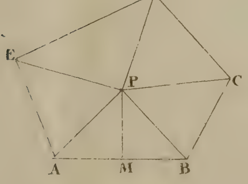

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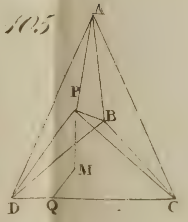

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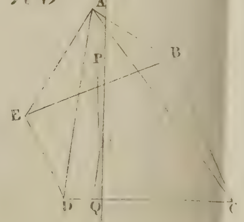

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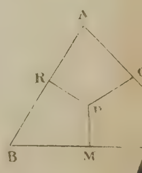

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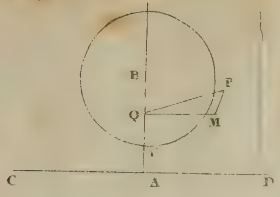

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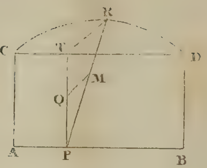

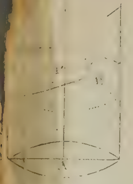

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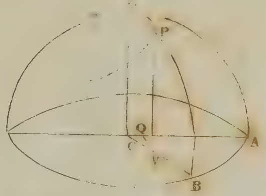

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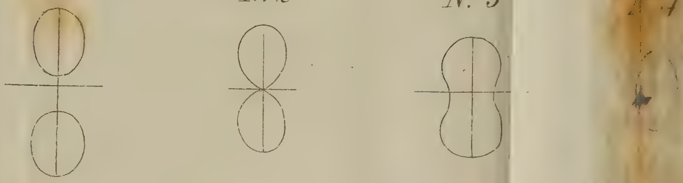

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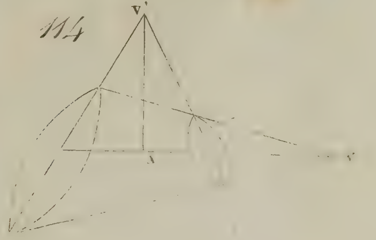

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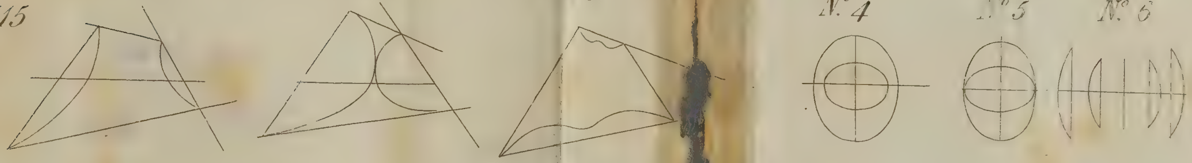

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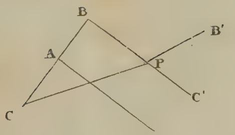

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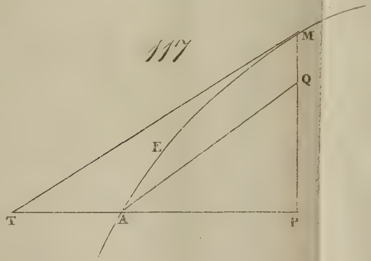

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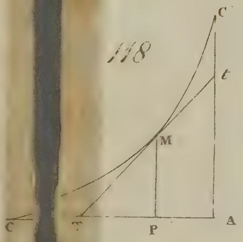

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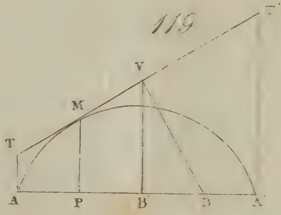

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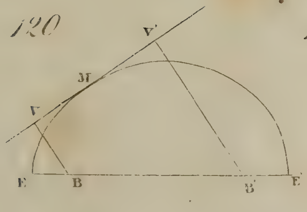

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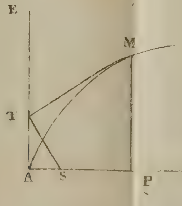

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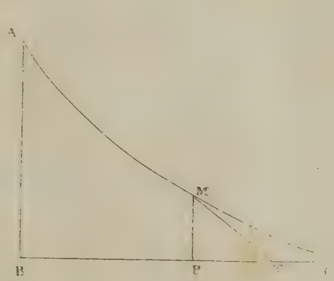

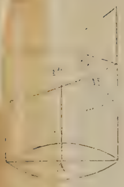

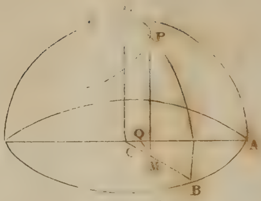

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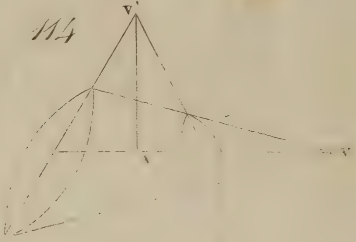

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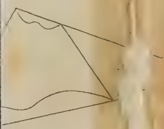

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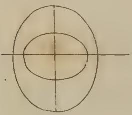

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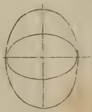

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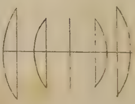

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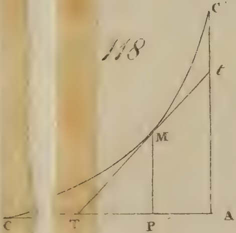

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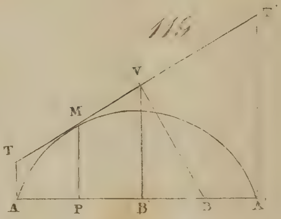

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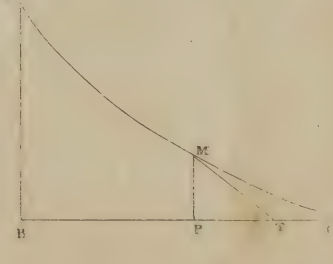

A
COLLECTION OF EXAMPLES
OF
THE APPLICATIONS
OF THE
CALCULUS OF FINITE DIFFERENCES.

BY

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PREFACE.

In applying the general principles of the Calculus of Differences, laid down in the Appendix to the Translation of Lacroix, we have assigned the first place to Examples purely analytical of the methods themselves, endeavouring always to select such as may not only be useful to the student as exercises, but also as results, which he may have occasion to refer to in his future enquiries, and will thus be valuable in themselves, as materials which he will find it advantageous in a more advanced state of his knowledge to have ready at hand. Many of these results are theorems of some generality which we believe will not be found elsewhere, or at least, are not in common use, and where these occur, the leading steps of the demonstration are either set down, or the principle on which it depends mentioned, and in both cases the student will find a useful exercise for his invention in supplying what is omitted. We have then proceeded to questions of a more mixed nature, illustrative of the application of the Calculus of Differences, to a variety of subjects in which it may be employed with advantage as an instrument of investigation, such as the determination of the general terms of the series when the law of their formation is given, the theory of circulating functions, continued fractions, the determination of curves from such properties as involve a series of consecutive points separated by finite intervals, the doctrine of Interest and Annuities, and such other subjects as can be properly treated within our limits.

The want of a regular treatise, on the Calculus of Differences in our language, has long been a serious obstacle to the progress of the enquiring student. The Appendix annexed to the translation of Lacroix's Differential and Integral Calculus, although from the necessity of studying compression it is not so complete as its author could have wished, will, it is hoped, remove this obstacle in some degree, and at least put the analytical principles of the pure theoretical part of the calculus in the reader's power. But the method of applying those principles, and the formulæ derived from them, to the various cases and questions of pure, as well as mixed mathematics, in which they may be advantageously introduced, also demands some degree of explanation, and accordingly, in such of our examples as have this for their object, the reader will find the successive steps of the processes fully detailed, till the questions are reduced to such analytical difficulties, as the Appendix, or the preceding problems will enable him to surmount by himself.

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PART III.

SECTION I.

EXAMPLES OF THE DIRECT METHOD OF DIFFERENCES.

In the following questions, x is supposed to be the independent variable, so that Δx is unity throughout.

(1). Required the differences of $\sin x\theta$ and $\cos x\theta$, θ being constant,

$$\Delta \sin x\theta = 2 \,.\, \sin \frac{\theta}{2} \,.\, \cos \left(x\theta + \frac{\theta}{2}\right)$$

$$\Delta \cos x\theta = -2 \,.\, \sin \frac{\theta}{2} \sin \left(x\theta + \frac{\theta}{2}\right).$$

(2). Required the differences of $\sin(h + x\theta)$ and $\cos(h + x\theta)$

$$\Delta \sin(h + x\theta) = 2 \,.\, \sin \frac{\theta}{2} \,.\, \cos \left\{h + \left(x + \frac{1}{2}\right)\theta\right\}$$

$$\Delta \cos(h + x\theta) = -2 \,.\, \sin \frac{\theta}{2} \,.\, \sin \left\{h + \left(x + \frac{1}{2}\right)\theta\right\}.$$

(3). To find the $(2n)^{\text{th}}$ and $(2n-1)^{\text{th}}$ differences of the same functions,

$$\Delta^{2n} \sin(h + x\theta) = \left(2 \sin \frac{\theta}{2}\right)^{2n} . \sin\{h + (x + n)\theta\}$$

$$\Delta^{2n} \cos(h + x\theta) = \left(2 \sin \frac{\theta}{2}\right)^{2n} . \cos\{h + (x + n)\theta\}$$

$$\Delta^{2n-1} \sin(h + x\theta) =$$

$$= \left(2 \sin \frac{\theta}{2}\right)^{2n-1} . \cos \left\{ h + \left(x + \frac{2n-1}{2} \theta \right) \right\}$$

$$\Delta^{2n-1} \cos (h + x\theta) =$$

$$= - \left(2 \sin \frac{\theta}{2}\right)^{2n-1} . \sin \left\{ h + \left(x + \frac{2n-1}{2} \theta \right) \right\}.$$

In like manner we may obtain the following differences:

(4). $$\Delta \tan x\theta = \frac{\sin \theta}{\cos x\theta \,.\, \cos (x+1)\theta}.$$

(5). $$\Delta \cot\text{an}\, x\theta = \frac{-\sin \theta}{\sin x\theta \,.\, \sin (x+1)\theta}.$$

(6). $$\Delta \frac{a^{2^x}+1}{a^{2^x}-1} = - \frac{2\, a^{2^x}}{a^{2^{x+1}}-1}.$$

(7). $$\Delta \frac{2^x (a^{2^x}+1)}{a^{2^x}-1} = 2^x . \frac{a^{2^x}-1}{a^{2^x}+1}.$$

(8). $$\Delta . 2^x \sin \frac{\theta}{2^x} = 2^{x+2} . \sin \frac{\theta}{2^{x+1}} . \left(\sin \frac{\theta}{2^{x+2}} \right)^2.$$

(9). $$\Delta . 2^{2x} \left(\sin \frac{\theta}{2^x} \right)^2 = 2^{2x+2} . \left(\sin \frac{\theta}{2^{x+1}} \right)^4.$$

(10). $$\Delta \tan \frac{\theta}{2^x} = - \frac{\tan \frac{\theta}{2^{x+1}}}{\cos \frac{\theta}{2^x}},$$

(11). $$\Delta . \tan \frac{\theta}{2^x} = - 2^x . \tan \frac{\theta}{2^x} . \left(\tan \frac{\theta}{2^{x+1}} \right)^2.$$

(12). $$\Delta . (-2)^x . \sin \frac{\theta}{2^x}$$

$$= (-1)^{x+1} . 2^{x+3} . \sin \frac{\theta}{2^{x+2}} \left(\cos \frac{\theta}{2^{x+2}} \right)^3.$$

(13). $$\Delta \frac{1}{2^x \,.\, \sin \frac{\theta}{2^x}} = -2 \,.\, \frac{\sin \frac{\theta}{2^{x+2}}}{2^x \,.\, \sin \frac{\theta}{2^x}}.$$

(14). $$\Delta \frac{1}{2^x \,.\, \tan \frac{\theta}{2^x}} = \frac{\tan \frac{\theta}{2^{x+1}}}{2^{x+1}}.$$

(15). $$\Delta \left\{ \frac{1}{2^x \,.\, \tan \frac{\theta}{2^x}} \right\}^2 = \frac{1}{2^{x+1}} - \left\{ \frac{\tan \frac{\theta}{2^{x+1}}}{2^{x+1}} \right\}.$$

(16). $$\Delta \frac{\cos \frac{\theta}{2^x}}{\left\{ 2^x \,.\, \sin \frac{\theta}{2^x} \right\}^3} = \frac{\sin \frac{\theta}{2^x + 1}}{\left\{ 2^{x+1} \,.\, \cos \frac{\theta}{2^x + 1} \right\}^3}.$$

(17.) $$\Delta \cot 2^x \theta = -\frac{1}{\sin 2^{x+1} \theta}.$$

(18.) $$\Delta \frac{\log (2 \sin 2^x \theta)}{2^x} = -\frac{\log \tan 2^x \theta}{2^{x+1}}.$$

(19.) $$\Delta \frac{1}{\left(2^x \,.\, \sin \frac{\theta}{2^x}\right)^2} = -\frac{1}{\left(2^{x+1} \,.\, \cos \frac{\theta}{2^{x+1}}\right)^2}.$$

(20.) Let $s_x = \sin (a + x\,\theta)$; $c_x = \cos (u + x\,\theta)$, then

$$\Delta \frac{1}{c_x \,.\, c_{x+1} \ldots c_{x+2n+1}}$$

$$= 2 \sin (n+1)\,\theta \,.\, \frac{s_{x+n+1}}{c_x \,.\, c_{x+1} \ldots c_{x+2n+2}}.$$

(21.) The same notation being employed,

$$\Delta \frac{(-1)^x}{s_x \,.\, s_{x+1}} = 2 \,.\, \cos \theta \,.\, \frac{(-1)^{x+1}}{s_x \,.\, s_{x+2}}$$

(22). $$\Delta \frac{(-1)^x}{s_x \,.\, s_{x+1} \cdots s_{x+2n+1}} = 2 \,.\, \cos(n+1)\theta \,.\, \frac{(-1)^{x+1} \,.\, s_{x+n+1}}{s_x \,.\, s_{x+1} \cdots s_{x+2n+2}}.$$

(23). $$\Delta \frac{(-1)^x}{c_x \,.\, c_{x+1}} = 2 \,.\, \cos\theta \,.\, \frac{(-1)^{x+1}}{c_x \,.\, c_{x+2}}.$$

(24). $$\Delta \frac{(-1)^x}{c_x \,.\, c_{x+1} \cdots c_{x+2n+1}} = 2 \,.\, \cos(n+1)\theta \,.\, \frac{(-1)^{x+1} \,.\, c_{x+2n+2}}{c_x \,.\, c_{x+1} \cdots c_{x+n+1}}.$$

(25). Required the difference of arc tan $x\,\theta$, or (as we shall in future designate this function) $\tan^{-1} x\,\theta$.

$$\Delta \tan^{-1} x\,\theta = \tan^{-1} \left\{ \frac{\theta}{1 + \theta^2 x + \theta^2 x^2} \right\}.$$

The expression $\tan^{-1} x\,\theta$ must not be confounded with $(\tan x\,\theta)^{-1}$ or $\frac{1}{\tan x\,\theta}$. The reasons for employing this mode of expression instead of the geometrical circumlocution arc $(\tan = x\,\theta)$ or arc tan $x\,\theta$ are these.

We have already in the differential calculus as well as in that of differences, experienced the great advantage not only in point of brevity, but of clearness and symmetry which arises from denoting the repetition of the operations expressed by d and Δ, by annexing the number of repetitions as an exponent to the characteristic, and we have already seen (Appendix, Art 378.) that the inverse operation of integration in the two calculi is rightly represented on this principle by the same characteristics d and Δ with negative exponents. The same notation may be used to denote the repetition of any operation, whether it be such as modifies the *form* of a function, which is the case with those just noticed or such as expresses the nature of the function itself, as log, cos, tan,

&c. We may use $\log^2 x$, $\cos^3 x$, $\tan^n x$, &c. for $\log \log x$, $\cos \cos \cos x$, $\tan \tan \ldots \tan x$, &c. respectively, and in general

$$f(f(x)) \text{ or } ff(x) \text{ may be written } f^2(x)$$

$$f(f(f(x))) \text{ or } fff(x) \text{ may be written } f^3(x)$$

and so on, which gives in general $f^m f^n(x) = f^{m+n}(x)$.

If we now enquire the meaning of $f^0(x)$, we need only make $n=0$, $m=1$, which gives

$$ff^0(x) = f(x),$$

and consequently $f^0(x) = x$. If now we make $m=1$ and $n=-1$ we find

$$ff^{-1}(x) = f^0(x) = x,$$

so that $f^{-1}(x)$ must denote "that quantity whose function f is x," or rather, that function of x which operated on in the manner denoted by f shall produce simply x. $f^{-1}(x)$ then denotes the *inverse function* of $f(x)$: thus $\tan^{-1} x$ will stand for arc $(\tan = x)$, $\sin^{-1} x$, $\cos^{-1} x$, $\log^{-1} x$ respectively for arc $(\sin = x)$, arc $(\cos = x)$, e^x, or number whose logarithm is x*. The symmetry of this notation and above all the new and most extensive views it opens of the nature of analytical operations seem to authorize its universal adoption, not to mention the real inconvenience which more than one author of eminence has been put to for want of some notation founded on principle to express any inverse function without introducing a new characteristic.

---

* The notations $f^n(x)$, $f^{-n}(x)$, $\sin^{-1} x$ &c. were explained by the author of these latter pages in a paper, "On a remarkable application of Cotes's theorem" in the Philosophical Transactions, 1813, as he then supposed for the first time. The work of a German Analyst, Burmann, has, however, within these few months come to his knowledge, in which the same is explained at a considerably earlier date. He, however, does not seem to have noticed the convenience of applying this idea to the inverse functions $\tan^{-1}$, &c. nor does he appear at all aware of the *inverse calculus of functions* to which it give rise. Burmann is a zealous partizan of the combinatory analysis of Hindenburg, the very principle of which is the exclusion of all *analytical artifice*, which is no where so strongly called for as in the calculus here alluded to.

The equation which gave rise to this digression is easily proved if we call to mind that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \, . \tan B};$$

for, if we take the inverse function $\tan^{-1}$ on both sides, we have

$$A - B = \tan^{-1} \left\{ \frac{\tan A - \tan B}{1 + \tan A \, . \tan B} \right\}$$

and for A and B writing $\tan^{-1} A$ and $\tan^{-1} B$,

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1} \left\{ \frac{A - B}{1 + A B} \right\}$$

which the student is left to apply to the case in question.

(26). Required the difference of $\tan^{-1}(h + x\theta)$.

$$\Delta \tan^{-1}(h + x\theta) = \tan^{-1} \left\{ \frac{\theta}{1 + \theta(h + x\theta) + (h + x\theta)^2} \right\}.$$

(27.) Required the difference of $\tan^{-1} \left\{ \frac{a + bx}{A + Bx} \right\}$.

$$\Delta \tan^{-1} \left\{ \frac{a + bx}{A + Bx} \right\} =$$

$$= \tan^{-1} \left\{ \frac{Ab - aB}{\left\{ \begin{matrix} A^2 + AB \\ + a^2 + ab \end{matrix} \right\} + x \left\{ \begin{matrix} 2AB + B^2 \\ + 2ab + b^2 \end{matrix} \right\} + x^2 (B^2 + b^2)} \right\}.$$

(28). Required the difference of $\tan^{-1} . u_x$

$$\Delta \tan^{-1} u_x = \tan^{-1} \frac{\Delta u_x}{1 + u_x u_{x+1}}.$$

(29). Required the difference of $2^x . \tan^{-1}\left(\frac{\theta}{2^x}\right)$

$$\Delta . 2^x \tan^{-1}\left(\frac{\theta}{2^x}\right) = 2^x . \tan^{-1}\left\{\frac{\theta^3}{4 . 2^{3x} + 3\,\theta^2 . 2^x}\right\}.$$

(30). It is required to demonstrate the truth of the two following theorems in which p represents $2 \sin \theta$,

$$\cos 2\,n\,\theta = 1 - n\,p . \sin \theta - \frac{n\,(n-1)}{1\,.\,2}\,p^2 . \cos 2\,\theta$$

$$+ \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,p^3 . \sin 3\,\theta + \&c.$$

$$\sin 2\,n\,\theta = \frac{n}{1}\,p . \cos \theta - \frac{n\,(n-1)}{1\,.\,2}\,p^2 . \sin 2\,\theta$$

$$- \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,p^3 . \cos 3\,\theta + \&c.$$

the sign being alternately $+ + - -$ and the series breaking off at p^n whenever n is a positive integer.

These may be deduced from the general expression for u_{x+n} in Art. 345, by substituting $\cos 2\,x\,\theta$ or $\sin 2\,x\,\theta$ for u_x, developing their successive differences and finally making $x = 0$.

(31). Required the successive differences of x^2, x^3, x^4, x^m, expressed in powers of x, and the law of their coefficients.

$\Delta . x^2 = 2\,x + 1,\quad \Delta^2 . x^2 = 2,$

$\Delta . x^3 = 3\,x^2 + 3\,x + 1,\quad \Delta^2 . x^3 = 6\,x + 6,\quad \Delta^3 . x^3 = 6,$

$\Delta . x^4 = 4\,x^3 + 6\,x^2 + 4\,x + 1,\quad \Delta^2 . x^4 = 12\,x^2 + 24\,x + 14,$

$\Delta^3 . x^4 = 24\,x + 36,\quad \Delta^4 . x^4 = 24,$

$$\Delta^n . x^m = \Delta^n o^m + \frac{m}{1}\,\Delta^n o^{m-1} . x + \frac{m\,(m-1)}{1\,.\,2}\,\Delta^n o^{m-2} . x^2 + \ldots$$

$$\ldots\ldots\ldots + \frac{m\,(m-1)\ldots\ldots(n+1)}{1\,.\,2\ldots\ldots(m-n)}\,\Delta^n o^n . x^{m-n}.$$

this last equation may be derived from the value of $\Delta^n . x^m$ given in the Appendix, Art. 350, by developing its several terms by the binomial theorem, and collecting the coefficients of similar powers of x. If we then call to mind the definition given in that article of the expression $\Delta^n o^m$ the truth of the above equation will be apparent. But the most regular as well as the easiest mode of obtaining it, is from the general theorem of Art. 361.

(32). To prove the truth of the following theorems

$$(\sin\theta)^n . \cos n\theta =$$

$$\theta^n . \left\{ 1 - \frac{\Delta^n o^{n+2}}{1.2....(n+2)} (2\theta)^2 + \frac{\Delta^n o^{n+4}}{1....(n+4)} (2\theta)^4 - \&c. \right\}$$

$$(\sin\theta)^n . \sin n\theta =$$

$$\theta^n . \left\{ \frac{\Delta^n o^{n+1}}{1....(n+1)} (2\theta) - \frac{\Delta^n o^{n+3}}{1....(n+3)} (2\theta)^3 + \&c. \right\}$$

these may also be deduced from the same general expression for $\Delta^n u_x$ by substituting for u_x, $\sin x\theta$ and $\cos x\theta$, and replacing $\Delta^n u_x$ by the value given in Example (3).

The numbers comprised in the form $\Delta^n o^m$ enter so extensively into the theory of series, and afford such remarkable facilities in the developement of functions that we shall take occasion to annex in this place a short table of their values as far as $\Delta^{10} o^{10}$ and hereafter to present the most remarkable of their properties, sufficient to explain the manner of their employment with reference to this object.

(33). To calculate the actual numerical values of $\Delta^n o^m$ for all values of m and n from 1 up to 10.

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}
0^1	1									
0^2	1	2								
0^3	1	6	6							
0^4	1	14	36	24						
0^5	1	30	150	240	120					
0^6	1	62	540	1560	1800	720				
0^7	1	126	1806	8400	16800	15120	5040			
0^8	1	254	5796	40824	126000	191520	141120	40320		
0^9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
0^{10}	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

The inspection of this table affords room for one or two remarks. The value of the function $\Delta^n o^m$ increases with the indices n and m, but is more influenced by the latter than the former. The expressions $\Delta^n o^n$, $\Delta^n o^{n+1}$, $\Delta^n o^{n+2}$, &c. go on perpetually increasing in a very high ratio, and end in surpassing any assignable number. Their rate of increase too surpasses at last that of any assignable geometric progression, as we shall soon see.

(34). To assign the approximate values of $\Delta^n o^n$, $\Delta^n o^{n+1}$, $\Delta^n o^{n+2}$, &c. when n is a very high number.

This may easily be done, if we actually develope $(e^t - 1)^n$, or

$$t^n \left\{ 1 + \left(\frac{t^2}{1 \,.\, 2} + \frac{t^3}{1 \,.\, 2 \,.\, 3} + \&c. \right) \right\}^n$$

by the binomial theorem, and compare the coefficients of the powers of t so produced with those of the same powers in the series,

$$t^n \left\{ \frac{\Delta^n o^n}{1 \ldots n} + \frac{\Delta^n o^{n+1}}{1 \ldots (n+1)} t + \&c. \right\}; \text{ Art. 361. App.}$$

for we thus obtain

$$\Delta^n o^n = 1 \,.\, 2 \ldots n,$$

$$\Delta^n o^{n+1} = 1 \,.\, 2 \ldots (n+1) \times \frac{n}{2},$$

$$\Delta^n o^{n+2} = 1 \,.\, 2 \ldots (n+2) \times \frac{3n^2 + n}{24},$$

$$\Delta^n o^{n+3} = 1 \,.\, 2 \ldots (n+3) \times \frac{n^3 + n^2}{48},$$

$$\Delta^n o^{n+4} = 1 \,.\, 2 \ldots (n+4) \times \frac{15n^4 + 30n^3 + 5n^2 - 2n}{5760}, \ \&c.$$

As n increases, these functions therefore increase ultimately at the same rate with those progressions spoken of in the Appendix under the name of hyper-geometrical series (Art. 414.) and when n is very large, we get, by applying the formulæ of Art. 411.

$$\Delta^n o^n = \sqrt{(2\pi)} \,.\, n^{\frac{1}{2}} \,.\, \left(\frac{n}{e}\right)^n$$

$$\Delta^n o^{n+1} = \frac{\sqrt{2\pi} \,.\, n^{\frac{3}{2}}}{2} \,.\, \left(\frac{n+1}{e}\right)^{n+1}$$

$$\Delta^n o^{n+2} = \frac{\sqrt{(2\pi)} \,.\, n^{\frac{5}{2}}}{8} \,.\, \left(\frac{n+2}{e}\right)^{n+2}; \text{ \&c.}$$

or, if we consider that when n is a very high number,

$$(n+1)^n = e \,.\, n^n,\ (n+2)^n = e^2 \,.\, n^n,\ \text{\&c.}$$

$$\Delta^n o^n = \sqrt{(2\pi)} \,.\, n^{\frac{1}{2}} \,.\, \left(\frac{n}{e}\right)^n$$

$$\Delta^n o^{n+1} = \frac{\sqrt{(2\pi)}}{2} \,.\, n^{\frac{5}{2}} \left(\frac{n}{e}\right)^n$$

$$\Delta^n o^{n+2} = \frac{\sqrt{(2\pi)}}{8} \,.\, n^{\frac{9}{2}} \,.\, \left(\frac{n}{e}\right)^n$$

$$\Delta^n o^{n+3} = \frac{\sqrt{(2\pi)}}{48} \,.\, n^{\frac{13}{2}} \,.\, \left(\frac{n}{e}\right)^n; \text{ \&c.}$$

If more exact formulæ be required, the series of powers of $\frac{1}{n}$ must be taken into the account.

(35). To shew that

$$u_{x+n} = u_x + \frac{n}{1} \Delta u_{x-1} + \frac{n(n+1)}{1\,.\,2} \Delta^2 u_{x-2} +$$
$$+ \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3} \Delta^3 u_{x-3} + \text{\&c.}$$

The generating function of u_x being $\phi(t)$, that of u_{x+n} is $\frac{1}{t^n}\phi(t)$. Let this be thrown into the form

$$\left\{1-t\left(\frac{1}{t}-1\right)\right\}^n.\phi(t)$$

and developed by the binomial theorem. If we then redescend from the generating functions to the coefficients of t^x in their developements, the theorem in question will result.

(36). To prove in general that

$$u_{x+n}=u_x+\frac{n}{1}\Delta u_{x-r}+\frac{n(n+2r-1)}{1.2}\Delta^2 u_{x-2r}+$$
$$+\frac{n(n+3r-1)(n+3r-2)}{1.2.3}\Delta^3 u_{x-3r}+\&c.$$

The generating function $\frac{1}{t^n}\phi(t)$ of u_{x+n} must be transformed into a series of terms of the form

$$\left\{t^r\left(\frac{1}{t}-1\right)\right\}^i.\phi(t),$$

and their coefficients shewn to coincide with those of the proposed series. To develope t^{-n} in powers of $t^r\left(\frac{1}{t}-1\right)$ put the latter function equal to z and we have to develope $\left(\frac{1}{t}\right)^n$ in powers of z, $\frac{1}{t}$ being a function of z given by the equation

$$t^r\left(\frac{1}{t}-1\right)=z$$

$$\text{or } \frac{1}{t}=1+z.\left(\frac{1}{t}\right)^r,$$

Lagrange's theorem demonstrated in Note E enables us to do this. If we put y for $\frac{1}{t}$, we have

$$1 - y + z y^r = 0$$

which gives

$$y^n = \frac{1}{t^n} = 1 + \frac{n}{1} z + \frac{n(n+2r-1)}{1.2} z^2 + \&c.$$

whence the theorem in question results.

SECTION II.

Exercises in the resolution of Functions into Factorials, to prepare them for Integration.

(1). To resolve x^2, x^3, x^4, x^n, into products of the factors x, $x-1$, $x-2$, &c., or as it is sometimes expressed, to reduce them to x and its *preceding values,*

$$x^2 = x + x(x-1)$$

$$x^3 = x + 3x(x-1) + x(x-1)(x-2)$$

$$x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3)$$

$$x^n = \frac{\Delta o^n}{1} x + \frac{\Delta^2 o^n}{1.2} x(x-1) + \frac{\Delta^3 o^n}{1.2.3} x(x-1)(x-2) + \&c.$$

The general expression may be deduced from the equation

$$u_{y+x} = u_y + \frac{x}{1} \Delta u_y + \frac{x(x-1)}{1.2} \Delta^2 u_y + \&c.$$

(See Appendix, Art. 345), by making $u_y = y^n$ and then supposing $y=0$. The values of Δo^n, $\Delta^2 o^n$, &c. when n does not exceed 10 may be taken from the table given in page (9).

(2). To resolve x, x^2, x^3, x^4, x^n into products of the successive factors $x+1$, $x+2$, &c. or to reduce them to *succeeding values* of x.

$$x = -1 + (x+1)$$

$$x^2 = +1 - 3(x+1) + (x+1)(x+2)$$

$$x^3 = -1 + 7(x+1) - 6(x+1)(x+2) + (x+1)(x+2)(x+3)$$

$$x^4 = +1 - 15(x+1) + 25(x+1)(x+2) - 10(x+1)\,.\,.\,(x+3) + \\ + (x+1)\,.\,.\,.\,.\,(x+4)$$

$$x^n = (-1)^n \left\{ \frac{\Delta o^{n+1}}{1} - \frac{\Delta^2 o^{n+1}}{1\,.\,2}(x+1) \right. \\ \left. + \frac{\Delta^3 o^{n+1}}{1\,.\,2\,.\,3}(x+1)(x+2) - \&c. \right\}$$

This may be deduced from the same general expression for u_{y+x} by putting $-x$ for x and supposing $u_y = y^{n+1}$ and then proceeding as above.

(3). It has been remarked (App. Art. 370.) that "to keep the numerical coefficients as low as possible in these reductions to the form of factorials is an object of importance," and that "this may be done by a proper disposition of the preceding and succeeding factors." The following examples will shew how this is to be performed.

$$x^3 = (x-1)\,x\,(x+1) + x.$$

$$x^5 = (x-2)(x-1)\,x\,(x+1)(x+2) + 5(x-1)\,x\,(x+1) + x.$$

$$x^7 = (x-3)\,.\,.\,.\,(x+3) + 14(x-2)\,.\,.\,.\,(x+2) + \\ + 21(x-1)\,x\,(x+1) + x.$$

In these instances it will be observed that no factorials with an even number of factors enter into the expression.

This is a simplification of considerable moment, and that it takes place in general for x^{2n+1} may be demonstated without difficulty as follows.

(4). To resolve x^{2n+1} in the same manner, and to determine the law of the coefficients.

Assume $x^2 = v$. Then $x^{2n+1} = v^n x$, suppose now

$$v^n = A_0 + A_1 (v-1) + A_2 (v-1)(v-4) +$$
$$+ A_3 (v-1)(v-4)(v-9) + \dots A_n (v-1)(v-4)\dots(v-n^2)$$

This assumption is *possible* because the second member is a rational integral function of v of the nth degree, and being reduced to powers of v, and compared with v^n will afford $n+1$ equations of the first degree for the determination of the indeterminate coefficients $A_0, A_1, \dots A_n$. The following is however a readier and more elegant method. The above equation being identical in v must hold good whatever numbers are substituted for v, hence if for v we write in succession 1, 4, 9, 25, &c., we get

$$1^{2n} = A_0,$$
$$2^{2n} = A_0 + (2^2 - 1^2) A_1,$$
$$3^{2n} = A_0 + (3^2 - 1^2) A_1 + (3^2 - 1^2)(3^2 - 2^2) A_2$$
$$\&c. = \&c.$$

Whence we derive

$$A_0 = 1^{2n},$$

$$A_1 = \frac{2^{2n}}{2^2 - 1^2} + \frac{1^{2n}}{1^2 - 2^2},$$

$$A_2 = \frac{3^{2n}}{(3^2 - 1^2)(3^2 - 2^2)} + \frac{2^{2n}}{(2^2 - 1^2)(2^2 - 3^2)} + \frac{1^{2n}}{(1^2 - 2^2)(1^2 - 3^2)},$$

and so on, the law being evident and the general value of A_x susceptible of direct expression in functions of x.

Such then being the values of A_0, A_1, &c., we have

$$x^{2n+1} = xv^n = x\,\{A_0 + A_1(x^2 - 1^2) + A_2(x^2 - 1^2)(x^2 - 2^2) + \&c.\}$$

$$= A_0\,x + A_1\,(x-1)\,x\,(x+1) + \&c.$$

v being equal to x^2.

The law observed by the coefficients is not a little remarkable. It extends too with a slight modification to cases of much greater generality, and it will hardly be thought irrelevant to the present subject to propose and resolve the following problem.

(5). To develope $F(x)$ in a series of factorial terms of the following form

$$F(x) = A_0 + A_1\,(x - f_1) + A_2\,(x - f_1)\,(x - f_2) + \&c.$$

$F(x)$ being any function whatever of x, and f_1, f_2, &c. particular values of any other function f_x corresponding to the values 1, 2, &c. of x.

A process exactly similar to the foregoing, viz., the substitution of f_1, f_2, f_3, &c. in the succession for x and the determination of the coefficients one from another by means of the equations thence arising gives $A_0 = F(f_1)$ or, for brevity's sake omitting the parentheses

$$A_0 = Ff_1,$$

$$A_1 = \frac{Ff_1}{f_1 - f_2} + \frac{Ff_2}{f_2 - f_1},$$

$$A_2 = \frac{Ff_1}{(f_1 - f_2)(f_1 - f_3)} + \frac{Ff_2}{(f_2 - f_1)(f_2 - f_3)} + \frac{Ff_3}{(f_3 - f_1)(f_3 - f_2)},$$

• • • • • • • • • • • • •

$$A_{n-1} = \frac{Ff_1}{(f_1 - f_2)(f_1 - f_3)\ldots(f_1 - f_n)} + \frac{Ff_2}{(f_2 - f_1)(f_2 - f_3)\ldots(f_2 - f_n)} + \ldots$$

$$\ldots\ldots + \frac{Ff_n}{(f_n - f_1)(f_n - f_2)\ldots(f_n - f_{n-1})}.$$

Several consequences follow from this theorem; 1st, If $F(x)$ be any rational integral function of x of the n^{th} degree, and f_x any function of x whatever, it is easily seen that all the values of the general expression for the coefficients after A_n must vanish of themselves, giving $A_{n+1}=0$, $A_{n+2}=0$, &c. to infinity, which is one of the most general and singular properties of rational integral functions: (2dly), If $F(x)=x^n$ and $f(x)=x^2$, we get the series of coefficients investigated in the last question: (3dly), If $F(x)=$ any rational integral function of x, and $f(x)=x^2$ we obtain a set of coefficients proper for the resolution of such a function as

$$x\,.\,F(x^2)$$

into the form

$$A_0\,x+A_1\,(x-1)\,x\,(x+1)+\&\text{c}.$$

which will often be found exceedingly convenient, and of which we shall give examples (See Ex. 9, 10, of this section). (4thly), If $F(x)=$any rational integral function of x and $f(x)=\pm x$, we obtain general formulæ for the resolution of any rational function into preceding or succeeding values of x. (5thly), If $F(x)=(1+a)^x$, and $f(x)=x$, we get the binomial theorem.

These instances will suffice to shew that this mode of developing $F(x)$ is not a mere matter of idle speculation. Other and still more extensive applications of it will shortly appear.

(6). To resolve x^2, x^4, x^6, x^{2n}, the even powers of x into factorials where the preceding and succeeding factors occur symmetrically.

This cannot be done, it is evident, by resolving them into factors x, $x\pm 1$, $x\pm 2$, &c. because the degree of any expression such as

$$A_0\,x+A_1\,(x-1)\,x\,(x+1)+\&\text{c}.$$

must necessarily be odd. The object may however be accomplished by taking $x \pm \frac{1}{2}$, $x \pm \frac{3}{2}$, &c. for the factors, when we shall find

$$x^2 = \frac{1}{4}\{1+(2x-1)(2x+1)\},$$

$$x^4 = \frac{1}{16}\{1+10(2x-1)(2x+1)$$

$$+(2x-3)(2x-1)(2x+1)(2x+3)\}$$

$$x^6 = \frac{1}{64}\{1+91(2x-1)(2x+1)+35(2x-3)\ldots(2x+3)+$$

$$+(2x-5)\ldots\ldots(2x+5)\}.$$

For x^{2n};—put $(2x)^2 = v$, and supposing $F(v) = v^n$, and $f(v) = (2v-1)^2$, we have

$$f_1 = 1^2,\ f_2 = 3^2,\ f_3 = 5^2,\ \&c.$$

$$Ff_1 = 1^{2n},\ Ff_2 = 3^{2n},\ Ff_3 = 5^{2n},\ \&c.,$$

and we shall therefore have by the general theorem above demonstrated (Art. 5. Sect. 2.)

$v^n = A_0 + A_1(v-1^2) + A_2(v-1^2)(v-3^2) + \&c$; where $A_0 = 1^{2n}$,

$$A_1 = \frac{1^{2n}}{1^2-3^2} + \frac{3^{2n}}{3^2-1^2},$$

$$A_2 = \frac{1^{2n}}{(1^2-3^2)(1^2-5^2)} + \frac{3^{2n}}{(3^2-1^2)(3^2-5^2)} + \frac{5^{2n}}{(5^2-1^2)(5^2-3^2)}.$$

Now since $(2x)^2 = v$, $x^{2n} = \frac{v^n}{2^{2n}}$, and we therefore have since

$$v-1^2 = \{\surd(v)-1\}\{\surd(v)+1\} = (2x-1)(2x+1),$$

and so on, the following final result, where A_0, &c. have the values above written

$$x^{2n} = \frac{1}{2^{2n}} \{ A_0 + A_1 (2x-1)(2x+1)$$

$$+ A_2 (2x-3) \ldots (2x+3) + \&c. \} .$$

In the same way may such a function as $F(x^2)$, any *even* function of x be treated.

(7). To complete the factorials

$$x(x+1)(x+3), \text{ and } (2x-1)(2x+1)(2x+5).$$

N. B. By completing them is meant reducing them to the others in which the terms follow the order of an arithmetical progression.

They are respectively equal to

$$x(x+1) + x(x+1)(x+2), \text{ and}$$

$$2(2x-1)(2x+1) + (2x-1)(2x+1)(2x+3).$$

(8). Reduce $(2x+3)^3$ to preceding values and $(x+a)^3$ to succeeding ones of x.

$$(2x-1)(2x-3)(2x-5) + 18(2x-1)(2x-3) +$$

$$+ 76(2x-1) + 64 = (2x+3)^2$$

$$(x+a)^2 = (a-1)^2 + (2a-3)(x+1) + (x+1)(x+2).$$

(9). Reduce $x(x^2+2)^2$ to factorials in which the preceding and succeeding factors occur symmetrically. The application of the theorem in (Art. 5. Sect. 2.) gives

$$x(x^2+2)^2 = 9x + 9(x-1)(x)(x+1)$$

$$+ (x-2)(x-1)x(x+1)(x+2).$$

(10). Reduce $(x^3+x)(2x^2-6)^3$ in the same manner

$$(x^3+x)(2x^2-6)^3 = -128x + 56(x-1)x(x+1)$$

$$+\,424\,(x-2)\,(x-1)\,x\,(x+1)(x+2)$$

$$+\,176(x-3)(x-2)(x-1)x(x+1)(x+2)(x+3(+\,+(x-4)\ldots\ldots$$

$$\ldots\ldots(x+4).$$

(11). Resolve $(x^4+1)^2$ into the smallest possible number of complete factorials.

$$(x^4+1)^2=\frac{1}{2^8}\{\,289+1140\,(2\,x-1)\,(2\,x+1)$$

$$+\,998\,(2\,x-3)\,(2\,x-1)\,(2\,x+1)\,(2\,x+3)$$

$$+\,84\,(2\,x-5)\ldots\ldots(2\,x+5)+(2\,x-7)\ldots\ldots(2\,x+7)\,\}.$$

(12). Affect $u_{x+1}\,.\,u_{x+3}$ with u_x and reduce it to succeeding values of u_x. *N.B.* By affecting it with u_x is meant introducing u_x as one of the factors of the result. In this and the following examples u_x is understood to have its difference constant, so that

$$u_x=a+h\,x,\ \Delta\,u_x=h.$$

$$\text{Then, } u_{x+1}\,.\,u_{x+3}=u_x\,u_{x+1}+3\,h\,u_x.$$

(13). Affect $u_{x-3}\,.\,u_{x+2}$ with u_x

$$u_x\,u_{x+1}-2\,h\,.\,u_x+6\,h^2=u_{x-3}\,u_{x+2}.$$

(14). Reduce $(u_{x-1})^2\,.\,(u_{x+1})^2$ to succeeding values of u_x,

$$(u_{x-1})^2\,.\,(u_{x+1})^2=$$

$$u_{x+1}\ldots\ldots u_{x+4}-10\,h\,.\,u_{x+1}\ldots\ldots u_{x+3}$$

$$+\,23\,h^2\,.\,u_{x+1}\,u_{x+2}-9\,h^3\,.\,u_{x+1}.$$

(15). Affect $u_{x-1}\,u_{x+2}\,u_{x+3}$ with u_x and reduce it to factorials consisting of u_x and its succeeding values.

Ans. $u_x\,u_{x+1}\,u_{x+2}+h\,.\,u_x\,u_{x+1}-2\,h^2\,.\,u_x-6\,h^3.$

(16). Reduce $u_x\ u_{x+1}\ u_{x+2}\ u_{x+4}\ u_{x+5}$, to complete factorials.

Ans. $u_x \dots u_{x+4} + 2\,h\,u_x \dots u_{x+3} + 2\,h^2 u_x \dots u_{x+2}$.

(17). Reduce $u_x \dots u_{x+n-1} \times u_{x+n+1} \dots u_{x+m}$ to complete factorials.

$$u_x \dots u_{x+m-1} + (m-n)\,h\,.\,u_x \dots u_{x+m-2}$$

$$+(m-n)\,(m-n-1)\,h^2\,.\,u_x \dots u_{x+m-3}$$

$$+ \dots .(m-n) \dots 3\,.\,2\,.\,1\;h^{m-n}\,u_x \dots u_{x+n-1}.$$

(18). Affect $u_{x-2}\,u_{x-1}\,u_{x+1}$ with u_x.

$$u_x u_{x+1} u_{x+2} - 5\,h\,u_x\,u_{x+1} + 2\,h^2\,u_x + 2\,h^3.$$

(19). Reduce $u_x^2\,.\,u_{x+1}^3$ to factorials in such a manner that all the terms shall be positive.

$$u_{x-2} \dots u_{x+2} + 3\,h\,u_{x-1} \dots u_{x+2}$$

$$+2\,h^2\,u_{x-1}\,u_x\,u_{x+1} + 4\,h^3\,u_x\,u_{x+1}.$$

(20). Affect $u_{x+1} \dots u_{x+n}$ with u_x.

$$u_{x+1} \dots u_{x+n} = u_x \dots u_{x+n-1} + n\,h\,u_x \dots u_{x+n-2} +$$

$$+\,n\,(n-1)\,h^2\,u_x \dots u_{x+n-3} + \dots\dots n\,(n-1) \dots 1\,.\,h^n.$$

(21). Reduce $u_x \dots u_{x+n}$ to factorials ending with any value u_{x+m+n}.

$$u_x \dots u_{x+n} = u_{x+m} \dots u_{x+m+n}$$

$$-\frac{(n+1)\,.\,m}{1}\,h\,u_{x+m+1} \dots u_{x+m+n}$$

$$+\frac{(n+1)\,.\,n \times m\,(m+1)}{1\,.\,2}\,h^2\,u_{x+m+2} \dots u_{x+m+n}$$

$$-\frac{(n+1)\,n\,(n-1) \times m\,(m+1)\,(m+2)}{1\,.\,2\,.\,3}\,h^3\,.\,\&c. + \&c.$$

The investigation is easy if we employ the principle explained in (Prob. 4. Sect. 2.) by assuming

$$u_x \dots u_{x+n} = A\, u_{x+m} \dots u_{x+m+n}$$
$$+ B\, u_{x+m+1} \dots u_{x+m+n} + \&c.,$$

and making u_{x+m+1}, u_{x+m+2}, &c. vanish in succession, which will give so many equations for determining A, B, &c. one from the other. The two last expressions are from Emerson's Increments.

(22). Affect $u_{x+\alpha}\, u_{x+\beta}$ with u_x,

$$u_{x+\alpha}\, u_{x+\beta} = u_x\, u_{x+1} + (\alpha + \beta - 1)\, h\, u_x + \alpha\, \beta\, .\, h^2.$$

(23). Affect $u_{x+\alpha}\, u_{x+\beta}\, u_{x+\gamma}$, with u_x,

$$u_{x+\alpha}\, u_{x+\beta}\, u_{x+\gamma} = u_x\, u_{x+1}\, u_{x+2} + (\alpha + \beta + \gamma - 3)\, h\, u_x\, u_{x+1} +$$
$$+ (\alpha\beta + \alpha\gamma + \beta\gamma - \alpha - \beta - \gamma + 1)\, h^2\, u_x + \alpha\, \beta\, \gamma\, .\, h^3.$$

(24). To determine the general law observed in these reductions, or, to reduce the function

$$u_{x+\alpha}\, u_{x+\beta}\, u_{x+\gamma} \times \&c.$$

where the number of factors is n to a series of complete factorials commencing with u_x and proceeding according to the succeeding values u_{x+1}, u_{x+2}, &c. u_x being as before equal to $a + h\,x$.

Assume $A_0 + A_1 u_x + A_2\, u_x\, u_{x+1} +$ &c. for the series and make x in succession $-\frac{a}{h}$, $-\left(\frac{a}{h} + 1\right)$, $-\left(\frac{a}{h} + 2\right)$, &c. so as to cause the factors u_x, u_{x+1}, &c. to vanish in succession, and the resulting equation will give the coefficients in succession just as in (Prob. 4. Sect. 2.) Or we may proceed by supposing $v = h\,x$, $F(v) = u_{x+\alpha}\, .\, u_{x+\beta}$, &c.

$$\text{or, } F(v)=(v+a+\alpha h)(v+a+\beta h).\ \&c.$$

$$\text{and } f_x = -a-(x-1)\,h,$$

and at once substituting these values in the general expressions there given for the coefficients and reducing. In either way we shall obtain for a final result the following general and useful

THEOREM.

Let $S_0 = 1$,

$$S_1 = \alpha+\beta+\gamma+\&c.$$

$$S_2 = \alpha\beta+\alpha\gamma+\beta\gamma+\&c.$$

. . . .

$$S_n = \alpha\beta\gamma \times \&c.$$

Then will

$$u_{x+\alpha}\cdot u_{x+\beta}\cdot u_{x+\gamma}\cdot \&c. = h^n S_n$$

$$+\frac{h^{n-1}}{1}u_x\{\Delta o\,.\,S_{n-1}-\Delta o^2\,.\,S_{n-2}+\Delta o^3\,.\,S_{n-3}-\&c.\}$$

$$+\frac{h^{n-2}}{1\,.\,2}u_x u_{x+1}\{\Delta^2 o^2\,.\,S_{n-2}-\Delta^2 o^3\,.\,S_{n-3}+\&c.\}$$

$$+\frac{h^{n-3}}{1\,.\,2\,.\,3}u_x u_{x+1} u_{x+2}\{\Delta^3 o^3\,.\,S_{n-3}-\&c.\}+\&c.$$

The terms $\Delta o^0\,.\,S_n$ in the coefficient of $\frac{h^{n-1}}{1}u_x$, $\Delta^2 o^0\,.\,S_n - \Delta^2 o^1\,.\,S_{n-1}$ in that of $\frac{h^{n-2}}{1\,.\,2}u_x u_{x+1}$ and so on being of themselves equal to zero are for brevity omitted in the respective series within the brackets, to which they belong. Nevertheless to preserve the symmetry of the equation, they ought, if not set down, at least to be understood, being included in the law of the other terms, and *this remark is to be considered as applying to all similar cases,* $\Delta^0 o^0$ *being unity and* Δo^0, $\Delta^2 o^0$, $\Delta^2 o^1$, $\Delta^3 o^0$, $\Delta^3 o^1$, $\Delta^3 o^2$, &c. *respectively zero.*

Some particular cases of the general theorem deserve to be stated separately, as they afford transformations which which we may have occasion to use.

(25). To resolve u^n_{x+1} into succeeding factors affected with u_x. Here $\alpha = \beta = \&c. = 1$: and

$$u^n_{x+1} = h^n + \frac{h^{n-1}}{1} u_x \times$$

$$\left\{ \frac{n}{1} \Delta o - \frac{n(n-1)}{1 \,.\, 2} \Delta o^2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} \Delta o^3 - \&c. \right\}$$

$$+ \frac{h^{n-2}}{1 \,.\, 3} u_x u_{x+1} \times$$

$$\left\{ \frac{n(n-1)}{1 \,.\, 2} \Delta^3 o^2 - \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} \Delta^2 o^3 - \&c. \right\}$$

$$+ \frac{h^{n-3}}{1 \,.\, 2 \,.\, 3} u_x u_{x+1} u_{x+2} \times$$

$$\left\{ \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} \Delta^3 o^3 - \&c. \right\} + \&c.$$

But this is not the simplest or most elegant form in which this equation can be expressed. If we complete the series within the brackets by inserting the deficient terms at their commencement, and then separate the symbols of operation from those of quantity they will become

$$- \Delta \left\{ 1 - \frac{n}{1} o + \frac{n(n-1)}{1 \,.\, 2} o^2 - \&c. \right\} = - \Delta (1 - 0)^n$$

$$+ \Delta^2 \left\{ 1 - \frac{n}{1} o + \frac{n(n-1)}{1 \,.\, 2} o^2 - \&c. \right\} = + \Delta^2 (1 - 0)^n, \&c.$$

Now the reader will find it demonstrated in Art. 17. Sect. 7. that the expressions $\Delta(1-0)^n$, $\Delta^2(1-0)^n$, &c. are respectively equivalent to

$$(-1)^n . \{ \Delta o^n - \Delta^2 o^n + \Delta^3 o^n . . . . \pm \Delta^n o^n \},$$

$$(-1)^n . \{ \Delta^2 o^n - \Delta^3 o^n + ... \mp \Delta^n o^n \},$$

$$(-1)^n . \{ \Delta^3 o^n . . . . \pm \Delta^n o^n \}, \text{ \&c.}$$

or, inverting the order of writing their terms, to

$$- \Delta^n o^n + \Delta^{n-1} o^n . . . . \pm \Delta o^n$$

$$+ \Delta^n o^n - \Delta^{n-1} o^n . . . . \pm \Delta^2 o^n, \text{ \&c.}$$

so that by substitution we shall have

$$u^n_{x+1} = h^n + \frac{\Delta^n o^n - \Delta^{n-1} o^n + ... \pm \Delta o^n}{1} h^{n-1} u_x$$

$$+ \frac{\Delta^n o^n - \Delta^{n-1} o^n + ... \mp \Delta^2 o^n}{1 . 2} h^{n-2} u_x u_{x+1}$$

$$+ \frac{\Delta^n o^n - \Delta^{n-1} o^n + ... \pm \Delta^3 o^n}{1 . 2 . 3} h^{n-3} u_x u_{x+1} u_{x+2}$$

$$+ \text{ \&c.}$$

The reader must not be startled by the employment of o as an algebraic symbol in such expressions as $\Delta(1-0)^n$. He will call to mind that this and similar expressions are mere abbreviations and have no meaning beyond what is expressed by their developement. The transformation in the latter part of this problem cannot, however, be comprehended without a previous knowledge of those more general properties of the functions $\Delta^m o^n$ which will be hereafter demonstrated, and is only inserted in this place that things relating to the same subject may be kept together.

(26). To resolve u^n_x into u_x and succeeding factors

$$u_x^n = (-1)^{n+1} \left\{ \frac{\Delta o^n}{1} h^{n-1} u_x - \frac{\Delta^2 o^n}{1 . 2} h^{n-2} u_x u_{x+1} + \text{ \&c.} \right\} ; \; ... (a.)$$

If in this we put $n+1$ for n and divide by u_x we get as follows:

$$u^n_x = (-1)^n \left\{ \frac{\Delta\, 0^{n+1}}{1} h^n - \frac{\Delta^2\, 0^{n+1}}{1\,.\,2} h^{n-1}\, u_{x+1} \right.$$

$$\left. + \frac{\Delta^3\, 0^{n+1}}{1\,.\,2\,.\,3} h^{n-2}\, u_{x+1}\, u_{x+2} - \&c. \right\}; \ldots . (b).$$

(27). Let $u_x = a + h\,x$, $u'_x = a' + h'\,x$. To resolve $(u_x)^n$ into u'_x and succeeding factors, u'_{x+1}, &c.

$$(a + h\,x)^n = \left(\frac{h}{h'}\right)^n . \left(\frac{a\,h'}{h} + h'\,x\right)^n$$

$$= \left(\frac{h}{h'}\right)^n . \left\{ (a' + h'\,x) + h' . \frac{a\,h' - a'\,h}{h\,h'} \right\}^n$$

or, $$(u_x)^n = \left(\frac{h}{h'}\right)^n . (u'_x + \alpha)^n$$

where $$\alpha = \frac{a\,h' - h\,a'}{h\,h'} = \frac{a}{h} - \frac{a'}{h'}.$$

Therefore in the general expression (24) writing u'_x for u_x and making β, γ, &c. equal to α, we get as follows:

$$u_x^n = \left(\frac{h}{h'}\right)^n \left\{ h'^n . \alpha^n \right.$$

$$+ \frac{h'^{n-1}}{1} u'_x \left(\frac{n}{1} \alpha^{n-1} \Delta\, 0 - \frac{n\,(n-1)}{1\,.\,2} \alpha^{n-2} \Delta\, 0^2 + \&c. \right)$$

$$+ \frac{h'^{n-2}}{1\,.\,2} u'_x\, u'_{x+1} \left(\frac{n\,(n-1)}{1\,.\,2} \alpha^{n-2} \Delta^2\, 0^2 - \&c. \right)$$

$$\left. + \&c. \right\}$$

Now, it will be proved in Art. 17. Sect. 7. that the series

$$-\Delta\left\{\alpha^n - \frac{n}{1}\alpha^{n-1}.o + \frac{n(n-1)}{1.2}\alpha^{n-2}.o^2 - \&c.\right\},$$

$$+\Delta^2\left\{\alpha^n - \frac{n}{1}\alpha^{n-1}.o + \frac{n(n-1)}{1.2}\alpha^{n-2}.o^2 - \&c.\right\}, \&c.$$

which the separation of symbols of operation from those of quantity (as in the last example but one) produces from the coefficients of the several terms $\frac{h'^{n-1}}{1} u_x$, &c. and which have for their abbreviations respectively

$$-\Delta(\alpha - o)^n, \quad +\Delta^2(\alpha - o)^n, \ \&c.$$

are equivalent to other series, the respective abbreviations of which are

$$(-1)^{n+1}\frac{\Delta}{(1+\Delta)^{\alpha}}o^n, \quad (-1)^{n+2}.\frac{\Delta^2}{(1+\Delta)^{\alpha}}o^n,$$

$$(-1)^{n+3}.\frac{\Delta^3}{(1+\Delta)^{\alpha}}o^n, \ \&c.$$

the series themselves being

$$(-1)^{n+1}\left\{\Delta\, o^n - \frac{\alpha}{1}\Delta^2 o^n + \frac{\alpha(\alpha+1)}{1.2}\Delta^3 o^n \ldots\ldots\right.$$

$$\left.\pm\frac{\alpha(\alpha+1)\ldots.(\alpha+n-2)}{1.2\ldots.(n-1)}\Delta^n o^n\right\},$$

$$(-1)^{n+2}\left\{\Delta^2 o^n - \frac{\alpha}{1}\Delta^3 o^n \ldots\ldots\right.$$

$$\left.\pm\frac{\alpha(\alpha+1)\ldots.(\alpha+n-3)}{1.2\ldots.(n-2)}\Delta^n o^n\right\}, \ \&c.$$

and the term α^n itself is shewn by (Art. 17. Sect. 7.) to be equivalent to $(-1)^n.\frac{1}{(1+\Delta)^{\alpha}}o^n$, or to the series

$$(-1)^n\left\{o^n - \frac{\alpha}{1}\Delta o^n + \ldots. \pm \frac{\alpha(\alpha+1)\ldots.(\alpha+n-1)}{1.2\ldots.n}\Delta^n o^n\right\}.$$

Thus we arrive at the following equation

$$u_x^n = \left(-\frac{h'}{h}\right)^n \cdot \left\{ \frac{1}{(1+\Delta)^a} 0^n \cdot h'^n - \frac{\Delta}{(1+\Delta)^a} 0^n \cdot \frac{h'^{n-1}}{1} u'_x \right.$$
$$\left. + \frac{\Delta^2}{(1+\Delta)^a} 0^n \cdot \frac{h'^{n-2}}{1 . 2} u'_x u'_{x+1} - \&c. \right\}$$

which affords an indefinite number of different ways of transforming any given expression of this kind. For example:

$$(28). \quad (x+a)^n = (-1)^n \cdot \left\{ \frac{1}{(1+\Delta)^a} 0^n - \frac{\Delta}{(1+\Delta)^a} 0^n \cdot \frac{x}{1} + \right.$$
$$\left. + \frac{\Delta^2}{(1+\Delta)^a} 0^n \cdot \frac{x(x+1)}{1 . 2} - \&c. \right\}.$$

SECTION III.

Exercises in the Reduction of Fractional Expressions to Integrable Forms.

(1). Reduce $\frac{x}{(x+1)(x+2)(x+3)}$ to an integrable form.

It becomes

$$\frac{1}{(x+1)(x+2)} - \frac{3}{(x+1)(x+2)(x+3)}.$$

$$(2). \quad \frac{1}{(x+1)(x+3)} =$$

$$= \frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+2)(x+3)}.$$

(3). $$\frac{u_{x+1}}{u_x u_{x+2} u_{x+3}} = \frac{1}{u_x u_{x+1}} - \frac{3h}{u_x u_{x+1} u_{x+2}} + \frac{4h^2}{u_x \ldots u_{x+3}} \text{ where } u_x = a + hx.$$

(4). $$\frac{(x+1)^2}{x(x+2)(x+3)(x+4)} = \frac{1}{x(x+1)} - \frac{6}{x(x+1)(x+2)} + \frac{19}{x\ldots\ldots(x+3)} - \frac{27}{x\ldots\ldots(x+4)}.$$

(5). $$\frac{3x-4}{(4x^2-1)(2x-7)} = \frac{3}{2} \cdot \frac{1}{(2x-1)(2x+1)} + \frac{13}{2} \cdot \frac{1}{(2x-3)(2x-1)(2x+1)} + \frac{26}{(2x-5)\ldots\ldots(2x+1)} + \frac{52}{(2x-7)\ldots\ldots(2x+1)}.$$

(6). Theorem (from Emerson's increments,)

$$\frac{1}{u_{x+n}} = \frac{1}{u_x} - \frac{nh}{u_x u_{x+1}} + \frac{n(n-1)h^2}{u_x u_{+1} u_{x+2}} - \&c.$$

This is immediately deducible from (Prob. 21. Sect. 2.) by making $m = 1$, writing $n - 1$ for n, and dividing the whole equation so prepared by $u_x \ldots . u_{x+n}$.

(7). A rather more general theorem (given by the same author) is the following:

$$\frac{1}{u_{x+n}} = \frac{1}{u_{x+m}} - \frac{(n-m)h}{u_{x+m} u_{x+m+1}} + \frac{(n-m)(n-m-1)h^2}{u_{x+m} u_{x+m+1} u_{x+m+2}} - \&c.$$

which may be proved by a process almost precisely the same.

These two transformations are not without their use in facilitating reductions of a certain class. For instance,

(8). To reduce $\frac{1}{u_{x-1}\,u_{x+2}}$ to an integrable form,

$$\frac{1}{u_{x+2}} = \frac{1}{u_x} - \frac{2\,h}{u_x\,u_{x+1}} + \frac{2\,h^2}{u_x\,u_{x+1}\,u_{x+2}}$$

therefore

$$\frac{1}{u_{x-1}\,u_{x+2}} = \frac{1}{u_{x-1}\,u_x} - \frac{2\,h}{u_{x-1}\ldots u_{x+1}} + \frac{2\,h^2}{u_{x-1}\ldots u_{x+2}}.$$

(9). To reduce $\frac{1}{u_x\,.\,u_{x+1}\,.\,u_{x+5}}$ in like manner

$$\frac{1}{u_{x+5}} = \frac{1}{u_{x+2}} - \frac{3\,h}{u_{x+2}\,.\,u_{x+3}} + \frac{6\,h^2}{u_{x+2}\,u_{x+3}\,u_{x+4}}$$
$$- \frac{6\,h^3}{u_{x+2}\,u_{x+3}\,u_{x+4}\,u_{x+5}},$$

and

$$\frac{1}{u_x\,u_{x+1}\,u_{x+5}} = \frac{1}{u_x\ldots u_{x+2}} - \frac{3\,h}{u_x\ldots u_{x+3}} + \frac{6\,h^2}{u_x\ldots u_{x+4}}$$
$$- \frac{6\,h^3}{u_x\ldots u_{x+5}}.$$

SECTION IV.

Exercises in the Integration of Equations of Differences.

(1). To integrate the equation

$$u_{x+1} - (x+1)u_x = 1 . 2 . . . . (x+1).$$

The complete integral is

$$u_x = (C + x) \times 1 . 2 . . . . . . x$$

C being an arbitary constant quantity.

(2). $$u_{x+1} - p\,a^{2x}\,u_x = q\,a^{x^2}$$

$$u_x = C . p^{x-1} . a^{x^2-x} + \frac{q\,a^{(x-1)^2}}{1-ap}.$$

(3). $$(x+1)^2 \{ u_{x+1} - a u_x \} = a^x$$

$$u_x = C . a^x + a^{x-1} . \Sigma \frac{1}{(x+1)^2}$$

$$= C . a^x + a^{x-1} \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \frac{1}{x^2} \right\}.$$

In this Example the integration of $\frac{1}{(1+x)^2}$ cannot be performed in finite terms, unless we express it in a series the number of whose terms is variable. This we have done, and in many cases (as we shall see) such a result is useful and satisfactory.

(4). $$u_{x+2} - p\,u_{x+1} + q\,u_x = 0$$

$$u_x = {}^1C \cdot \left\{\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}\right\}^x$$

$$+ {}^2C \cdot \left\{\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}\right\}^x$$

${}^1C$ and ${}^2C$ being two arbitrary constants.

(5). $u_{x+2} - a(x+2)u_{x+1} + b(x+1)(x+2)u_x = 0$

$$u_x = 1 \, . \, 2 \ldots . \, x \, \{ {}^1C \, . \, \alpha^x + {}^2C \, . \, \beta^x \}$$

α and β being the two roots of $u^2 + a\,u + b = 0$. The integration is performed by assuming

$$u_x = 1 \, . \, 2 \ldots \ldots x \times v_x.$$

(6). $u_{x+2} + a\,p^{2x}\,u_{x+1} + b\,p^{4x}\,u_x = c\,q^x \, . \, p^{x^2}$

$$u_x = p^{x^2+x} \{ {}^1C \, . \, \alpha^x + {}^2C \, . \, \beta^x \} + \frac{c\,q^x\,p^{(x-2)^2}}{q^2 + a\,p\,q + b\,p^4}$$

α and β being the roots of

$$u^2 + \frac{a}{p^4}u + \frac{b}{p^6} = 0.$$

(7). $u_{x+2} + a \, . \, \phi(x+1)u_{x+1} + b \, . \, \phi(x) \, . \, \phi(x+1)\,u_x = c,$

$\phi(x)$ being any function of x whatever;

$$u_x = P\,\phi(x-1) \times$$

$$\left\{ {}^1C\,\alpha^x + {}^2C\,\beta^x + \alpha^{x-1}\,\Sigma \, . \frac{\beta^{x-1}}{\alpha^x}\,\Sigma\,\frac{c}{\beta^x \, . \, P\,\phi(x+1)} \right\},$$

$P\,\phi(x)$ denoting $\phi(1) \, . \, \phi(2) \ldots . \, \phi(x)$, and α and β being roots of $u^2 + a\,u + b = 0$.

(8). $u_{x+3} + a\,p^x\,u_{x+2} + b\,p^{2x}\,u_{x+1} + c\,p^{3x}\,u_x = 0$

$$u_x = p^{\frac{x(x+1)}{2}} \{ {}^1C \, . \, \alpha^x + {}^2C \, . \, \beta^x + {}^3C \, . \, \gamma^x \}$$

α, β, γ, being the three roots of

$$u^3 + \frac{a}{p^3}u^2 + \frac{b}{p^5}u + \frac{c}{p^6} = 0.$$

(9). $u_{x+n} + a p^{x} u_{x+n-1} + b p^{2x} u_{x+n-2} + \dots k p^{nx} u_x = 0,$

$$u_x = p^{\frac{x(x+1)}{2}} \{ {}^1C . \alpha^x + {}^2C . \beta^x + \dots {}^nC . \nu^x \}$$

$\alpha, \beta, \dots \nu$ being the n roots of

$$u^n + \frac{a}{p^n} u^{n-1} + \frac{b}{p^{2n-1}} u^{n-2} + \frac{c}{p^{3n-3}} u^{n-3} + \dots \frac{k}{p^{\frac{n(n+1)}{2}}} = 0.$$

(10). $u_{x+2} + a u_{x+1} + b u_x = X_x,$

$$u_x = {}^1C . \alpha^x + {}^2C . \beta^x + \frac{1}{b} \alpha^x . \Sigma . \left(\frac{\beta}{\alpha}\right)^x \Sigma \frac{X_x}{\beta^x};$$

α and β being the roots of $u^2 + au + b = 0.$

(11). $\Delta^n u_x = X_x,$

$$u_x = {}^1C + {}^2C . x + {}^3C . x^2 + \dots {}^nC . x^{n-1} + \Sigma^n X_x .$$

(12). $\Delta u_x + \Delta^2 u_x = 42 u_x$

$$u_x = {}^1C . 7^x + {}^2C . (-6)^x.$$

(13). $(x+3)^2 . u_{x+2} - 2\frac{(x+2)^2}{x+1} u_{x+1}$

$$+ \frac{(x+1)^2 (x+2)}{x} u_x = 0.$$

is to be integrated by the help of a given particular integral $\left(\frac{x}{x+1} = u_x\right)$.

$$u_x = {}^1C . \frac{x}{x+1} + {}^2C . \left(\frac{x}{x+1}\right)^2.$$

(14.) $u_{x+1} - 2u_x^2 + 1 = 0,$

$$u_x = \tfrac{1}{2} \{ C^{2^x} + C^{-2^x} \};$$

The substitution to be used is $u_x = \cos v_x$.

(15). $u^2_{x+1} - 4\,u^2_x\,(u^2_x + 1) = 0,$

$$u_x = \tfrac{1}{2}\left\{ C^{2^x} - C^{-2^x} \right\}.$$

The substitution is $u_x = \sqrt{(-1)} \,.\, \sin v_x$.

(16). $u^2_x - 4\,u_x\,u_{x+1} + a\,(2\,u_x + u_{x+1}) = 0,$

Substitute $\frac{a}{2} \,.\, \frac{2\,v_x - 1}{v_x + 1}$ for u_x and again, in the resulting equation, $-\cos w_x$ for v_x, when it will assume an integrable form.

(17). $u^2_x + 2\,u_x\,u_{x+1} - a\,(4\,u_x - u_{x+1}) = 0,$

Substitute $2\,a \,.\, \frac{v_x + 1}{2\,v_x - 1}$ for u_x

(18). $u_{x+2}\,u_{x+1}\,u_x = a\,(u_{x+2} + u_{x+1} + u_x),$

Assume $u_x = \sqrt{(a)} \,.\, \tan v_x$ then will the equation for determining v_x be

$$v_{x+2} + v_{x+1} + v_x = 0,$$

whence finally

$$u_x = \sqrt{(a)} \,.\, \tan\left\{ {}^1C \,.\, \cos\frac{2\,\pi\,x}{3} + {}^2C \,.\, \sin\frac{2\,\pi\,x}{3} \right\}.$$

(19). The same substitution will be found to succeed in the equations

$$u_{x+2}\,u_{x+1}\,u_x + a\,(u_{x+2} - u_{x+1} + u_x) = 0,$$

$$u_{x+2}\,u_{x+1}\,u_x - a\,(u_{x+2} - u_{x+1} - u_x) = 0,$$

$$u_{x+2}\,u_{x+1}\,u_x + a\,(u_{x+2} + u_{x+1} - u_x) = 0.$$

(20). To integrate the equation

$$u_{x+1}\,u_x - a\,(u_{x+1} - u_x) + 1 = 0.$$

Laplace, *Journ. de l'Ecole Polytechnique*, Cah. 15.

Differentiate it relative to x, and it becomes

$$(a + u_{x+1})\frac{d\,u_x}{d\,x} - (a - u_x)\frac{d\,u_{x+1}}{d\,x} = 0.$$

But the proposed equation gives

$$a = \frac{u_{x+1}\,u_x + 1}{u_{x+1} - u_x};$$

so that by eliminating a, we find

$$\frac{d\,u_{x+1}}{1 + u^2_{x+1}} - \frac{d\,u_x}{1 + u^2_x} = 0,$$

and integrating

$$\int \frac{d\,u_{x+1}}{1 + u^2_{x+1}} - \int \frac{d\,u_x}{1 + u^2_x} = A$$

A being a certain function of a to be hereafter determined. In fact, since both this and the proposed, are each of them complete integrals of one and the same differential equation, the one can be nothing more than a transformation of the other. Now this latter is equivalent to

$$\Delta \int \frac{d\,u_x}{1 + u^2_x} = A$$

because $\int \frac{d\,u_{x+1}}{1 + u^2_{x+1}}$ is the same function of u_{x+1} that $\int \frac{d\,u_x}{1 + u^2_x}$ is of x. This equation is immediately integrable (relative to the characteristic Δ) and gives

$$\int \frac{d\,u_x}{1 + u^2_x} = A\,x + C$$

C being an arbitrary constant. To determine the function A of a we first have to assign $\int \frac{d\,u_x}{1 + u^2_x}$. Now this is arc $(\tan = u_x)$ or, $\tan^{-1}(u_x)$, hence

$$\tan^{-1} u_x = A x + C$$

$$u_x = \tan (A x + C)$$

therefore,

$$u_{x+1} = \tan (A x + C + A)$$

$$= \frac{u_x + \tan A}{1 - u_x . \tan A}.$$

But the proposed equation gives

$$u_{x+1} = \frac{u_x + \frac{1}{a}}{1 - u_x . \frac{1}{a}},$$

which compared with the foregoing gives $\tan A = \frac{1}{a}$, or $A = \tan^{-1} \frac{1}{a}$. The complete integral then of the proposed equation is

$$u_x = \tan \left\{ x . \tan^{-1} \left(\frac{1}{a}\right) + C \right\}$$

which, although in appearance transcendental, is easily freed from that form, and reduced to a particular case of the integral found in the Appendix. Art. 386. of the more general equation there discussed. The same method applies to certain other equations, for which see the author cited.

(21). To integrate the more general equation

$$u_{x+1} u_x - a_x (u_{x+1} - u_x) + 1 = 0,$$

The proposed equation gives

$$\frac{1}{a_x} = \frac{u_{x+1} - u_x}{1 + u_x u_{x+1}} = \frac{\Delta u_x}{1 + u_x u_{x+1}}$$

but by (Art. 28. Sect. 1.) we have

$$\Delta \tan^{-1} u_x = \tan^{-1} \left(\frac{\Delta u_x}{1 + u_x u_{x+1}} \right)$$

Therefore this becomes by substitution

$$\Delta \tan^{-1} u_x = \tan^{-1} \frac{1}{a_x}$$

and integrating

$$\tan^{-1} u_x = C + \Sigma \tan^{-1} \frac{1}{a_x}$$

or taking the tangent of each member

$$u_x = \tan \left\{ C + \Sigma \tan^{-1} \frac{1}{a_x} \right\},$$

The preceding is a particular case of this, a_x being there an absolute constant.

SECTION V.

Exercises in the Integration of Equations of Mixed Differences.

It will be necessary in the following cases to adhere to the notation of partial differences we have before employed (See Appendix. Art. 357, 364.) viz.

$$\frac{d}{dx} u_{x,y} = \frac{d u_{x,y}}{dx}; \qquad \frac{d}{dy} u_{x,y} = \frac{d u_{x,y}}{dy},$$

and so on for the higher orders of the differential coefficients, thus

$$\left(\frac{d}{dx}\right)^m \left(\frac{d}{dy}\right)^n u_{x,y} = \frac{d^{m+n} u_{x,y}}{dx^m . dy^n}.$$

The same mode of referring the symbols of operation to their proper independent variables may also be conveniently extended to the characteristic Δ, as follows:

$$\frac{\Delta}{\Delta x} u_{x,y} = u_{x+1,y} - u_{x,y}\,;\ \frac{\Delta}{\Delta y} u_{x,y} = u_{x,y+1} - u_{x,y},$$

$$\left(\frac{\Delta}{\Delta x}\right)^m \left(\frac{\Delta}{\Delta y}\right)^n u_{x,y} =$$

$$\left(\frac{\Delta}{\Delta y}\right)^n u_{x+m,y} - \frac{m}{1}\left(\frac{\Delta}{\Delta y}\right)^n u_{x+m-1,y} + \&c.$$

These two different modes of varying the independent quantities x, y, may occur together, as in the expression

$$\frac{\Delta}{\Delta x}\,\frac{d}{d y} u_{x,y} = \frac{d\,u_{x+1,y}}{d y} - \frac{d\,u_{x,y}}{d y}.$$

and others of the like sort, and it is manifestly a matter of indifference in what order the operations denoted by Δ and d are performed. Equations of mixed differences determine the form of a function by assigning a relation between these derivatives. In Appendix. Art. 387. we have considered mixed differential equations with one independent variable. We shall here give a few instances where more than one are involved.

(1). To integrate the equation

$$u_{x+1,y} = \frac{d}{d y} u_{x,y}$$

This gives $\qquad u_{x,y} = \left(\frac{d}{d y}\right)^x \phi\,(y),$

$\phi\,(y)$ being an arbitrary function of y. The reason is evident; for if we take $u_{0,y} = \phi\,(y)$, we have $u_{1,y} = \frac{d}{d y}\phi\,(y)$, whence we derive $u_{2,y} = \left(\frac{d}{d y}\right)^2 \phi\,(y)$, and so on.

(2). Suppose $\frac{\Delta}{\Delta x} u_{x,y} = a \,.\, \frac{d}{d y} u_{x,y},$

Assume $$u_{x,y}=a^x \,.\, e^{-\frac{y}{a}} \,.\, v_{x,y},$$

and

$$\frac{\Delta}{\Delta x} u_{x,y} = e^{-\frac{y}{a}} \{ a^{x+1} v_{x+1,y} - a^x v_{x,y} \}$$

$$= a^x \,.\, e^{-\frac{y}{a}} \{ a v_{x+1,y} - v_{x,y} \}$$

also,

$$a \,.\, \frac{d}{dy} u_{x,y} = a^x \,.\, e^{-\frac{y}{a}} \,.\, \left\{ a \frac{d}{dy} v_{x,y} - v_{x,y} \right\}.$$

The equation then becomes by substitution and reduction

$$v_{x+1,y} = \frac{d}{dy} v_{x,y}$$

which has already been integrated, and thus we get

$$u_{x,y}=a^x \,.\, e^{-\frac{y}{a}} \,.\, \left(\frac{d}{dy}\right)^x \phi(y)$$

ϕ denoting an arbitrary function.

(3). Given $u_{x+1,y} = a \,.\, \left(\frac{d}{dy}\right)^n u_{x,y}$,

Assume $u_{x,y}=a^x \,.\, v_{x,y}$ and we find for the integral

$$u_{x,y}=a^x \,.\, \left(\frac{d}{dy}\right)^{nx} \phi(y)$$

(4). $u_{x+2,y}-a\left(\frac{d}{dy}\right) u_{x+1,y} + b \left(\frac{d}{dy}\right)^2 u_{x,y}=0,$

Assume, $u_{x,y}=p^x \,.\, v_{x,y}$, and we get by substitution and division of the whole by p^x,

$$p^2 v_{x+2,y}-a p \,.\, \frac{d}{dy} v_{x+1,y} + b \,.\, \left(\frac{d}{dy}\right)^2 v_{x',y}=0.$$

In this equation the sum of the indices $x+2$, $x+1$, x, below the v in each term, and the corresponding exponents 0, 1, 2, of the symbols of differentiation $\frac{d}{dy}$ is the same: if then we suppose

$$v_{x,y} = \left(\frac{d}{dy}\right)^x \phi(y)$$

we shall have

$$v_{x+2,y} = \frac{d}{dy} v_{x+1,y} = \left(\frac{d}{dy}\right)^2 v_{x,y} = \left(\frac{d}{dy}\right)^{x+2} \phi(y),$$

and the whole equation is divisible by this function, leaving

$$p^2 - ap + b = 0,$$

to determine p. Let α and β be its roots, then since the proposed equation is of the first degree, and either $\alpha^x . \left(\frac{d}{dy}\right)^x \phi_1(y)$ or $\beta^x . \left(\frac{d}{dy}\right)^x \phi_2(y)$ separately satisfy it, their sum is the general expression for $u_{x,y}$,

$$u_{x,y} = \alpha^x . \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . \left(\frac{d}{dy}\right)^x \phi_2(y)$$

$\phi_1(y)$ and $\phi_2(y)$ being two arbitrary functions of y.

$$(5). \quad u_{x+n,y} - a\frac{d}{dy} u_{x+n-1,y} + b\left(\frac{d}{dy}\right)^2 u_{x+n-2,y} - \ldots$$
$$\ldots\ldots \pm k . \left(\frac{d}{dy}\right)^n u_{x,y} = 0.$$

Let α, β, γ, &c. be the roots of

$$u^n - a u^{n-1} + b u^{n-2} - \ldots \pm k = 0,$$

then

$$u_{x,y} = \alpha^x . \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . \left(\frac{d}{dy}\right)^x \phi_2(y) + \text{\&c.}$$

$\phi_1(y)$, $\phi_2(y)$, $\ldots \phi_n(y)$ denoting n arbitrary functions of y.

But here a remark of considerable importance offers itself. It will immediately be observed that the process by which the above two equations are integrated is entirely independent of the nature of the operation denoted by $\frac{d}{dy}$. It might have been any other, and thus we might by the same process integrate

$$u_{x+n,y} - a\frac{\Delta}{\Delta y}u_{x+n-1,y} + b\left(\frac{\Delta}{\Delta y}\right)^2 u_{x+n-2,y} \cdots$$
$$\cdots \pm k\left(\frac{\Delta}{\Delta y}\right)^n u_{x,y} = 0,$$

or yet more generally

$$u_{x+n,y} - a\nabla u_{x+n-1,y} + b\nabla^2 u_{x+n-2,y} - \cdots$$
$$\cdots \pm k\nabla^n u_{x,y} = 0,$$

where $\nabla u_{x,y}$ denotes any linear combination of the differences or differential coefficients of $u_{x,y}$ relative to y, of whatever order we please, nor does its generality stop here.

(6). To integrate the equation

$$\left(\frac{\Delta}{\Delta x}\right)^2 u_{x,y} - a\frac{\Delta}{\Delta x}\frac{d}{dy}u_{x,y} + b\left(\frac{d}{dy}\right)^2 u_{x,y} = 0,$$

Assume $u_{x,y} = \alpha^x . e^{-\frac{y}{a}} . v_{x,y}$,

and we get by substitution, and division by $\alpha^x . e^{-\frac{a}{y}}$,

$$0 = \left\{\alpha^2 v_{x+2,y} - a\,\alpha . \frac{d}{dy}v_{x+1,y} + b . \left(\frac{d}{dy}\right)^2 v_{x,y}\right\}$$
$$-\frac{2}{a}\left\{\alpha^2 v_{x+1,y} - a\,\alpha . \tfrac{1}{2}\left(v_{x+1,y} + \frac{d}{dy}v_{x,y}\right) + b . \frac{d}{dy}v_{x,y}\right\}$$
$$+\frac{1}{\alpha^2}\left\{\alpha^2 - a\,\alpha + b\right\} v_{x,y}.$$

Let us for a moment suppose $v_{x,y} = \left(\frac{d}{dy}\right)^x \phi(y)$, then it is evident by the preceding problems that this equation will become

$$0 = (\alpha^2 - a\,\alpha + b)\, v_{x+2,y} - \frac{2}{\alpha}(\alpha^2 - a\,\alpha + b)\, v_{x+1,y}$$

$$+ \frac{1}{\alpha^2}(\alpha^2 - a\,\alpha + b)\, v_{x,y} :$$

the factor $\alpha^2 - a\,\alpha + b$ being found in each term, if this be made to vanish the whole is satisfied, so that provided α be assumed a root of the equation

$$\alpha^2 - a\,\alpha + b = 0,$$

any error we may have made in our value of $v_{x,y}$ is corrected, and calling α and β its two roots we see that

$$\alpha^x . e^{-\frac{\alpha}{y}} . \left(\frac{d}{dy}\right)^x \phi_1(y) \text{ and } \beta^x . e^{-\frac{\beta}{y}} . \left(\frac{d}{dy}\right)^x \phi_2(y),$$

each satisfy the proposed equation, so that

$$u_{x,y} = \alpha^x . e^{-\frac{\alpha}{y}} \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . e^{-\frac{\beta}{y}} \left(\frac{d}{dy}\right)^x \phi_2(y).$$

(7). $$\left(\frac{\Delta}{\Delta x}\right)^n u_{x,y} - a . \left(\frac{\Delta}{\Delta x}\right)^{n-1} \frac{d}{dy} u_{x,y} +$$

$$+ b . \left(\frac{\Delta}{\Delta x}\right)^{n-2} \left(\frac{d}{dy}\right)^2 u_{x,y} -$$

$$\dots \pm k . \left(\frac{d}{dy}\right)^n u_{x,y} = 0.$$

A process similarly conducted will be found to lead to the following result:

$$u_{x,y} = \alpha^x . e^{-\frac{y}{\alpha}} . \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . e^{-\frac{y}{\beta}} . \left(\frac{d}{dy}\right)^x \phi_2(y) + \&c.$$

Where α, β, γ, &c. are the n roots of

$$u^n - a\,u^{n-1} + \ldots \pm k = 0,$$

and $\phi_1(y), \ldots \phi_n(y)$ are as many arbitrary functions of y.

(8). $$u_{x,y,z} = a\left(\frac{d}{dy}\right)^{\alpha} u_{x-1,y,z} + b\left(\frac{d}{dz}\right)^{\beta} u_{x-2,y,z},$$

$$u_{x,y,z} = a^x \,.\left(\frac{d}{dy}\right)^{\alpha x} \phi(y) + b^{\frac{x}{2}} \,.\left(\frac{d}{dz}\right)^{\frac{\beta x}{2}} \psi(z)$$

$\phi(y)$ and $\psi(z)$ denoting arbitrary functions of y and z.

SECTION VI.

Exercises in the Summation of Series by the Integration of their general Terms.

As the integration of any function leads directly to the sum of the series of which it is the general term (Appendix. Art 389.) the following examples may be looked upon as exercises in that part of the inverse calculus of differences which relates to the integration of explicit functions. To sum then the following series,

(1). $1.2.4 + 2.3.5 + 3.4.6 + \text{\&c.};$

S_x denoting the sum to x terms, we shall find

$$S_x = \frac{x(x+1)(x+2)(3x+13)}{12}.$$

(2). $1^4 + 3^4 + 5^4 + \ldots (2x-1)^4 = S_x$

$$S_x = \frac{x(4x^2-1)(12x^2-7)}{15}.$$

(3). $S_x = 1 + 2p + 3p^2 + 4p^3 + \&c.$

$$S_x = \frac{1 - p^x(1 + x - xp)}{(1-p)^2}.$$

Sum to infinity (when $p < 1$) $= S = \dfrac{1}{(1-p)^2}$.

(4). $$\frac{10}{1.2.3.4} + \frac{14}{2.3.4.5} + \frac{18}{3.4.5.6} + \&c.$$

$$S_x = \frac{2}{3} - \frac{2}{(x+1)(x+3)}; \quad S = \frac{2}{3}.$$

(5). $$\frac{1}{\sqrt{2}(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})}$$

$$+ \frac{1}{(2+\sqrt{2})(3+\sqrt{2})} + \&c.$$

$$S_x = \frac{x}{\sqrt{2}(x+\sqrt{2})}; \quad S = \frac{1}{\sqrt{2}}.$$

(6). $1 . 3^2 + 3 . 5^2 + 5 . 7^2 + \&c.$

$$S_x = \frac{x(6x^3 + 16x^2 + 9x - 4)}{3}.$$

(7). $2 . 5 . 8 + 4 . 8 . 14 + 8 . 14 . 26 + 16 . 26 . 50 + \&c.$

$$S_x = \frac{36 . 2^{3x} + 84 . 2^{2x} + 56 . 2^x - 176}{7}.$$

See Appendix. Art. 374.

(8). $$\frac{3}{5.7} - \frac{9}{7.29} + \frac{27}{29.79} - \frac{81}{79.245} + \&c.$$

$$S_x = \frac{1}{20} - \frac{1}{4\{2 + (-3)^x\}}; \quad S = \frac{1}{20}.$$

(9). $$\frac{1}{2.5.8.11}+\frac{3.19}{5.8.11.14}+\frac{5.19^2}{8.11.14.17}+\&c.$$

$$S_x=\frac{19^x}{27(3x+2)(3x+5)(3x+8)}-\frac{1}{2160}.$$ See Ex. 11.

(10). $$\frac{2}{1.3.5}-\frac{5.11}{3.5.7}+\frac{8.11^2}{5.7.9}-\&c.$$

$$S_x=\frac{1}{24}-\frac{(-11)^x}{8(2x+1)(2x+3)}.$$ See Ex. 11.

(11). To sum the series

$$\frac{p}{u_0\,u_1\,u_2\ldots u_{m-1}}+\frac{(p+q)\cdot s}{u_1\,u_2\ldots u_m}+\frac{(p+2q)\,s^2}{u_2\,u_3\ldots u_{m+1}}+\&c.$$

whenever it can be done, and to determine the condition which must be satisfied to render it practicable, (u_0, u_1, u_2, &c. forming an arithmetical progression).

The $(x+1)^{\text{th}}$ term is

$$\frac{(p+qx)\,s^x}{u_x\,u_{x+1}\ldots\ldots u_{x+m-1}}.$$

Now the difference of a function

$$\frac{A\,.\,s^x}{u_x\ldots\ldots u_{x+m-2}}$$

is easily found to be

$$\frac{A\,(s\,u_x-u_{x+m-1})\,s^x}{u_x\ldots\ldots u_{x+m-1}}.$$

For u_x substitute $a+hx$ and the numerator becomes

$$(A\,\{(s-1)\,a-(m-1)\,h\}-A\,(s-1)\,h\,x)\,s^x,$$

which compared with $(p+qx)\,s^x$, that of the function in question gives

$$A=\frac{q}{(s-1)\,h},$$

and thus we get for the sum of the series

$$S_x = \frac{q}{(s-1)\,h}\left\{\frac{s^x}{u_x\ldots\ldots u_{x+m-2}} - \frac{1}{u_0\ldots\ldots u_{m-2}}\right\},$$

and for the equation of condition

$$\frac{p}{q} = \frac{a}{h} - \frac{m-1}{s-1},$$

also when $s < 1$

$$S = \frac{q}{(1-s)\,h\,.\,u_0\ldots\ldots u_{m-2}}.$$

The two series immediately preceding this example are particular cases of this.

(12). $\frac{1^2\,.\,4}{2\,.\,3} + \frac{2^2\,.\,4^2}{3\,.\,4} + \frac{3^2\,.\,4^3}{4\,.\,5} + \&c.$

$$S_x = \frac{2}{3} + \frac{4}{3}\,.\,\frac{x-1}{x+2}\,.\,4^x.$$

(13). $\frac{1^2\,.\,9}{3\,.\,5} + \frac{2^2\,.\,9^2}{5\,.\,7} + \frac{3^2\,.\,9^3}{7\,.\,9} + \&c.$

$$S_x = \frac{3}{32}\left\{1 + \frac{2x-1}{2x+3}\,.\,3^{2x+1}\right\}.$$

(14). To determine in what cases the function

$$\frac{(p + qx + rx^2)\,s^x}{u_x\ldots\ldots u_{x+m-1}}$$

is integrable, and in those cases to perform the integration.

Assume for the integral

$$\frac{(A + Bx)\,s^x}{u_x\ldots\ldots u_{x+m-2}} \qquad (a.)$$

and by comparing the difference of this with the proposed, we shall find

$$A = \frac{q}{(s-1)\,h} - \left(\frac{a}{h} + \frac{s-m+1}{s-1}\right) \cdot \frac{r}{(s-1)\,h}$$

$$B = \frac{r}{(s-1)\,h},$$

and for the equation of condition

$$p - q \cdot \left(\frac{a}{h}\right) + r\left(\frac{a}{h}\right)^2 =$$

$$= \frac{m-1}{s-1}\left\{ -q + 2\,r\left(\frac{a}{h}\right) + \frac{s-m+1}{s-1}\,r \right\}; \qquad (b).$$

Whenever this holds good, the proposed function is directly integrable the integral being as above (*a*).

If we resolve the equation of condition (*b*) with respect to s, two values of s (real or imaginary) will be found which render the proposed function integrable. This may be yet farther extended, and by a process of the same nature the following theorem may be proved.

(15). THEOREM. *It is always practicable to assign such values of* s, *real or imaginary, being the roots of an equation of the* n^{th}, *degree that the function*

$$\frac{(\alpha + \beta\,x + \gamma\,x^2 + \ldots\ldots \nu\,x^n)\,.\,s^x}{u_x\,u_{x+1}\ldots\ldots\ldots u_{x+m-1}}$$

shall be directly integrable, α, β, γ, ... ν, *being any given quantities, and* u_x *being of the form* $a + h\,x$.

(16). To integrate the function

$$\frac{x^2\left(1 + \frac{h}{a}\right)^{2x}}{(h\,x + a)\,(h\,x + a + h)},$$

or to sum the series

$$\frac{1^2}{(a+h)\,(a+2\,h)} \cdot \left(\frac{a+h}{a}\right)^2 +$$

$$+ \frac{2^2}{(a+2\,h)\,(a+3\,h)} \cdot \left(\frac{a+h}{a}\right)^4 + \&c.$$

This function satifies the equation of condition (b) of (14), and we shall therefore find for the value of the integral, or the sum of the series to $x-1$ terms

$$C+\frac{a^2}{h^3(2a+h)}\cdot\frac{h(x-1)-a}{hx+a}\left(\frac{a+h}{a}\right)^{2x},$$

or, determining the constant, and writing $x+1$ for x,

$$S_x=\frac{a(a+h)}{h^3(2a+h)}\left\{1+\frac{hx-a}{h(x+1)+a}\left(\frac{a+h}{a}\right)^{2x+1}\right\}.$$

This comprehends as a particular case the two series summed in (12) and (13). As series of this sort are not without their interest, especially when considered in an extended point of view, a few more cases are subjoined by way of farther exercise.

(17). $\frac{2}{1\,.\,3}\times\frac{1}{3}+\frac{3}{3\,.\,5}\times\frac{1}{3^2}+\frac{4}{5\,.\,7}\times\frac{1}{3^3}+\text{\&c.}$

$$S_x=\frac{1}{4}\left\{1-\frac{1}{(2x+1)\,.\,3^x}\right\};\quad S=\frac{1}{4}.$$

(18). $\frac{3}{1\,.\,2}\times\frac{1}{2}+\frac{4}{2\,.\,3}\times\frac{1}{2^2}+\text{\&c.}$

$$S_x=1-\frac{1}{(x+1)\,.\,2^x},\qquad S=1.$$

(19). $\frac{5}{1\,.\,2\,.\,3}\times\frac{1}{2}+\frac{6}{2\,.\,3\,.\,4}\times\frac{1}{2^2}+\frac{7}{3\,.\,4\,.\,5}\times\frac{1}{2^3}+\text{\&c.}$

$$S_x=\frac{1}{2}-\frac{1}{(x+1)(x+2)\,.\,2^x};\quad S=\frac{1}{2}.$$

(20). $1+2+3+\ldots\,x=\frac{x(x+1)}{2}.$

(21). $1^2+2^2+3^2+\ldots\,x^2=\frac{x(x+1)(x+2)}{3}-\frac{x(x+1)}{2}.$

(22). $1^3 + 2^3 + 3^3 + \ldots\ldots x^3$

$$= \frac{x(x+1)(x+2)(x+3)}{4} - \frac{x(x+1)(x+2)}{1} + \frac{x(x+1)}{2}$$

$$= \left(\frac{x(x+1)}{2}\right)^2.$$

(23). $1^n + 2^n + 3^n + \ldots . x^n = S_x$

$$S_x = (-1)^{n+1} \left\{ \Delta 0^n . \frac{x(x+1)}{1.2} - \Delta^2 0^n . \frac{x(x+1)(x+2)}{1.2.3} + \ldots \right.$$
$$\left. \ldots . \pm \Delta^n 0^n . \frac{x(x+1)\ldots .(x+n)}{1.2\ldots .(n+1)} . \right\}$$

See Equation (*a*) of Art. 26. Sect. 2. where $h = 1$, $u_x = x + 1$.

The same sums may also be exhibited as follows:

(24). $1 + 2 + \ldots x = \dfrac{x(x+1)}{1.2}.$

(25). $$1^2 + 2^2 + \ldots . x^2 = \frac{1}{8} \left\{ \frac{2x+1}{1} + \right.$$
$$\left. + \frac{(2x-1)(2x+1)(2x+3)}{3} \right\}.$$

(26). $$1^3 + 2^3 + \ldots . x^3 = \frac{(x-1)x(x+1)(x+2)}{4} +$$
$$+ \frac{x(x+1)}{2}.$$

(27). $$1^4 + 2^4 + \ldots . x^4 = \frac{1}{32} \left\{ \frac{2x+1}{1} + \right.$$
$$+ 10 . \frac{(2x-1)(2x+1)(2x+3)}{3} +$$
$$\left. + \frac{(2x-3)(2x-1)(2x+1)(2x+3)(2x+5)}{5} \right\}$$

(28). $$1^5 + \ldots . x^5 = \frac{(x-2)(x-1)\ldots .(x+3)}{6} +$$
$$+ 5 . \frac{(x-1)\ldots .(x+2)}{4} + \frac{x(x+1)}{2}.$$

(29). In general; $1^{2n+1} + 2^{2n+1} + \ldots . x^{2n+1} =$
$$= A_0 . \frac{x(x+1)}{2} + A_1 . \frac{(x-1)\ldots .(x+2)}{4} +$$
$$+ A_2 . \frac{(x-2)\ldots .(x+3)}{6} + \&c.$$

where the coefficients are those given in (Art. 4. Sect. 2.), and
$$1^{2n} + 2^{2n} + \ldots . x^{2n} = \frac{1}{2^{2n+1}}$$
$$\left\{\begin{array}{c} A_0 . \frac{2x+1}{1} + A_1 . \frac{(2x-1)(2x+1)(2x+3)}{3} + \&c. \\ - A_0 + 1^2 . A_1 - 1^2 . 3^2 . A_2 + \&c. \end{array}\right\}$$
the coefficients here being those determined in (Art. 6. Sect. 2.)

(30). $1 + 3 + 5 + \ldots . (2x-1) = x^2.$

(31). $1^2 + 3^2 + 5^2 + \ldots . (2x-1)^2 =$
$$= \frac{(2x-1) . 2x . (2x+1)}{6}.$$

(32). $1^3 + 3^3 + 5^3 + \ldots . (2x-1)^3 = 2x^4 - x^2.$

(33). The general expression including all these is
$$1^n + 3^n + 5^n + \ldots . (2x-1)^n =$$
$$= (-1)^n . \left\{ \frac{\Delta 0^{n+1}}{1} . 2^{n-1} . \frac{(2x+1)-1}{1} - \frac{\Delta^2 0^{n+1}}{1 . 2} . 2^{n-2} . \right.$$
$$\left. . \frac{(2x+1)(2x+3) - 1 . 3}{2} + \&c. \right\}$$

(34). $(a+h)^n+(a+2h)^n+(a+3h)^n+\ldots\ldots(a+xh)^n=$

$$=u_1^{\,n}+u^n_{\,2}+\ldots\ldots\ldots u^n_{\,x}$$

where $u_x=a+hx$;

$$S_x=(-1)^{n+1}\left\{\frac{\Delta\, o^n}{1}\,.\,h^{n-2}\,.\,\frac{u_x\,u_{x+1}-u_0\,u_1}{2}-\frac{\Delta^2\,o^n}{1\,.\,2}\,h^{n-3}\,.\,\frac{u_x\,u_{x+1}\,u_{x+2}-u_0\,u_1\,u_2}{3}+\&c.\right\}$$

Expressions for the sums of the powers of the natural numbers were first given by Wallis in his *Arithmetica infinitorum* for the purpose of applying Cavalerius's method of indivisibles to the quadrature of curvilinear spaces whose ordinates are rational integral functions of their abscissæ. Their theory was treated in a more general way by John Bernouilli, and after him by Euler, to whom we are indebted for the general theorem for the expression of $\Sigma\, u_x$ in a series, whose numerical coefficients (from their identity with those found by John Bernouilli in the case of $\Sigma\, x^n$) he called by the name of that Geometer. The expressions given in the above and some following examples for these sums are different we believe from any yet noticed, and seem to be the simplest their nature admits.

(35). $\cos\theta+\cos 3\,\theta+\cos 5\,\theta+\ldots\cos(2\,x-1)\,\theta=$

$$=\frac{\sin x\,\theta}{\sin\theta}\,.\cos x\,\theta.$$ (Append. Art. 373.)

(36). $\sin\theta+\sin 3\,\theta+\sin 5\,\theta+\ldots\sin(2\,x-1)\,\theta=$

$$=\frac{\sin x\,\theta}{\sin\theta}\,.\sin x\,\theta.$$

(37). $\cos\theta+\cos(\theta+h)+\cos(\theta+2\,h)+\ldots$

$$\cos\{\theta+(x-1).h\}=\cos\left\{\theta+\frac{x-1}{2}\,h\right\}.\frac{\sin\left(\frac{x}{2}\,h\right)}{\sin\left(\frac{1}{2}\,h\right)}.$$

(38). $\sin\theta + \sin(\theta+h) + \sin(\theta+2h) + \ldots.$

$$\sin\{\theta + (x-1).h\} = \sin\left\{\theta + \frac{x-1}{2}h\right\}.\frac{\sin\left(\frac{x}{2}h\right)}{\sin\left(\frac{1}{2}h\right)}.$$

(39). $1.(\cos\theta)^2 + 2(\cos 2\theta)^2 + 3(\cos 3\theta)^2 + \ldots.$

$$\ldots.x(\cos x\theta)^2 = \frac{x(x+1)}{4} +$$

$$\frac{x}{4.\sin\theta}.\sin(2x+1)\theta - \frac{1}{4}\left(\frac{\sin x\theta}{\sin\theta}\right)^2.$$

(40). $\sin\theta.\cos\psi + \sin 2\theta.\cos 3\psi + \ldots..$

$$\sin x\phi.\cos(2x-1)\psi =$$

$$= \sin\left\{\frac{\theta}{2} + x\left(\frac{\theta}{2}+\psi\right)\right\}.\frac{\sin x\left(\frac{\theta}{2}+\psi\right)}{2\sin\left(\frac{\theta}{2}+\psi\right)} +$$

$$+ \sin\left\{\frac{\theta}{2} + x\left(\frac{\theta}{2}-\psi\right)\right\}.\frac{\sin x\left(\frac{\theta}{2}-\psi\right)}{2\sin\left(\frac{\theta}{2}-\psi\right)}.$$

(41). $\dfrac{1}{\cos\theta.\cos 2\theta} + \dfrac{1}{\cos 2\theta.\cos 3\theta} +$

$$\frac{1}{\cos 3\theta.\cos 4\theta} + \&c.$$

$S =_x \dfrac{\tan(x+1)\theta - \tan\theta}{\sin\theta}$; See (Art. 4. Sect 1.)

(42). $\dfrac{1}{\sin\theta.\cos 2\theta} - \dfrac{1}{\cos 2\theta.\sin 3\theta} +$

$$+ \frac{1}{\sin 3\theta.\cos 4\theta} - \&c.$$

$$S_x = \frac{1}{\cos\theta \,.\, \sin 2(x+1)\theta}\{(-1)^{x+1} - \cos 2(x+1)\theta\} - \frac{1}{\sin\theta}.$$

(43). $$\frac{1}{\sin\theta \,.\, \sin 2\theta} + \frac{1}{\sin 2\theta \,.\, \sin 3\theta} + \frac{1}{\sin 3\theta \,.\, \sin 4\theta} + \&c.$$

$$S_x = \frac{-\operatorname{cotan}(x+1)\theta + \operatorname{cotang}\theta}{\sin\theta}.$$

(44). $$\frac{1}{\cos\theta \,.\, \sin 2\theta} - \frac{1}{\sin 2\theta \,.\, \cos 3\theta} + \frac{1}{\cos 3\theta \,.\, \sin 4\theta} - \&c.$$

$$S_x = \frac{(-1)^{x+1} + \cos 2(x+1)\theta}{\cos\theta \,.\, \sin 2(x+1)\theta} + \frac{\tan\theta}{\cos\theta}.$$

(45). $$\frac{1}{\sin\theta \,.\, \sin 3\theta} - \frac{1}{\sin 2\theta \,.\, \sin 4\theta} + \frac{1}{\sin 3\theta \,.\, \sin 5\theta} - \&c.$$

$$S_x = \frac{1}{2 \,.\, \cos\theta}\left\{\frac{1}{\sin\theta \,.\, \sin 2\theta} + \frac{(-1)^{x+1}}{\sin(x+1)\theta \,.\, \sin(x+2)\theta}\right\}.$$

See (Art. 20. Sect. 1.)

(46). $$\frac{1}{\cos\theta \,.\, \cos 3\theta} - \frac{1}{\cos 2\theta \,.\, \cos 4\theta} + \frac{1}{\cos 3\theta \,.\, \cos 5\theta} - \&c.$$

$$S_x = \frac{1}{2 \,.\, \cos\theta}\left\{\frac{1}{\cos\theta \,.\, \cos 2\theta} + \frac{(-1)^{x+1}}{\cos(x+1)\theta \,.\, \cos(x+2)\theta}\right\}.$$

See Art. 22. Sect. 1.

(47). $\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} + \&c.$

See Art. 14. Sect. 1.

$$S_x = \frac{1}{2^{x-1}}\cot\frac{\theta}{2^{x-1}} - 2\,.\cot 2\,\theta,$$

$$S = \frac{1}{\theta} - 2\,.\cot 2\,\theta.$$ See Lacroix, Translation, Art. 57.

(48). $(\tan\theta)^2 + \frac{1}{2}\left(\tan\frac{\theta}{2}\right)^2 + \left(\frac{1}{4}\tan\frac{\theta}{4}\right)^2 + \&c.$

$$S_x = \frac{8}{3}\left(1 - \frac{1}{2^{2x}}\right) + \frac{4}{(\tan 2\,\theta)^2} - \frac{1}{\left(2^{x-1}\,.\tan\frac{\theta}{2^{x-1}}\right)^2},$$

$$S = \frac{8}{3} + \frac{4}{(\tan 2\,\theta)^2} - \frac{1}{\theta^2},$$

See Art. 15. Sect. 1. Also Lacroix. Translation, Art. 57.

(49). $\tan\theta\,.\,(\sec\theta)^2 + \left(\frac{1}{2}\tan\frac{\theta}{2}\right)\left(\frac{1}{2}\sec\frac{\theta}{2}\right)^2 +$

$$+ \left(\frac{1}{4}\tan\frac{\theta}{4}\right)\left(\frac{1}{4}\sec\frac{\theta}{4}\right)^2 + \&c.$$

$$S_x = \frac{\cos\frac{\theta}{2^{x-1}}}{\left(2^{x-1}\,.\sin\frac{\theta}{2^{x-1}}\right)^3} - 8\,.\frac{\cos 2\,\theta}{(\sin 2\,\theta)^3}.$$

$$S = \frac{1}{\theta^3} - 8\,.\frac{\cos 2\,\theta}{(\cos 2\,\theta)^3};$$ See Art. 16. Sect. 1. and Translation, Art. 57.

(50). $\frac{a-1}{a+1} + 2\,.\frac{a^2-1}{a^2+1} + 4\,.\frac{a^4-1}{a^4+1} + \&c.$

$$S_x = 2^x \cdot \frac{a^{2^x}+1}{a^{2^x}-1} - \frac{a+1}{a-1}; \qquad \text{(Art. 7. Sect. 1.)}$$

(51). $$\frac{a}{a^2-1} + \frac{a^2}{a^4-1} + \frac{a^4}{a^8-1} + \&c.$$

$$S_x = \tfrac{1}{2}\left\{\frac{a+1}{a-1} - \frac{a^{2x}+1}{a^{2^x}-1}\right\}; \qquad \text{(Art. 6. Sect. 1.)}$$

(52). $$\frac{1}{\sin\theta} + \frac{1}{\sin 2\theta} + \frac{1}{\sin 4\theta} + \frac{1}{\sin 8\theta} + \&c.$$

$$S_x = \cot\frac{\theta}{2} - \cot. 2^{x-1}\theta; \qquad \text{(Art. 18. Sect. 1.)}$$

(53). $$\frac{1}{\left(2\,.\cos\frac{\theta}{2}\right)^2} + \frac{1}{\left(4\,.\cos\frac{\theta}{4}\right)^2} + \frac{1}{\left(8\,.\cos\frac{\theta}{8}\right)^2} + \&c.$$

$$S_x = \frac{1}{(\sin\theta)^2} - \frac{1}{\left(2^x\,.\sin\frac{\theta}{2^x}\right)^2}$$

$$S = \frac{1}{(\sin\theta)^2} - \frac{1}{\theta^2}; \qquad \text{(Art. 17. Sect. 1.)}$$

(54). $$\sin\theta\left(\sin\frac{\theta}{2}\right)^2 + 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2 +$$

$$+ 4\sin\frac{\theta}{4}\left(\sin\frac{\theta}{8}\right)^2 + \&c.$$

$$S_x = \frac{1}{4}\left(2^x\,.\sin\frac{\theta}{2^{x-1}} - \sin 2\theta\right)$$

$$S = \frac{\theta}{2} - \frac{\sin 2\theta}{4}. \qquad \text{(See Art. 8. Sect. 8.)}$$

(55). $(\sin\theta)^4 + 4\left(\sin\frac{\theta}{2}\right)^4 + 4^2.\left(\sin\frac{\theta}{4}\right)^4 + \&c.$

$$S_x = \left(2^{x-1}\sin\frac{\theta}{2^{x-1}}\right)^2 - \left(\frac{\sin 2\theta}{2}\right)^2,$$

$$S = \theta^2 - \left(\frac{\sin 2\theta}{2}\right)^2; \quad \text{(Art. 9. Sect. 1.}$$

(56). $\tan\theta\left(\tan\frac{\theta}{2}\right)^2 + 2\tan\frac{\theta}{2}\left(\tan\frac{\theta}{4}\right)^2 +$

$$+ 4\tan\frac{\theta}{4}\left(\tan\frac{\theta}{8}\right)^2 + \&c.$$

$$S_x = \tan\theta - 2^x.\tan\frac{\theta}{2^x},$$

$$S = \tan\theta - \theta. \quad \text{(Art. 11. Sect. 1.)}$$

(57). $\left(\frac{1}{\cos\theta}\right)^2 + \left(\frac{2}{\cos 2\theta}\right)^2 + \left(\frac{4}{\cos 4\theta}\right) +$

$$+ \left(\frac{8}{\cos 8\theta}\right)^2 + \&c.$$

$$S_x = \left(\frac{2^x}{\sin 2^x\theta}\right)^2 - \frac{1}{\sin\theta^2}.$$

(58). To assign the value of the continued product,

$$P_x = \tan\theta\,(\tan 2\theta)^{\frac{1}{2}}\,(\tan 4\theta)^{\frac{1}{4}}\ldots.(\tan 2^x\theta)^{\frac{1}{2^x}}.$$

If we sum the series

$$\log\tan\theta + \tfrac{1}{2}\log\tan 2\theta + \tfrac{1}{4}\log\tan 4\theta + \&c.$$

to $x+1$ terms by the help of (Art. 19. Sect. 1.) and then transform the logarithmic equation into an equation of factors by the well known property

$$a.\log A + b.\log B + \&c. = \log(A^a.B^b.\&c.)$$

we shall find for the value of P_x

$$P_x = \frac{4 . \sin \theta^2}{(2 \sin 2^{x+1} \theta)^{\frac{1}{2^x}}},$$

and if P be the product to infinity, $P = 4 . \sin \theta^2$.

(59). To sum the series

$$\tan^{-1} \frac{1}{1 + 1 + 1^2} + \tan^{-1} \frac{1}{1 + 2 + 2^2}$$

$$+ \tan^{-1} \frac{1}{1 + 3 + 3^2} + \&c.$$

or, as it would stand in the ordinary notation,

$$\text{arc} \left(\tan = \frac{1}{1+1+1^2}\right) + \text{arc} \left(\tan = \frac{1}{1+2+2^2}\right) + \&c.$$

to x terms and to infinity

$$S_x = \frac{\pi}{4} - \tan^{-1} \left(\frac{1}{x+1}\right); \quad S = \frac{\pi}{4}; \quad \text{(Art. 25. Sect. 1.)}$$

(60). $\tan^{-1} \frac{1}{2 . 1^2} + \tan^{-1} \frac{1}{2 . 2^2} + \tan^{-1} \frac{1}{2 . 3^2} + \&c.$

$$S_x = \frac{\pi}{4} - \tan^{-1} \frac{1}{2x + 1}; \quad S = \frac{\pi}{4};$$

Deducible from 26. Sect. 1. by taking $h + x\theta = 2x + 1$.

(61). $\tan^{-1} \frac{1}{7} + 2 . \tan^{-1} \frac{1}{38} + 2^2 . \tan^{-1} \frac{1}{268} + \&c.$

the progression of the denominators being

$$4 + 3, \quad 4 . 8 + 3 . 2, \quad 4 . 8^2 + 3 . 2^2, \ \&c.$$

$$S_x = 2^x . \tan^{-1} \frac{1}{2^x} - \frac{\pi}{4}; \quad S = 1 - \frac{\pi}{4}; \quad \text{(See 29. Sect. 1.)}$$

(62). $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{27} + \ldots\ldots$

$$\ldots\ldots + \tan^{-1}\frac{2}{3\,x+5\,x^2},$$

$$S_x = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{2}{5\,x+4}, \quad S = \tan^{-1}\frac{1}{2}.$$

(63). To determine in what cases the function

$$\tan^{-1}\frac{1}{p + q\,x + r\,x^2}$$

is immediately integrable, and in such cases to sum the series

$$\tan^{-1}\frac{1}{p+q+r} + \tan^{-1}\frac{1}{p+q\,.\,2+r\,.\,2^2} +$$

$$+ \tan^{-1}\frac{1}{p+q\,.\,3+r\,.\,3^2} + \&c.$$

Let the function in question be compared with the second member of the equation in (27. Sect. 1.) and we find

$$a = \frac{1}{\surd(r)}, \quad b = 0, \quad A = -\frac{q-r}{2\surd(r)}, \quad B = -\surd(r),$$

besides which there remains an equation of condition to be satisfied, viz.

$$q^2 - r^2 = 4\,(p\,r - 1);\ldots\ldots(a)$$

Whenever this equation then holds good, we have

$$\Sigma \tan^{-1}\frac{1}{p+q\,x+r\,x^2} = C - \tan^{-1}\left\{\frac{2}{q-r+2\,r\,x}\right\}.$$

The series 59, 60, 62. are all particular cases of this, and their general terms will be found to satisfy the equation (*a*). Other examples are the following.

(64). Let the general or x^{th} term of a series be

$$u_x = \tan^{-1} \frac{1}{10\,x^2 - 24\,x + 12},$$

$$S_x = \tan^{-1}(7) + \tan^{-1}(10\,x - 7); \quad S = \frac{\pi}{2} + \tan^{-1}(7).$$

(65). Let $u_x = \tan^{-1} \dfrac{1}{34\,x^2 - 8\,x - 8}$,

then

$$S_x = \tan^{-1}(34\,x + 13) - \tan^{-1}(13); \quad S = \tan^{-1} \frac{1}{13}.$$

(66). Let $u_x = \tan^{-1} \dfrac{1}{74\,x^2 - 12\,x - 18}$,

then

$$S_x = \tan^{-1}(74\,x + 31) - \tan^{-1}(31); \quad S = \tan^{-1} \frac{1}{31}.$$

(67). Let $u_x = \tan^{-1} \dfrac{1}{26\,x^2 - 16\,x - 4}$,

$$S_x = \tan^{-1}(26\,x + 5) - \tan^{-1}(5); \quad S = \tan^{-1} \frac{1}{5}.$$

The series (47, 48, 49, 50, 53.) are due to Mr. Wallace, who gave them under a somewhat different form, among a variety of similar ones in a paper communicated to the Royal Society of Edinburgh in 1808, as formulæ of approximation to the arc of a circle (when continued to infinity) to which purpose their rapid convergence, even in the most unfavourable cases, well adapts them. In fact we have by (47)

$$\frac{1}{\theta} = (2\,.\cot 2\,\theta + \tan\theta) + \tfrac{1}{2}\tan\tfrac{1}{2}\,\theta + \&c.$$

in which if we notice that $2 \cot 2\theta + \tan\theta = \dfrac{1}{\tan\theta}$ we have the reciprocal of the arc, expressed in the form delivered in the paper alluded to. The series (50) when continued backwards by writing $-x$ for x gives

$$\frac{1}{2} \cdot \frac{a^{\frac{1}{2}} - 1}{a^{\frac{1}{2}} + 1} + \frac{1}{4} \cdot \frac{a^{\frac{1}{4}} - 1}{a^{\frac{1}{4}} + 1} + \ldots\ldots \frac{1}{2^x} \cdot \frac{a^{2^{-x}} - 1}{a^{2^{-x}} + 1} =$$

$$= \frac{a+1}{a-1} - \frac{a^{2^{-x}} + 1}{2^x \left(a^{2^{-x}} - 1\right)},$$

and this sum, when the series is continued to infinity will be found to reduce itself to

$$\frac{a+1}{a-1} - \frac{2}{\log a}.$$

This expression is accordingly given by Mr. Wallace in the same paper, as affording means of computing the logarithm of an insulated number (a high prime for instance), or at least its reciprocal, at once. It is true the operations are laborious on account of the multiplied extractions of roots and decimal divisions they require, but they are not on that account less valuable. Regarded in the light of elegant formulæ in the inverse method of differences, these series assume a higher rank in the scale of analytical estimation, in proportion to the difficulty of that field of research, and the little reason we have to hope for any farther progress in it. For this reason, I have added the series (52, 54, 55, 56, 57.) which are of a similar nature, but have not been noticed by him. Of these (54, 55, and 56.) afford in like manner, formulæ of approximation to the arc of a circle, viz.

$$\theta = \frac{1}{2} \sin 2\theta +$$

$$+ 2 \left\{ \sin\theta \,.\, \left(\sin\frac{\theta}{2}\right)^2 + 2 \sin\frac{\theta}{2} \left(\sin\frac{\theta}{4}\right)^2 + \&c. \right\}$$

$$\theta^2 = \left(\frac{\sin 2\,\theta}{2}\right)^2 +$$

$$+ \left\{(\sin \theta)^4 + 4\left(\sin \frac{\theta}{2}\right)^4 + 4^2\left(\sin \frac{\theta}{4}\right)^4 + \&c.\right\}; \qquad (a).$$

$$\theta = \tan \theta - \left\{\tan \theta \left(\tan \frac{\theta}{2}\right)^2 + 2 \tan \frac{\theta}{2}\left(\tan \frac{\theta}{4}\right)^2 + \&c.\right\}$$

These all converge with the same degree of rapidity after a few of the first terms, viz. nearly according to the powers of $\frac{1}{4}$, but for actual computation, the formula (*a*) far surpasses the rest in convenience. They differ from Mr. Wallace's in giving the immediate values of the arc and its square instead of their reciprocals.

The continued product (58) is due to Mr. Babbage: the summation of the series of reciprocal sines (52) may be obtained from it, by taking the logarithmic differential relative to θ and *vice versâ*, the latter may be derived from the former by integration: The method, however, in which we have here presented them has the advantange of exhibiting the principle on which all transformations of the same kind ultimately depend.

The series (59) and (60) are noticed by Euler in the Comm. Acad. Petropol. ix. 1737. p. 234., as well as by Spence in his Logarithmic Transcendents. By the former they are given as particular instances of a general formula of great neatness, at which he arrives by a kind of tentative method, but which may be obtained very shortly thus:

$$\text{Since } \Delta \tan^{-1} u_x = \tan^{-1}\left(\frac{\Delta u_x}{1 + u_x u_{x+1}}\right); \text{ 28. Sect. 1.}$$

therefore we have

$$\Sigma \tan^{-1} \frac{\Delta u_x}{1 + u_x u_{x+1}} = C + \tan^{-1} u_x.$$

Thus we get for the sum of the series

$$\tan^{-1}\left\{\frac{u_1 - u_0}{1 + u_0 u_1}\right\} + \tan^{-1}\left\{\frac{u_2 - u_1}{1 + u_1 u_2}\right\} + \ldots\ldots$$

$$\ldots\tan^{-1}\left\{\frac{u_x - u_{x-1}}{1 + u_{x-1} u_x}\right\},$$

the following expression

$$\tan^{-1} u_x - \tan^{-1} u_0.$$

Which is in fact Euler's formula, for u_x being a function of x of a form perfectly arbitrary, its particular values u_0, u_1, u_2, ... u_x to any extent we please, may be looked upon as so many arbitrary and independent constants, and may be represented by separate letters, a, b, c, &c., which done, a very trifling reduction will give the formula in question. These series of inverse tangents in which the numerators of the fractions under the characteristic $\tan^{-1}$ are unity, and the denominators integer numbers (as 59, 60, 61, 62, 64, 65, 66, 67.) are extremely remarkable on account of the facilities they afford for extending the integer evaluation of the function $\tan^{-1}(x)$ or as Spence denotes it ${}^1C(x)$. (64. . . . 7) have not, I believe, been noticed, nor has (61) which is not included in Euler's general formula, but may be derived without difficulty by a method similar to that by which that formula was originally obtained, from the following equation

$$\tan^{-1}\frac{1}{n} + \tan^{-1}\frac{1}{4n^3+3n} = 2\tan^{-1}\frac{1}{2n}; \qquad (a).$$

which the reader will have no difficulty in verifying and which is analogous to a theorem of Borda for calculating the logarithm of a number, by means of three preceding logarithms and a series. The same remark applies to this class of series as to the rest: they are properly and naturally examples of the application of the inverse method of differences, however they may have been originally obtained, and it may not be amiss to shew how any equation such as (a)

expressing a relation between three values of a particular function may become the origin of a similar series.

(68). To sum the series

$$\tan^{-1}\left(\frac{1}{4\,n^3+3\,.\,n}\right) + 2\,.\tan^{-1}\left(\frac{1}{32\,n^3+6\,n}\right) +$$

$$+ 4\,.\tan^{-1}\left(\frac{1}{256\,n^3+12\,n}\right) \text{ \&c.}$$

The $(x+1)^{\text{th}}$ term is

$$2^x\,.\tan^{-1}\left\{\frac{1}{4\,(2^x\,n)^3 + 3\,.\,2^x\,n}\right\} = u_{x+1}.$$

Now in (*a*), let $2^x n$ be written for n, and the whole multiplied by 2^x, and we get

$$2^x\,.\tan^{-1}\left\{\frac{1}{4\,(2^x\,n)^3 + 3\,.\,2^x\,n}\right\} =$$

$$= 2^{x+1}\tan^{-1}\frac{1}{2^{x+1}\,n} - 2^x\tan^{-1}\frac{1}{2^x\,n}.$$

The second member of this equation is evidently the complete difference of $2^x\tan^{-1}\dfrac{1}{2^x\,n}$, so that integrating both members,

$$\Sigma\,u_{x+1} = C + 2^x\,.\tan^{-1}\frac{1}{2^x\,n}$$

and if the constant be determined as usual we find

$$S_x = 2^x\,.\tan^{-1}\frac{1}{2^x\,n} - \tan^{-1}\frac{1}{n}, \quad S = \frac{1}{n} - \tan^{-1}\frac{1}{n}.$$

(69). Let $f^0(n) = n$, $f(n) = 4\,n^3 + 3\,n$,

$$f^2(n) = 4f(n)^3 + 3\,.f(n), \text{ \&c.}$$

Required the sum of the series

$$\tan^{-1}\frac{1}{2f^0(n)} - \tan^{-1}\frac{1}{2f(n)} + \tan^{-1}\frac{1}{2f^2(n)} - \&c.$$

The $(x+1)^{\text{th}}$ term, or u_{x+1} is

$$u_{x+1} = (-1)^x . \tan^{-1}\frac{1}{2f^x(n)}.$$

Now if in the equation (a) we substitute $f^x(n)$ for n and multiply the whole by $(-1)^x$ we obtain

$$(-1)^x . \tan^{-1}\frac{1}{f^{x+1}(n)} + (-1)^x . \tan^{-1}\frac{1}{f^x(n)} =$$

$$= 2(-1)^x . \tan^{-1}\frac{1}{2f^x(n)}.$$

The first member of this equation is the exact difference of

$$(-1)^{x+1} . \tan^{-1}\frac{1}{f^x(n)}$$

because the two terms of which it consists are the successive values of this function due to the variation of x, with contrary signs, and that without any regard to the form of the function $f^x(n)$ considered as a function of n; the second member is equal to $2\,u_{x+1}$; Hence

$$2\,u_{x+1} = \Delta . (-1)^{x+1}\tan^{-1}\frac{1}{f^x(n)}$$

$$\Sigma\,u_{x+1} = \frac{(-1)^{x+1}}{2}\tan^{-1}\frac{1}{f^x(n)} + C,$$

whence we obtain for the sum of the series

$$S_x = \tfrac{1}{2}\left\{\tan^{-1}\frac{1}{n} + (-1)^{x+1}\tan^{-1}\frac{1}{f^x(n)}\right\};$$

$$S = \tfrac{1}{2}\tan^{-1}\frac{1}{n},$$

for, whatever be the value of n, the quantities n, $f(n)$, $f^2(n)$, $f^3(n)$, &c. form an increasing series which diverges with extreme rapidity. Thus, if $n = 1$, these successive values are

$$1,\ 7,\ 1393,\ 10812186007,\ \&c.$$

If n then be any integer number, $\frac{1}{f^3(n)}$ and the following values may even be altogether disregarded, in a numerical point of view. If we have detained the reader too long on this point, its close connexion with the quadrature of the circle, will induce him to pardon the digression. We will now resume the subject.

(70). The series 1, 5, 17, 53, 161, 485, &c. is a recurring one.—What are—its scale of relation,—its general term, and its sum to x terms?

The scale is............

The general term is......

The sum is $3^x - x - 1$.

(71). Which of the two series

$$1,\ 0,\ 3,\ 2,\ 6,\ 11,\ 23,\ 49,\ 223,\ \&c.$$
$$1,\ 0,\ 3,\ 2,\ 5,\ 10,\ 24,\ 51,\ 247,\ \&c.$$

is a recurring one, and what is its scale of relation?

(72). To shew that

$$\sin\theta,\ \sin 2\theta,\ \sin 3\theta, \ldots\ldots \sin x\theta,$$

and $\cos\theta,\ \cos 2\theta,\ \cos 3\theta, \ldots\ldots \cos x\theta,$

form two recurring series, and to find their scales of relation.

The reader will remember in order to prove this, that the character of a recurring series consists in the possibility of expressing any term by one or more of the preceding terms, multiplied by invariable quantities.

SECTION VII.

Problems and Theorems relating to the developement of exponential Functions, and the properties of the numbers comprised in the form $\Delta^m o^n$.

THE equation

$$\Delta^n u_x = (e^{\frac{dx}{d}} - 1)^n u_x$$

discovered by Lagrange (See Appendix, Art. 358.) and the yet more general theorem of Arbogast demonstrated in the following article, render it desirable to possess some general formula, to facilitate the developement of these and similar expressions. We have already seen some of the uses to which the numbers comprised in the form $\Delta^m o^n$ are applicable in the theory of series. In what follows we shall lay before the reader a connected view of their properties, which bear directly upon the point in question, and afford an easy and general solution of the difficulty. But their application is by no means confined to this, and before we quit this subject we shall point out their use in one or two other instances where they may be introduced with advantage.

(1). PROBLEM. To develope $f(e^t)$ any function whatever of e^t, in a series of the powers of t, or to determine A_0, A_1, A_2, &c, in the following equation,

$$f(e^t) = A_0 + A_1 t + A_2 t^2 + \ldots\ldots A_x t^x + \&c.$$

Let $f(1)$, $f'(1)$, $f''(1)$, &c. denote the values assumed by $f(x)$, $\frac{df(x)}{dx}$, $\frac{d^2 f(x)}{dx^2}$, &c. the several differential coefficients or derived functions of $f(x)$ when x becomes unity. Then by Taylor's theorem we shall have (h being any quantity)

$$f(1+h) = f(1) + \frac{f'(1)}{1} h + \frac{f''(1)}{1 \,.\, 2} h^2 + \&c.$$

Since this is true whatever be the value of h, suppose it equal to $e^t - 1$, and the above equation becomes

$$f(e^t) = f(1) + \frac{f'(1)}{1} (e^t - 1) + \frac{f''(1)}{1 \,.\, 2} (e^t - 1)^2 + \&c.$$

The coefficient of t^x therefore in the first member $f(e^t)$ is equal to the sum of its several coefficients in the terms of the second member. Now, the coefficient of t^x in $f(1)$ is $f(1) \times 0^x$ being $f(1)$ when $x = o$ and zero in all other cases.

In $\frac{f'(1)}{1}(e^t - 1)$ it is $\frac{f'(1)}{1} \,.\, \frac{1^x - 0^x}{1 \,.\, 2 \dots x} =$

$$= \frac{f'(1)}{1} \,.\, \frac{\Delta o^x}{1 \dots x}.$$

In $\frac{f''(1)}{1 \,.\, 2}(e^t - 1)^2$, or $\frac{f''(1)}{1 \,.\, 2}(e^{2t} - 2e^t + 1)$ it is

$$\frac{f''(1)}{1 \,.\, 2} \times \frac{2^x - 2 \,.\, 1^x + 0^x}{1 \dots x} = \frac{f''(1)}{1 \,.\, 2} \,.\, \frac{\Delta^2 o^x}{1 \dots x}$$

and so on; (as is evident if we consider that the developement of e^{nt} in general is

$$1 + \frac{n}{1} t + \frac{n^2}{1 \,.\, 2} t^2 + \dots \frac{n^x}{1 \dots x} t^x + \&c.)$$

Let these be collected together, and we find for the value of A_x or the coefficient in $f(e^t)$

$$A_x = \frac{1}{1 \,.\, 2 \dots x} \left\{ f(1) \,.\, o^x + \frac{f'(1)}{1} \Delta o^x + \frac{f''(1)}{1 \,.\, 2} \Delta^2 o^x + \&c. \right\}$$

Now let the symbols of operation be separated from those of quantity, and we get

$$A_x = \frac{1}{1\,.\,2\ldots\ldots x}\left\{f(1) + \frac{f'(1)}{1}\Delta + \frac{f''(1)}{1\,.\,2}\Delta^2 + \&c.\right\}o^x$$
$$= \frac{f(1+\Delta)\,o^x}{1\,.\,2\ldots\ldots x}.$$

$f(1+\Delta)\,o^x$ being understood (as in all similar cases) to have no other meaning than its developement, of which it is a mere abbreviated expression, each power of Δ being understood to be separately affixed *as if* by multiplication to the 0^x which follows. Hence this general

THEOREM.

$$f(e^t) = f(1) + \frac{t}{1}f(1+\Delta)\,o + \frac{t^2}{1\,.\,2}f(1+\Delta)\,o^2 + \&c.$$

which will be found to comprise all the properties of the numbers $\Delta^m\,o^n$ we shall have occasion to employ.

(2). Corollary. Hence, if by any of the usual methods the developement of $f(e^t)$ be obtained, or the value of A_x assigned, that of $f(1+\Delta)\,o^x$ may be obtained in functions of x and *vice versâ*, for we have

$$A_x = \frac{f(1+\Delta)\,o^x}{1\ldots\ldots x}$$

and,

$$f(1+\Delta)\,o^x = 1\,.\,2\ldots\ldots x\,.\,A_x$$

For example, we shall have,

(3). $(1+\Delta)\,o^x = 1^x, \quad (1+\Delta)^2\,o^x = 2^x, \ldots\ldots (1+\Delta)^n\,o^x = n^x,$

whatever value we assign to n, whether positive, negative, integral, fractional, or even imaginary. For,

let $f(1+\Delta) = (1+\Delta)^n$, then $f(e^t) = e^{nt}$, or since

$$e^{nt} = 1 + \frac{n}{1}t + \ldots\ldots \frac{n^x}{1\ldots\ldots x}t^x + \&c.$$

we have

$$1 \,.\, 2 \ldots . x \,.\, A_x = n^x$$

the value of $f(1 + \Delta)\, o^x$ to be found.

Suppose for instance $n = -1$, and we have

$$\frac{1}{1+\Delta}\, o^x = (-1)^x.$$

Now the first member of this (by the definition) has no other meaning than

$$\{ 1 - \Delta + \Delta^2 - \&c. \}\, o^x$$

or,

$$o^x - \Delta\, o^x + \Delta^2\, o^x - \&c.$$

But in this *(as in all other such series) we may omit all the terms after* $\Delta^x o^x$, *they being each separately zero, by the property of these numbers,* (Appendix, Art. 350.) so that we get,

$$o^x - \Delta\, o^x + \Delta^2\, o^x \ldots . \pm \Delta^x\, o^x = (-1)^x,$$

or, merely reversing the order of writing it,

$$\Delta^x\, o^x - \Delta^{x-1} o^x + \ldots . \pm \Delta\, o^x \mp o^x = 1,$$

whatever be the value of x. It may not be amiss to verify this result by a numerical example, suppose for instance $x = 5$, and taking the values of $\Delta^5 o^5$, $\Delta^4 o^5$, &c. from the Table (Ex. 33. Sect. 1.) we have

$$120 - 240 + 150 - 30 + 1 = 1.$$

(4.) Again, we may prove in like manner, that

$$\{ \log(1 + \Delta) \}^n\, o^x = 0,$$

unless $n = x$, in which case its value is $1 \,.\, 2 \ldots . n$. For since $(\log e^t)^n = t^n$, the coefficient of t^x in the developement of this function (regarded as a function of e^t) is zero unless $x = n$, in which case we have A_x or $A_n = 1$, and $1 \,.\, 2 \ldots . n\, A_n = 1 \,.\, 2 \ldots . n$.

(5). Thus taking $n=1$, we have

$$\frac{\Delta\, 0^x}{1} - \frac{\Delta^2\, 0^x}{2} + \ldots \pm \frac{\Delta^x\, 0^x}{x} = 0,$$

for every value of x greater than unity, which may in like manner be verified by numerical substitution.

(6). $$\{\log(1+\Delta)\}^n . f(\Delta)\, 0^x = x(x-1) \ldots\ldots$$
$$\ldots (x-n+1) f(\Delta)\, 0^{x-n}.$$

The coefficient of t^x in the developement of $t^n . f(e^t - 1)$ is evidently the same with that of t^{x-n} in that of $f(e^t - 1)$. The former coefficient is

$$\frac{\{\log(1+\Delta)\}^n f(\Delta)\, 0^x}{1.2\ldots x},$$

and the latter, $\dfrac{f(\Delta)\, 0^{x-n}}{1.2\ldots(x-n)}$,

which being equated, the proposition is apparent.

(7). $$\Delta^{-n}\, 0^x = \frac{1.2\ldots x}{1.2\ldots(x+n)} \left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n 0^{x+n}.$$

An immediate consequence of the foregoing, changing only x into $x+n$, and making $f(\Delta) = \Delta^{-n}$. This equation enables us to continue the series,

$$\Delta^{\circ}\, 0^x, \quad \Delta\, 0^x, \quad \Delta^2\, 0^x, \quad \&c.$$

backwards, to any extent, according to one uniform law, though it must rather be regarded as a definition of $\Delta^{-n} 0^x$ than in any other light, since the value of that expression (or its equivalent $\Sigma^n\, 0^x$) is not fixed by assigning only the superior limit (0) of the integral.

(8). Prop. To shew that whatever be the value of n,

$$f\{(1+\Delta)^n\}\, 0^x = n^x . f(1+\Delta)\, 0^x.$$

Take the identical equation

$$f\{(e^t)^n\} = f(e^{nt}).$$

The coefficient of t^x in the first member of this equation (regarded as a function of e^t) is by (1. Sect. 7.)

$$\frac{f\{1+\Delta)^n\}\, o^x}{1 . 2 \ldots . x}$$

but, in the second it is evidently equal to $n^x \times$ into that of the same power of t in $f(e^t)$, or to

$$n^x . \frac{f(1+\Delta)\, o^x}{1 . 2 \ldots . x}.$$

Equating these the transformation in question results, which is often of great use in eluding very troublesome developements. Thus for instance.

(9).
$$\{1+\sqrt{(1+\Delta)}\}^m o^x = \frac{1}{2^x} . (2+\Delta)^m o^x,$$

$$\{(1+\Delta)^n - 1\}^m o^x = n^x . \Delta^m o^x,$$

$$\{1+(1+\Delta)^n\}^m o^x = n^x . (2+\Delta)^m o^x.$$

(10). To prove the following very general properties of the numbers comprised in the form $\Delta^n o^m$,

$$\left\{f(1+\Delta) + f\left(\frac{1}{1+\Delta}\right)\right\} o^{2x-1} = 0 ; \ldots . (a),$$

$$\left\{f(1+\Delta) - f\left(\frac{1}{1+\Delta}\right)\right\} o^{2x} = 0 ; \ldots . (b).$$

whatever be the form of the function denoted by f.

Suppose

$$f(e^t) = A_0 + A_1 t + A_2 t^2 + \&c.$$

then will $$f(e^{-t}) = A_0 - A_1 t + A_2 t^2 - \&c.$$

Hence, it is evident that their sum $f(e^t)+f(e^{-t})$ contains no odd powers of x, and their difference $f(e^t)-f(e^{-t})$ no even ones. The coefficients therefore of t^{2x-1} in the developement of the former, and of t^{2x} in that of the latter expression are respectively zero. Now these expressions, put under the form $f(e^t) \pm f\left(\frac{1}{e^t}\right)$ and regarded as functions of e^t give, by applying the general theorem (Ex. 1. Sect. 7.), for the aforesaid coefficients, the first members respectively of the equations (*a*) and (*b*), whence the truth of the proposition is apparent. It may also be derived from (8) by making $n=-1$. Many particular cases of these theorems assume a very remarkable form, thus:

(11). If we take $f(1+\Delta) = \dfrac{1}{1+(1+\Delta)}$,

we have by (*b*)

$$\frac{\Delta}{2+\Delta}\,0^{2x}=0,$$

or

$$\frac{\Delta\, 0^{2x}}{2} - \frac{\Delta^2\, 0^{2x}}{2^2} + \frac{\Delta^3\, 0^{2x}}{2^3} \dots \pm \frac{\Delta^{2x}\, 0^{2x}}{2^{2x}} = 0.$$

(12). If we suppose $f(1+\Delta) = \{1-(1+\Delta)\}^n$, we have $f\left(\dfrac{1}{1+\Delta}\right) = \left\{1-\dfrac{1}{1+\Delta}\right\}^n = \left(\dfrac{\Delta}{1+\Delta}\right)^n$, and the above theorem gives the two equations

$$\left(\frac{\Delta}{1+\Delta}\right)^n 0^{2x-1} = (-1)^{n+1}\,\Delta^n\, 0^{2x-1},$$

and

$$\left(\frac{\Delta}{1+\Delta}\right)^n 0^{2x} = (-1)^n\,\Delta^n\, 0^{2x},$$

both which may be included in one, by writing it as follows:

$$\left(\frac{\Delta}{1+\Delta}\right)^n 0^x = (-1)^{n+x}\,\Delta^n\,0^x.$$

(13). Let us take a transcendental form of f, and suppose

$$f(1+\Delta) = \int \frac{d\,\Delta}{\Delta} \log(1+\Delta),$$

then we have it demonstrated (in Note *N*, p. 683 to Lacroix, Engl. Transl.) that

$$f(1+\Delta) + f\left(\frac{1}{1+\Delta}\right) = \tfrac{1}{2}\log(1+\Delta)^2,$$

but by (4) it appears that $\log(1+\Delta)^2\,0^{2x}=0$, unless $2x=2$, or $x=1$, therefore (this case excepted)

$$f\left(\frac{1}{1+\Delta}\right)0^{2x} = -f(1+\Delta)\,0^{2x},$$

which substituted in (*b*) of (10) gives

$$f(1+\Delta)\,0^{2x} = 0.$$

Now the form of $f(1+\Delta)$ in this instance being the transcendent,

$$\int \frac{d\,\Delta}{\Delta}\log(1+\Delta) = \frac{\Delta}{1^2} - \frac{\Delta^2}{2^2} + \&c.$$

our equation becomes

$$\frac{\Delta\,0^{2x}}{1^2} - \frac{\Delta^2\,0^{2x}}{2^2} + \ldots \pm \frac{\Delta^{2x}\,0^{2x}}{(2x)^2} = 0,$$

which will be found verified in every case ($x=1$ excepted) by actual substitution of the numerical values given in the table. To such an extent may the separation of the symbols of operation from those of quantity be carried, without the possibility of error or misconception. What value the first member of the above equation assumes, when the exponent of 0 is odd, will be seen hereafter (22. Sect. 8.)

(14). THEOREM. Let $f_1(\Delta)$ and $f_2(\Delta)$ be any two functions of Δ, then will the following equation hold good

$$\{f_1(\Delta).f_2(\Delta)\}\, 0^x = f_1(\Delta).f_2(\Delta')\{0+0'\}^x,$$

where in the second member, the unaccented Δ is to be referred to the unaccented powers of 0, and the accented to the accented powers.

Let

$$f_1(e^t-1) = A_0 + A_1 t + A_2 t^2 + \&c.$$

$$f_2(e^t-1) = a_0 + a_1 t + a_2 t^2 + \&c.$$

then will

$$f_1(e^t-1)\times f_2(e^t-1) =$$

$$A_0 a_0 + \{A_1 a_0 + A_0 a_1\}\, t + \{A_2 a_0 + A_1 a_1 + A_0 a_2\}\, t^2 + \&c.$$

The coefficient of t^x in the first member of this, by the general theorem in (1. Sect. 7.) is represented by

$$\frac{\{f_1(1+\Delta-1)\times f_2(1+\Delta-1)\}\, 0^x}{1\ldots\ldots x} =$$

$$\frac{\{f_1(\Delta)\times f_2(\Delta)\}\, 0^x}{1\ldots\ldots x}, \qquad (a).$$

while in the second it is

$$A_x a_0 + A_{x-1} a_1 + \ldots . A_0 a_x, \qquad (b).$$

But, because A_x and a_x are the coefficients of t^x in the respective developements $f_1(e^t-1)$ and $f_2(e^t-1)$ we have by the same general theorem

$$A_x = \frac{f_1(\Delta)\, 0^x}{1\ldots\ldots x}, \quad a_x = \frac{f_2(\Delta)\, 0^x}{1\ldots\ldots x},$$

hence we find, $A_{x-1} = \dfrac{f_1(\Delta)\, 0^{x-1}}{1\ldots .(x-1)}$, &c.

$$a_0 = f_2(\Delta)\, 0^0, \quad a_1 = \frac{f_2(\Delta)\, 0^1}{1}, \ \&c.$$

and by substitution, the expression (b) becomes

$$\frac{1}{1\ldots x}\left\{f_1(\Delta)\,0^x . f_2(\Delta)\,0^0 + \frac{x}{1} f_1(\Delta)\,0^{x-1} . f_2(\Delta)\,0^1 + \right.$$

$$\left. + \frac{x(x-1)}{1.2} f_1(\Delta)\,0^{x-2} . f_2(\Delta)\,0^2 + \&c.\ldots\ldots\right\}$$

Now let the symbols of operation be separated from those of quantity, keeping the powers of 0 distinct from each other, by the system of accentuation explained in Appendix, Art. 355, and it becomes

$$\frac{1}{1\ldots x} f_1(\Delta) f_2(\Delta')\{0+0'\}^x,$$

which compared with the expression (a) renders the proposition evident.

(15). By a process precisely similar, we may prove in general that

$$\{f_1(\Delta) \times f_2(\Delta) \times f_3(\Delta) \times \&c.\}\,0^x =$$

$$f_1(\Delta) . f_2(\Delta') . \&c.\{0+0'+0''+\&c.\}^x.$$

(16). Hence also we may shew that whatever be the value of n,

$$\{(1+\Delta)^n f(\Delta)\}\,0^x = f(\Delta)\{n+0\}^x,$$

$(n+0)^x$ being developed in powers of 0 as a mere algebraic symbol, and $f(\Delta)$ being then applied to each separate power so produced. For, if in the foregoing proposition (14) we write $(1+\Delta)^n$ for $f_1(\Delta)$ and $f(\Delta)$ for $f_2(\Delta)$ we see that

$$(1+\Delta)^n f(\Delta)\,0^x =$$

$$(1+\Delta)^n 0^x . f(\Delta)\,0^0 + \frac{x}{1}(1+\Delta)^n\,0^{x-1} . f(\Delta)\,0^1 + \&c.$$

Now by 3, Sect. 7.) it appears that

$$(1+\Delta)^n\,0^x = n^x, \quad (1+\Delta)^n\,0^{x-1} = n^{x-1}, \&c.$$

so that the second member becomes

$$n^x . f(\Delta) o^0 + \frac{x}{1} n^{x-1} . f(\Delta) o^1 + \&c.$$

which separating the symbols of operation from those of quantity, takes the form

$$f(\Delta) \left\{ n^x o^0 + \frac{x}{1} n^{x-1} o^1 + \frac{x(x-1)}{1 . 2} n^{x-2} o^2 + \&c. \right\}$$
$$= f(\Delta) \{ n + o \}^x.$$

(17). As a particular instance of the application of this let $f(\Delta) = \Delta^m$ and let $n = -1$, and we shall get

$$\Delta^m (-1 + o)^x = \frac{\Delta^m}{1 + \Delta} o^x,$$

or since $\Delta^m (-1 + o)^x = (-1)^x . \Delta^m (1 - o)^x$,

$$\Delta^m (1 - o)^x = (-1)^x . \frac{\Delta^m}{1 + \Delta} o^x$$
$$= (-1)^x . \{ \Delta^m o^x - \Delta^{m+1} o^x + \ldots . \pm \Delta^x o^x \}.$$

This is the transformation we have already had occasion to employ in (25. Sect. 1.), and that made use of in (27. Sect. 1.) may be derived precisely in the same manner; for if instead of putting $n = -1$, in the theorem in (14), we put $n = -a$, still supposing $f(\Delta) = \Delta^m$, it becomes

$$\Delta^m (-a + o)^x = \frac{\Delta^m}{(1 + \Delta)^a} o^x,$$

whence,

$$\Delta^m (a - o)^x = (-1)^x . \frac{\Delta^m}{(1 + \Delta)^a} o^x.$$

and the use of these transformations in simplifying pretty complicated expressions, and reducing them to a manageable and even elegant form, is in the instances alluded to (and expecially the latter) by no means contemptible. If we make

$m=0$, the terms of $\Delta^m (a-o)^x$ after the first $(\Delta^0 a^x \,.\, o^0 = a^x \,.\, \Delta^0 o^0 = a^x)$ all vanish, and we have simply

$$a^x = (-1)^x \,.\, \frac{1}{(1+\Delta)^a}\, o^x$$

as we there asserted. Other uses of the transformations in this and the last number will shortly appear.

(18). THEOREM. $f(\Delta)\{(a+o)^x \,.\, (b+o)^y \,.\, (c+o)^z \,.\, \&c.\} =$

$$= f(\Delta) \,.\, (1+\Delta')^a \,.\, (1+\Delta'')^b \,.\, \&c.$$

$$\{ (o+o')^x \,.\, (o+o'')^y \,.\, (o+o''')^z \,.\, \&c. \}$$

The Δ's and their powers being referred by the accents over them to the powers of 0, affected with the same number of accents.

To demonstrate it, we have

$$f(\Delta)\,(a+o)^x \,.\, (b+o)^y =$$

$$a^x f(\Delta)\,(b+o)^y + \frac{x}{1}(a^{x-1} f(\Delta)\, o\,(b+o)^y + \&c.$$

$$= a^x \,.\, \Big\{ b^x f(\Delta)\, o^0 + \frac{y}{1} b^{y-1} f(\Delta)\, o^1 +$$

$$\frac{y\,(y-1)}{1\,.\,2}\, b^{y-2} f(\Delta)\, o^2 + \&c. \Big\}$$

$$+ \frac{x}{1}\, a^{x-1} \Big\{ b^y f(\Delta)\, o^1 + \frac{y}{1} b^{y-1} f(\Delta)\, o^2 + \&c. \Big\}$$

$$+ \frac{x\,(x-1)}{1\,.\,2}\, a^{x-2} \times \&c. + \&c.$$

Now we have, $b^y = (1+\Delta)^b o^y$; $b^{y-1} = (1+\Delta)^b o^{y-1}$; &c. by (3. Sect. 7.) and substituting these values in the above expression it becomes

$$a^x \Big\{ (1+\Delta)^b o^y \,.\, f(\Delta)\, o^0 + \frac{y}{1}(1+\Delta)^b o^{y-1} \,.\, f(\Delta)\, o^1 + \&c. \Big\}$$

$$+ \frac{x}{1} a^{x-1} \left\{ (1+\Delta)^b o^y \cdot f(\Delta) o^1 + \frac{y}{1} (1+\Delta)^b o^{y-1} f(\Delta) o^2 + \&c. \right\}$$

$$+ \frac{x(x-1)}{1 \cdot 2} \cdot \&c.$$

In this we may now separate the symbols of operation from those of quantity, by employing the system of accentuation, and we shall have for the value of our expression,

$$a^x \cdot f(\Delta) \cdot (1+\Delta'')^b \left\{ o''^y \cdot o^0 + \frac{y}{1} o''^{y-1} o^1 + \&c. \right\}$$

$$+ \frac{x}{1} a^{x-1} \cdot f(\Delta) \cdot (1+\Delta'')^b \left\{ o''^y \cdot o^1 + \frac{y}{1} o''^{y-1} o^2 + \&c. \right\} + \&c.$$

The series within the brackets have for their abbreviations respectively, $o^0 (o + o'')^y$, $o^1 (o + o'')^y$, $o^2 (o + o'')^y$, &c. and writing these in their places, and at the same time replacing a^x, a^{x-1}, &c. by their values (given by 3. Sect. 7.)

$$a^x = (1+\Delta')^a o'^x ; \; a^{x-1} = (1+\Delta')^a o'^{x-1}, \&c.$$

our formula once more transformed will be

$$f(\Delta) \cdot (1+\Delta'')^b \{ o^0 (o + o'')^y \} \cdot (1+\Delta')^a o'^x +$$

$$+ \frac{x}{1} \cdot f(\Delta) (1+\Delta'')^b \{ o^1 (o + o'')^y \} \cdot (1+\Delta')^a o'^{x-1} + \&c.$$

and again, finally separating the symbols of operation from those of quantity, it will become

$$f \cdot (\Delta) (1+\Delta')^a (1+\Delta'')^b \left\{ (o + o'')^y \left(o'^x o^0 + \frac{x}{1} o'^{x-1} o^1 + \&c. \right) \right\}$$

$$= f(\Delta) (1+\Delta')^a (1+\Delta'')^b \{ (o + o')^x \cdot (o + o'')^y \},$$

and by a similar train of operations the theorem in question may be proved, to the full extent of its enunciation.

SECTION VIII.

Application of the foregoing Theorems to the developement of particular Functions, the Summation of Series, &c.

(1). To develope $\frac{1}{1+e^t}$ in powers of t.

$$f(e^t) = \frac{1}{1+e^t} = A_0 + A_1 t + A_2 t^2 + \&c.$$

$$f(1+\Delta)\, 0^x = \frac{1}{1+(1+\Delta)}\, 0^x = \frac{1}{2+\Delta}\, 0^x.$$

Therefore, by Ex. 2. Sect. 7.

$$A_x = \frac{1}{1\,.\,2\,\ldots\ldots\,x} \times \frac{1}{2+\Delta}\, 0^x,$$

$$= \frac{1}{1\,.\,2\,\ldots\ldots\,x} \left\{ \frac{0^x}{2} - \frac{\Delta\, 0^x}{2^2} + \ldots\ldots \pm \frac{\Delta^x\, 0^x}{2^{x+1}} \right\}.$$

Thus we find $A_0 = 1\,.\,\frac{0^0}{2} = \frac{1}{2}$, $A_1 = \frac{1}{1}\left(0 - \frac{1}{4}\right) = -\frac{1}{4}$,

$A_2 = 0$, and so on, so that

$$\frac{1}{1+e^t} = \frac{1}{2} - \frac{t}{4} + t^2\,.\,\&c.$$

This is the great advantage resulting from the employment of these functions: any series of them such as

$$a\,.\,0^x + b\,\Delta\, 0^x + c\,\Delta^2\, 0^x + \&c.$$

however complicated, necessarily breaks off in a limited number $(x+1)$ of terms, and thus enables us to assign in a comparatively simple form, the general terms of an unlimited variety of developements, which would otherwise be scarcely expressible without having recourse to the combinatory analysis, which ought never, in my opinion, to be employed, till every artifice of abbreviation, and every refinement of analysis has been found unavailing.

(2). To develope $(e^t-1)^n$ in powers of t,

$$f(e^t)=(e^t-1)^n;\quad f(1+\Delta)\,o^x=(1+\Delta-1)^n\,o^x=\Delta^n\,o^x;$$

$$(e^t-1)^n=\Delta^n\,o^0+\frac{\Delta^n\,o^1}{1}\,t+\frac{\Delta^n\,o^2}{1\,.\,2}\,t^2+\&c.$$

$$=\frac{\Delta^n\,o^n}{1\,.\,.\,.\,.\,n}\,t^n+\frac{\Delta^n\,o^{n+1}}{1\,.2\,.\,.\,.(n+1)}\,t^{n+1}+\&c.$$

This elegant expression was originally given by Mr. Ivory (Leybourne's Repository, 1804. Quest. 60.) and afterwards by Dr. Brinkley, Phil. Trans. 1807. i. Both these Geometers arrive at it, however, by a different method from that above pursued. The former mentions it only incidentally, nor does the latter, who pursued the subject much farther, seem to have perceived the *system* of which it, and several more of the truly beautiful theorems he has given in that paper, form a part. To him, however, belongs the merit of introducing the numbers comprised in the form $\Delta^m\,o^n$ among the *data* of analysis, as objects of ultimate reference. I ought too, to notice that the developement of $\frac{1}{1+e^t}$ in the form above given (1. Sect. 8.) is also to be found for the first time in his paper.

(3). To develope $\frac{t}{e^t-1}$ in powers of t,

$$\frac{t}{e^t-1}=\frac{\log\,(e^t)}{e^t-1}=f(e^t);$$

$$f\,(1+\Delta)o^x=\frac{\log\,(1+\Delta)}{(1+\Delta)-1}\,o^x=\left\{\frac{o^x}{1}-\frac{\Delta\,o^x}{2}+\,.\,.\,.\,.\,\pm\frac{\Delta^x\,o^x}{x+1}\right\},$$

and consequently the coefficient of t^x, or

$$A_x = \frac{1}{1\,.\,2\ldots\ldots x}\left\{\frac{0^x}{1} - \frac{\Delta\, 0^x}{2} + \frac{\Delta^2\, 0^x}{3} \ldots\ldots \pm \frac{\Delta^x\, 0^x}{x+1}\right\}.$$

This expression for the coefficient of t^x is given in the Phil. Trans. 1815. in a paper, "On the developement of Exponential Functions."

It has been already shewn (Appendix, Art. 408.) that the odd values of A_x in the developement of $\frac{t}{e^t - 1}$ (A_1 excepted) all vanish, hence we see that the following equation must hold good for every value of x except unity,

$$\frac{\Delta\, 0^{2x-1}}{2} - \frac{\Delta^2\, 0^{2x-1}}{3} + \ldots\ldots + \frac{\Delta^{2x-1}\, 0^{2x-1}}{2x} = 0.$$

From the same article, it also appears that the coefficient of t^{2x} in the same developement is

$$(-1)^{x+1}\,.\,\frac{B_{2x-1}}{1\,.\,2\ldots\ldots 2x},$$

which compared with $\frac{f(1+\Delta)\, 0^{2x}}{1\,.\,2\ldots\ldots 2x}$, gives the following very simple expression for the numbers of Bernouilli,

$$B_{2x-1} = (-1)^{x+1}\,.\,f(1+\Delta)\, 0^{2x}$$

$$= (-1)^{x+1}\,.\,\frac{\log\,(1+\Delta)}{\Delta}\, 0^{2x}$$

or, $$B_{2x-1} = (-1)^{x+1}\left\{\frac{0^{2x}}{1} - \frac{\Delta\, 0^{2x}}{2} \ldots\ldots \pm \frac{\Delta^{2x}\, 0^{2x}}{2x+1}\right\}.$$

Thus we may calculate the numerical values of these numbers, with a degree of facility far surpassing that afforded by the expression demonstrated in that article, for instance we have

$$B_1 = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}; \quad B_3 = +\frac{1}{2} - \frac{14}{3} + \frac{36}{4} - \frac{24}{5} = \frac{1}{30},$$

and so on.

(4). To develope $\left(\frac{t}{e^t - 1}\right)^n$ in powers of t.

This function being equal to $\left(\frac{\log(e^t)}{e^t - 1}\right)^n$, the coefficient of t^x will be by (Ex. 1. Sect. 7.)

$$\frac{1}{1.2....x} \times \left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n o^x.$$

For the developement of this function into a formula adapted to numerical computation, the reader is referred to the paper "*on the Developement of Exponential Functions*" above cited. Or he will find the original function completely developed in a most elegant series, in Dr. Brinkley's paper above noticed. Our object is to remark that since by (Ex. 7. Sect. 7.) we have

$$\frac{1}{1....(x+n)} \cdot \left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n o^{x+n} = \frac{1}{1....x} \Delta^{-n} o^x,$$

therefore, the coefficient of t^{x+n} in the proposed function, or, which is the same thing, that of t^x in

$$(e^t - 1)^{-n}$$

is properly represented by $\frac{\Delta^{-n} o^x}{1.2....x}$, and therefore that the equation

$$(e^t - 1)^n = \frac{\Delta^n o^n}{1....n} t^n + \frac{\Delta^n o^{n+1}}{1....(n+1)} t^{n+1} + \&c.$$

proved by Dr. Brinkley to subsist for positive values of n, is now shewn to hold good also for negative.

(5). To develope e^{e^t} in powers of t.

$$f(e^t) = e^{e^t}, \quad f(1+\Delta) o^x = e^{1+\Delta} o^x = e \cdot e^{\Delta} o^x,$$

whence we deduce

$$A_x = \frac{e}{1\,.\,2\,.\,.\,.\,.\,x} \times e^{\Delta}\,0^x,$$

$$= \frac{e}{1\,.\,2\,.\,.\,.\,.\,x}\left\{0^x + \frac{\Delta\,0^x}{1} + \frac{\Delta^2\,0^x}{1\,.\,2} + \dots\frac{\Delta^x\,0^x}{1\,.\,.\,.\,.\,x}\right\}.$$

Thus $A_0 = e$, $A_1 = e$, $A_2 = \frac{2\,e}{1\,.\,2}$, $A_3 = \frac{5\,e}{1\,.\,2\,.\,3}$, &c.

and

$$e^{e^t} = e + e\,\frac{t}{1} + 2\,e\,.\,\frac{t^2}{1\,.\,2} + 5\,e\,.\,\frac{t^3}{1\,.\,2\,.\,3} + \&c.$$

(6). To develope $\frac{1}{\sqrt{(1+e^t)}}$ and in general $(a+e^t)^n$ in powers of t,

$f(e^t) = (a+e^t)^n$; $f(1+\Delta) = (1+a+\Delta)^n$,

$$A_x = \frac{1}{1\,.\,2\,.\,.\,.\,.\,x} \times \{(1+a)+\Delta\}^n\,0^x,$$

$$= \frac{(1+a)^n}{1\,.\,2\,.\,.\,.\,.\,x}\left\{0^x + \frac{n}{1}\,.\,\frac{\Delta\,0^x}{1+a} + \frac{n\,(n-1)}{1\,.\,2}\,.\,\frac{\Delta^2\,0^x}{(1+a)^2} + \,.\,.\,.\,.\,.\right.$$
$$\left.\dots\,.\, + \frac{n\,(n-1)\,.\,.\,.\,.\,(n-x+1)}{1\,.\,2\,.\,.\,.\,.\,x}\,.\,\frac{\Delta^x\,0^x}{(1+a)^x},\right\}$$

and in the case proposed where $a=1$ and $n = -\frac{1}{2}$, the value of A_x is

$$\frac{1}{\sqrt{2}\,.\,1\,.\,2\,.\,.\,.\,.\,x}\left\{0^x - \frac{1}{4}\,\Delta\,0^x + \frac{1\,.\,3}{4\,.\,8}\,\Delta^2\,0^x - \dots\dots\dots\right.$$
$$\left.\dots\,.\,\pm\,\frac{1\,.\,3\,.\,.\,.\,.\,(2\,x-1)}{4\,.\,8\,.\,.\,.\,.\,(4\,x)}\,\Delta^x\,0^x,\right\}$$

and in the same manner may any algebraic function of e^t be developed with little trouble.

(7). To extend the above mode of developing $f(e^t)$ to exponential functions of two or more variables of the form $f(e^t, e^{t'}, e^{t''}, \&c.)$

Let all but t be regarded as so many constants, entering into the composition of the given function, regarded as a function of e^t. Then will the coefficient of any power of t, as t^x be, by what has been proved,

$$\frac{f\{1+\Delta, e^{t'}, e^{t''}, \&c.\}\, o^x}{1.2\ldots x},$$

Into this function t does not enter, but, being a function of $e^{t'}$, &c. it is itself developable in a series of power of t', t'', &c. Let the coefficient of t'^y in this developement be sought by a similar process. It will be

$$\frac{f\{1+\Delta, 1+\Delta', e^{t''}, \&c.\}\, o^x . o'^y}{1.2\ldots x \times 1.2\ldots y},$$

the powers of Δ produced by this second process being kept distinct from those resulting from the first. and applied to their proper power o^y of o, by the accents affixed to them as in (Ex. 14. Sect. 7.). This then will be the coefficient of $t^x . t'^y$, in the developement of the proposed function, regarded as a function only of t and t'. Proceeding in this manner, and denoting by $A_{x, y, z, \&c.}$ the coefficient of the combination $t^x \cdot t'^y . t''^z . \&c.$ in the final result, we have

$$A_{x, y, z, \&c.} = \frac{f\{1+\Delta, 1+\Delta', 1+\Delta'', \&c.\}\, o^x . o'^y . o''^z . \&c.}{1.2\ldots x \times 1.2\ldots y \times 1.2\ldots z \times \&c.}.$$

(8). To develope $f\{e^{t+t'+t''+\&c.}\}$ in powers of t, t', &c.

In this case

$$A_{x, y, z, \&c.} = \frac{f\{(+\Delta)(1+\Delta')(1+\Delta'').\&c.\}\, o^x . o'^y . o''^z . \&c.}{1.2\ldots x \times 1.2\ldots y \times 1.2\ldots z \times \&c.}.$$

Now let $K(1+\Delta)^n . (1+\Delta')^n . \&c.$ be any term of the developement of $f\{(1+\Delta)(1+\Delta').\&c.\}$. Then when this is prefixed to $0^x . 0'^y . 0''^z$. &c. in the manner denoted by the accents, it becomes

$$K . (1+\Delta)^n 0^x . (1+\Delta)^n 0^y . (1+\Delta)^n 0^z . \&c.$$
$$= K . n^x . n^y . n^z . \&c. \text{ (by 3. Sect. 7.)}$$
$$= K . n^{x+y+z+\&c.} = K (1+\Delta)^n 0^{x+y+z+\&c.}.$$

Hence it appears that the same series of terms will result from

$$f\{(1+\Delta)(1+\Delta') . \&c.\}\, 0^x . 0'^y . \&c.$$

as would have arisen from

$$f(1+\Delta) 0^{x+y+z+\&c.},$$

and consequently, that

$$A_{x,y,z,\&c.} = \frac{f(1+\Delta) 0^{x+y+z+\&c.}}{1 . 2 \ldots . x \times 1 . 2 \ldots . y \times 1 . 2 \ldots . z \times \&c.},$$

Which is also deducible from the theorem for raising a polynomial to any power, combined with that in (Ex. 1. Sect. 7.) and *vice versâ*, the multinomial theorem is directly deducible from this.

Hence too it appears, that the developement of $f(e^{t+t'+\&c.})$ is directly deducible from that of $f(e^t)$; the coefficient of $t^x . t'^y . t''^z$. &c. in the former being equal to that of $t^{x+y+z+\&c.}$ in the latter, multiplied into

$$\frac{1 . 2 \ldots . (x+y+z+\&c.)}{1 \ldots . x \times 1 \ldots . y \times 1 \ldots . z + \&c.}.$$

(9). To prove that

$$\frac{1}{2+\Delta} 0^{2x-1} = (-1)^x . \frac{2^{2x}-1}{2x} . B_{2x-1},$$

B_{2x-1} being the x^{th} number of Bernouilli.

By (Ex. 1. Sect. 8.) $\dfrac{1}{2+\Delta} 0^{2x-1}$ is the coefficient of

t^{2x-1} in the devolopement of $\frac{1}{1+e^t}$, multiplied by $1 \,.\, 2 \,.\, 3 \ldots (2x-1)$, that is, to the coefficient of t^{2x} in $\frac{t}{1+e^t}$, multiplied by the same quantity. Again, by Appendix, Art. 408. it appears that this last coefficient is equal to $-(2^{2x}-1) \times$ into the coefficient of the same power in $\frac{t}{e^t-1}$, which coefficient is $(-1)^{x+1} \,.\, \frac{B_{2x-1}}{1\,.\,2 \ldots . 2x}$. Hence we must have

$$\frac{1}{2+\Delta}\, o^{2x-1} =$$

$$1\,.\,2 \ldots .(2x-1) \times -(-1)^{x+1} \,.\, \frac{2^{2x}-1}{1\,.\,2 \ldots . 2x} \,.\, B_{2x-1}$$

$$= (-1)^x \,.\, \frac{2^{2x}-1}{2x} \,.\, B_{2x-1}.$$

(10). To sum the series

$$1^n - 2^n + 3^n - 4^n + \&\text{c. } ad\ inf.$$

Since by (Ex. 3. Sect. 7.) we have $(1+\Delta)\, o^n = 1^n$, $(1+\Delta)^2\, o^n = 2^n$, &c. therefore the proposed series becomes by substitution

$$S = (1+\Delta)\, o^n - (1+\Delta)^2\, o^n + (1+\Delta)^3\, o^n - \&\text{c.}$$

or, separating the symbols of operation from those of quantity

$$S = \{ (1+\Delta) - (1+\Delta)^2 + (1+\Delta)^3 - \&\text{c.} \}\, o^n$$

$$= \frac{1+\Delta}{1+(1+\Delta)}\, o^n = \left\{ 1 - \frac{1}{2+\Delta} \right\} o^{2n},$$

because this function developed in powers of Δ will necessarily produce the same series as the other. To throw this into a calculable form, let it be actually developed in this manner, and we find

$$S = \frac{0^n}{2} + \frac{\Delta\, 0^n}{2^2} - \frac{\Delta^2\, 0^n}{2^3} + \ldots \pm \frac{\Delta^n\, 0^n}{2^{n+1}}.$$

Thus we have

$$1 - 1 + 1 - 1 + \&c. = \frac{1}{2},$$

$$1 - 2 + 3 - 4 + \&c. = \frac{1}{4},$$

$$1^2 - 2^2 + 3^2 - 4^2 + \&c. = 0,$$

$$1^3 - 2^3 + 3^3 - 4^3 + \&c. = \frac{1}{8},$$

$$1^4 - 2^4 + 3^4 - 4^4 + \&c. = 0;\ \&c.$$

All the even values vanish, and it ought to be so, for the substitution of $-\frac{1}{2\,\{1+(1+\Delta)\}}$ for $f(1 + \Delta)$ in (10. Sect. 7.) equation (*b*), gives

$$\frac{1+\Delta}{1+(1+\Delta)}\, 0^{2x} = \frac{0^{2x}}{2} = 0,$$

in every case except when $x=0$, when it becomes $\frac{1}{2}$. The odd values may be expressed by the numbers of Bernouilli, for writing $2x-1$ for n, the general expression for the odd values of S becomes

$$-\frac{1}{2+\Delta}\, 0^{2x-1} = (-1)^{x+1} \cdot \frac{2^{2x}-1}{2x} B_{2x-1}$$

by (Ex. 9. Sect. 8.): a result exactly agreeing with that deduced by Euler in his *Institutiones Calculi differentialis*, Cap. vii. p. 501. from a principle, it must be confessed, not at all satisfactory.

(11). The series

$$1^n + 2^n + 3^n + \ldots x^n = S_x$$

being proposed, if we treat it in the same way, we get

$$S_x = \{(1+\Delta) + (1+\Delta)^2 + \dots .(1+\Delta)^x\} o^n$$

$$= \frac{(1+\Delta)^{x+1} - (1+\Delta)}{\Delta} o^n$$

which, developed in powers of Δ, gives

$$S_x = x \,.\, o^n + \frac{(x+1)x}{1\,.\,2} \Delta o^n + \frac{(x+1)x(x-1)\Delta^2 o^n}{1\,.\,2\,.\,3} + \dots\dots$$

$$\dots\dots + \frac{(x+1)x \dots .(x-n+1)}{1\,.\,2 \dots .(n+1)} \Delta^n o^n.$$

This is different in its form, from any expression we have yet given for the sum of this series. It may, however, be obtained by resolving $(x+1)^n$ into preceding, instead of succeeding values, and integrating, as in (Ex. 26. Sect. 2. and Ex. 23. Sect. 6.)

(12). To sum the series of (10) viz.

$$1^n - 2^n + 3^n - \&c.$$

to x terms.

Here $S_x = \{(1+\Delta) - (1+\Delta)^2 + (1+\Delta)^3 \dots\dots$

$$\pm (1+\Delta)^x\} o^n$$

$$= \frac{(1+\Delta) - (-1)^x (1+\Delta)^{x+1}}{2+\Delta} o^n.$$

This expression, developed in powers of Δ affords a calculable value of S_x, the number of whose terms can never exceed $n+1$, and consequently the developement need never be carried farther. Thus we have, for instance,

$$1 - 2 + 3 - \dots . \pm x = (-1)^{x+1} \,.\, \left(\frac{2x+1}{4}\right) + \frac{1}{4},$$

$$1^2 - 2^2 + 3^2 - \dots . \pm x^2 = (-1)^{x+1} \,.\, \frac{x(x+1)}{2},$$

$$1^3 - 2^3 + 3^3 \dots \pm x^3 =$$

$$(-1)^{x+1} . \tfrac{1}{2}\left(x^3 + \frac{3}{2}x^2 - \frac{1}{4}\right) - \frac{1}{8};\ \&c.$$

(13). To sum the series

$$1^n . t + 2^n . t^2 + 3^n . t^3 + \&c.$$

to x terms and to infinity. This series treated exactly in the same manner, becomes

$$S_x = \{ t(1+\Delta) + t^2(1+\Delta)^2 + \dots t^x(1+\Delta)^x \}\, 0^n$$

$$= \frac{t^{x+1}(1+\Delta)^{x+1} - t(1+\Delta)}{t(1+\Delta) - 1}\, 0^n.$$

Let S represent the series to infinity, then

$$S = \frac{t(1+\Delta)}{(1-t) - t\Delta}\, 0^n = \left\{ \frac{t}{1-t} + \frac{t\Delta}{(1-t)(1-t-t\Delta)} \right\} 0^n$$

$$= \frac{t}{1-t}\left\{ 1 + \frac{\Delta}{1-t} + \frac{t\Delta^2}{(1-t)^2} + \&c. \right\} 0^n,$$

$$= \frac{1}{1-t}\left\{ t . 0^n + \left(\frac{t}{1-t}\right)\Delta 0^n + \left(\frac{t}{1-t}\right)^2 \Delta^2 0^n + \dots\dots \right.$$

$$\left. \dots\dots \left(\frac{t}{1-t}\right)^n \Delta^n 0^n \right\}.$$

Thus we have as particular instances

$$1 . t + 2 . t^2 + 3 . t^3 + \&c. = \frac{t}{(1-t)^2},$$

$$1^2 . t + 2^2 . t^2 + 3^2 . t^3 + \&c. = \frac{t}{(1-t)^2} + \frac{2t^2}{(1-t)^3}.$$

The expression for S_x may, in like manner, be easily developed in powers of Δ, and will then assume a calculable form, but we prefer leaving it in its present state. The reader may, if he please, supply this part of the operation.

(14). In general to sum the series

$$A_0 . 0^n + A_1 . 1^n + A_2 . 2^n + \&c.$$

having given

$$A_0 + A_1 . t + A_2 . t^2 + \&c. = F(t).$$

The series treated as before gives

$$S = \{ A_0 + A_1 (1 + \Delta) + \&c. \} 0^n$$

$$= F(1 + \Delta) 0^n$$

$$= F(1) . 0^n + \frac{F'(1)}{1} \Delta 0^n + \&c.$$

when adapted to actual computation by the developement of $F(1 + \Delta)$ in powers of Δ.

(15). To sum the series

$$S = 1^n + \frac{2^n . h}{1} + \frac{3^n . h^2}{1 . 2} + \frac{4^n . h^3}{1 . 2 . 3} + \&c.$$

This series being the same with

$$0^{n+1} + \frac{1^{n+1}}{1} h + \frac{2^{n+1}}{1 . 2} h^2 + \&c.$$

by substituting e^{ht} for $F(t)$ and $n + 1$ for n in the last problem we find

$$S = e^{h(1+\Delta)} 0^{n+1}$$

$$= e^h \left\{ 0^{n+1} + \frac{\Delta 0^{n+1}}{1} h + \frac{\Delta^2 0^{n+1}}{1 . 2} h^2 + \ldots\ldots \right.$$

$$\left. \ldots . \frac{\Delta^{n+1} 0^{n+1}}{1 \ldots . (n+1)} h^{n+1} \right\}.$$

Thus, if we suppose $h = 1$, we find

$$1 + \frac{2}{1} + \frac{3}{1 . 2} + \&c. = 2e,$$

$$1^2 + \frac{2^2}{1} + \frac{3^2}{1 \, . \, 2} + \&c. = 5\,e,$$

$$1^3 + \frac{2^3}{1} + \frac{3^3}{1 \, . \, 2} + \&c. = 15\,e, \text{ and so on.}$$

(16). $1^n - 3^n + 5^n - \&c. = S.$

Then,

$$S = \frac{1+\Delta}{1+(1+\Delta)^2}\, 0^n = \frac{1+\Delta}{2(1+\Delta)+\Delta^2}\, 0^n.$$

The developement of this in powers of Δ would be attended with some trouble, but this is not indispensable. It may be reduced to a limited number of *calculable* terms in many different ways, the simplest of which seems to be as follows:

$$S = \left\{ \frac{1+\Delta}{2(1+\Delta)} - \frac{(1+\Delta)\,\Delta^2}{4\,(1+\Delta)^2} + \frac{(1+\Delta)\,\Delta^4}{8\,(1+\Delta)^3} - \&c. \right\} 0^n.$$

$$= \frac{0^n}{2} - \frac{1}{4} \cdot \frac{\Delta^2}{1+\Delta}\, 0^n + \frac{1}{8} \cdot \frac{\Delta^4}{(1+\Delta)^2}\, 0^n - \&c.$$

the number of whose terms can never exceed $\frac{n}{2}+1$ and each term of which is easily calculated. In fact, whenever n is an odd number S vanishes, for, if in the equation (a) of (10. Sect. 7.) we suppose

$$f(1+\Delta) = \frac{1+\Delta}{1+(1+\Delta)^2},$$

there will result

$$\frac{1+\Delta}{1+(1+\Delta)^2}\, 0^{2x-1} = 0.$$

and when n is even (writing $2\,x$ for it) the value of S becomes

$$\frac{0^{2x}}{2} - \frac{1}{4} \cdot \frac{\Delta^2}{1+\Delta}\, 0^{2x} + \ldots \pm \frac{1}{2^{x+1}} \cdot \frac{\Delta^{2x}}{(1+\Delta)^x}\, 0^{2x};$$

Thus we find

$$1 - 1 + 1 - 1 + \&c. = \frac{1}{2}$$

$$1 - 3 + 5 - 7 + \&c. = 0,$$

$$1^2 - 3^2 + 5^2 - 7^2 + \&c. = -\frac{1}{2}$$

$$1^3 - 3^3 + 5^3 - 7^3 + \&c. = 0, \&c.$$

See also Note *D. Lacroix, Transl.* p. 360.

(17). To complete the theory of these sums we shall subjoin the following example, which will call for the employment of nearly all the transformations above demonstrated, and will thus, by illustrating their use and management, prove the more acceptable as the mode of investigation followed in this and the last section is, if we mistake not, perfectly novel in analysis.

To sum the series

$$\frac{1}{1^{2x+1}} - \frac{1}{3^{2x+1}} + \frac{1}{5^{2x+1}} - \&c. \textit{ ad inf.}$$

and to express its sum by means of the numbers of Bernoulli.

Let C_{2x+1} represent the sum of the series, or

$$C_1 = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \&c.$$

$$C_3 = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c.$$

Then Euler has shewn that, θ being any arc, the developement of sec θ will be

$$\sec\theta = \frac{2^2}{\pi} . C_1 + \frac{2^4}{\pi^3} C_3 . \theta^2 + \frac{2^6}{\pi^5} C_5 . \theta^4 + \&c.$$

So that C_{2x+1} is equal to $\left(\frac{\pi}{2}\right)^{2x+1} \times \frac{1}{2}$ the coefficient of θ^{2x} in the developement of sec θ. Call this coefficient A_{2x}, then since

$$\sec\theta = \frac{1}{\cos\theta} = \frac{2}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}},$$

we have by (1. Sect. 7.)

$$A_{2x} = \frac{1}{1\,.\,2\,.\,.\,.\,.\,(2\,x)} \times \frac{2}{(1+\Delta)^{\sqrt{-1}} + (1+\Delta)^{-\sqrt{-1}}}\,0^{2x},$$

but since by (8. Sect. 7.), writing $2\,x$ for x and $\sqrt{-1}$ for n

$$f\,\{(1+\Delta)^{\sqrt{-1}}\}\,0^{2x} = (\sqrt{-1})^{2x}\,.\,f\,(1+\Delta)\,0^{2x},$$
$$= (-1)^{x}\,.\,f\,(1+\Delta)\,0^{2x},$$

we have

$$A_{2x} = \frac{2\,.\,(-1)^{x}}{1\,.\,2\,.\,.\,.\,.\,(2\,x)} \times \frac{1}{(1+\Delta)+(1+\Delta)^{-1}}\,0^{2x},$$

which substituted in the value of C_{2x+1} gives

$$C_{2x+1} = \frac{(-1)^{x}\,.\left(\frac{\pi}{2}\right)^{2x+1}}{1\,.\,2\,.\,.\,.\,.\,(2\,x)}\,.\,\frac{1+\Delta}{(1+\Delta)^{2}+1}\,0^{2x}.$$

The latter factor of this expression coincides precisely with that obtained in the last number for the series

$$1^{2x} - 3^{2x} + 5^{2x} - \&c.$$

and its numerical value may readily be computed by the formula there given. Hence this remarkable relation between the two series

$$1\,.\,2\,.\,3\,.\,.\,.\,.\,(2\,x) \times \left\{\frac{1}{1^{2x+1}} - \frac{1}{3^{2x+1}} + \frac{1}{5^{2x+1}} - \&c.\right\} =$$
$$= (-1)^{x}\,.\left(\frac{\pi}{2}\right)^{2x+1} \times \{1^{2x} - 3^{2x} + 5^{2x} - \&c.\}$$

and as particular instances,

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \&c. = \frac{\pi}{4}$$

$$\frac{1}{1^{3}} - \frac{1}{3^{3}} + \frac{1}{5^{3}} - \&c. = \frac{\pi^{3}}{32},\ \&c.$$

The transformation into numbers of Bernouilli may be accomplished as follows.

$$\frac{1+\Delta}{(1+\Delta)^2+1} = \frac{(1+\Delta)^2}{(1+\Delta)\{(1+\Delta)^2+1\}} =$$

$$= \frac{1}{2(1+\Delta)}\left\{1+\frac{(1+\Delta)^2-1}{(1+\Delta)^2+1}\right\}$$

$$= \frac{1}{2(1+\Delta)} + \frac{1}{2}\cdot\frac{(1+\Delta)-(1+\Delta)^{-1}}{(1+\Delta)^2+1}.$$

Now, by (3. Sect. 7.) it appears that $\frac{1}{1+\Delta}o^{2x}=1$, and we therefore have

$$\frac{1+\Delta}{(1+\Delta)^2+1}o^{2x} = \frac{1}{2}+\frac{1}{2}\,\frac{(1+\Delta)-(1+\Delta)^{-1}}{(1+\Delta)^2+1}\,o^{2x}.$$

$$= \frac{1}{2} + 2^{2x-1}.\frac{(1+\Delta)^{\frac{1}{2}}-(1+\Delta)^{-\frac{1}{2}}}{(1+\Delta)+1}\,o^{2x}, \qquad \text{(by 8. Sect. 7.)}$$

$$= \frac{1}{2} + 2^{2x-1}.\frac{(1+\Delta)^{\frac{1}{2}}-(1+\Delta)^{-\frac{1}{2}}}{2+\Delta}\,o^{2x}.$$

Now, in the theorem (16. Sect. 7.) if we make $f(\Delta)=\frac{1}{2+\Delta}$, and $n=\frac{1}{2}$ and $-\frac{1}{2}$ in succession, it gives

$$\frac{(1+\Delta)^{\frac{1}{2}}}{2+\Delta}\,o^{2x} = \frac{1}{2+\Delta}\left(o+\frac{1}{2}\right)^{2x};$$

$$\frac{(1+\Delta)^{-\frac{1}{2}}}{2+\Delta}\,o^{2x} = \frac{1}{2+\Delta}\left(o-\frac{1}{2}\right)^{2x};$$

so that by substitution we get

$$\frac{1+\Delta}{(1+\Delta)^2+1}\,o^{2x} = \frac{1}{2}+2^{2x-1}.\frac{1}{2+\Delta}\left\{\left(o+\frac{1}{2}\right)^{2x}-\left(o-\frac{1}{2}\right)^{2x}\right\}$$

$$= \frac{1}{2}+\frac{2^{2x-1}}{2+\Delta}\left\{\frac{2x}{1}\,o^{2x-1}+\frac{2x(2x-1)(2x-2)}{1.2.3\times2^2}\,o^{2x-3}+\&c.\right\}$$

Again, we have, by (9. Sect. 8.)

$$\frac{1}{2+\Delta} o^{2x-1} = (-1)^x \cdot \frac{2^{2x}-1}{2x} B_{2x-1}$$

whence,

$$\frac{1}{2+\Delta} o^{2x-3} = -(-1)^x \cdot \frac{2^{2x-2}-1}{2x-2} B_{2x-3}; \text{ \&c.}$$

and consequently, by substituting these values, our expression is cleared of the symbols Δ and o, and reduces itself to

$$\frac{1}{2} + (-1)^x \cdot 2^{2x-1} \left\{ \frac{2^{2x}-1}{1} B_{2x-1} \right.$$
$$\left. - \frac{2x(2x-1)}{1 \cdot 2} \cdot \frac{2^{2x-2}-1}{3 \cdot 2^2} B_{2x-3} + \text{\&c.} \right\}$$

and hence we have, finally,

$$C_{2x+1} = \frac{\left(\frac{\pi}{2}\right)^{2x+1}}{1 \cdot 2 \ldots (2x)} \left\{ \frac{(-1)^x}{2} + \right.$$
$$+ 2^{2x-1} \left(\frac{2^{2x}-1}{1} B_{2x-1} - \frac{2x(2x-1)}{1 \cdot 2} \cdot \frac{2^{2x-2}-1}{3 \times 2^2} B_{2x-3} + \right.$$
$$\left. \left. + \ldots \frac{2x \ldots 3}{1 \cdot 2 \ldots (2x-2)} \cdot \frac{2^2-1}{(2x-1) \cdot 2^{2x-2}} B_1 \right) \right\}.$$

The practicability of expressing this function by means of the numbers of Bernouilli was first shewn, and the above expression demonstrated in the Phil. Trans. 1814. The demonstration there given is however very circuitous and rather obscure. The above has the advantage of connecting what was before an insulated result with the general theory of series of this sort.

(18). Given the sum of the series

$$A_0 + A_1 \cdot t + A_2 \cdot t^2 + \text{\&c.} = F(t)$$

or the generating function of A_r, required the sum of

$$A_1 . 1^n . t + A_2 (1^n + 2^n) . t^2 + A_3 (1^n + 2^n + 3^n) t^3 + \&c.$$

or the generating function of

$$A_x . (1^n + 2^n + \dots . x^n).$$

By substituting for $1^n, 2^n, \dots . x^n$, their values

$$(1+\Delta) 0^n, \ (1+\Delta)^2 0^n, \dots . (1+\Delta)^x 0^n$$

we find as in (11. Sect. 8.)

$$1^n + 2^n + \dots . x^n = \frac{(1+\Delta)^{x+1} - (1+\Delta)}{\Delta} 0^n,$$

So that our series becomes,

$$A_1 t . \frac{(1+\Delta)^2 - (1+\Delta)}{\Delta} 0^n + A_2 t^2 . \frac{(1+\Delta)^3 - (1+\Delta)}{\Delta} 0^n + \&c.$$

$$= \frac{1+\Delta}{\Delta} \{ A_1 t (1+\Delta) + A_2 t^2 (1+\Delta)^2 + \&c. \} 0^n -$$

$$- \frac{1+\Delta}{\Delta} 0^n . \{ A_1 t + A_2 t^2 + \&c. \}$$

$$= \frac{1+\Delta}{\Delta} \{ F . (1+\Delta) t - A_0 \} 0^n - \frac{1+\Delta}{\Delta} 0^n \{ F(t) - A_0 \}$$

$$= \frac{1+\Delta}{\Delta} \{ F(t + t\Delta) - F(t) \} 0^n$$

(19). For example

$$\frac{1^n}{1} + \frac{1^n + 2^n}{1 . 2} + \frac{1^n + 2^n + 3^n}{1 . 2 . 3} + \&c. =$$

$$= e . (1+\Delta) \frac{e^{\Delta} - 1}{\Delta} 0^n = e \left\{ 1 + \frac{3}{1 . 2} \Delta + \frac{4}{1 . 2 . 3} \Delta^2 + \&c. \right\} 0^n$$

$$= e \left\{ 0^n + \frac{3}{1 . 2} \Delta 0^n + \frac{4}{1 . 2 . 3} \Delta^2 0^n + \right.$$

$$\left. + \dots \dots \frac{n+2}{1 . 2 . \dots . (n+1)} \Delta^n 0^n \right\}.$$

(20). $1^n \cdot t + (1^n + 2^n)\, t^2 + (1^n + 2^n + 3^n)\, t^3 + \&c.$ ad inf.

$$S = \frac{1+\Delta}{\Delta}\left\{\frac{t}{1-t-t\,\Delta} - \frac{t}{1-t}\right\} 0^n$$

$$= \frac{t^2}{(1-t)^2}\left\{1 + \frac{\Delta}{(1-t) - t\,\Delta}\right\} 0^n$$

$$= \frac{t}{(1-t)^2}\left\{t \cdot 0^n + \left(\frac{t}{1-t}\right)\Delta\, 0^n + \left(\frac{t}{1-t}\right)^2 \Delta^2\, 0^n + \right.$$

$$\left. + \ldots\ldots \left(\frac{t}{1-t}\right)^n \Delta^n\, 0^n\right\}.$$

(21). It is required to prove the following expression for the numbers of Bernoulli,

$$B_{2x-1} = (-1)^{x+1} \cdot \left\{\frac{\Delta\, 0^{2x+1}}{1^2} - \frac{\Delta^2\, 0^{2x+1}}{2^2} + \right.$$

$$\left. \ldots\ldots + \frac{\Delta^{2x+1}\, 0^{2x+1}}{(2\,x+1)^2}\right\},$$

Take the equation (Append. Art. 411.)

$$\Sigma\, u_x = \int u_x\, d\,x - \frac{u_x}{2} + \frac{B_1}{1\,.\,2} \cdot \frac{d}{d\,x}\, u_x - \&c.$$

in which make $u_x = x^{2i}$, and we shall find for the coefficient of x in the developement of $\Sigma\,(x^{2i})$

$$(-1)^{i+1}\, B_{2i-1}.$$

But, if we integrate the expression given for x^n in (2. Sect. 2.) we find

$$\Sigma\,(x^n) = C + (-1)^n \cdot \left\{\frac{x}{1^2}\,\Delta\, 0^{n+1} - \frac{x\,(x+1)}{1\,.\,2^2}\,\Delta^2\, 0^{n+1} + \&c.\right\}$$

If this be developed in powers of x, and the coefficients of x collected together, we shall find for the whole coefficient of that term

$$(-1)^n \cdot \left\{\frac{\Delta\, 0^{n+1}}{1^2} - \frac{\Delta^2\, 0^{n+1}}{2^2} + \&c.\right\}$$

In this if we write $2i$ for n, we shall find by comparing it with the coefficient previously found

$$B_{2i-1} = (-1)^{i+1}\left\{\frac{\Delta\, o^{2i+1}}{1^2} - \frac{\Delta^2\, o^{2i+1}}{2^2} + \&c.\right\}$$

which is, in fact, the equation to be proved. If $2i - 1$ be put for n the term x vanishes form the integral $\Sigma\,(x^n)$, (See Appendix Art. 369.) which shews that we must have

$$\frac{\Delta\, o^{2i}}{1^2} - \frac{\Delta^2\, o^{2i}}{2^2} + \&c. = 0,$$

which has been already proved by a different process (13. Sect. 7.) It will not be amiss now to notice one more property of the numbers comprised in the form $\Delta^m\, o^n$ by the aid of which the table of their numerical values given in (33. Sect. 1.) may be continued to any extent, with very little trouble.

(22). The numbers

$$\Delta^n\, o^n, \quad \Delta^n\, o^{n+1}, \quad \Delta^n\, o^{n+2}, \ \&c.$$

form a recurring series. To shew this, and to determine its scale of relation.

Since (App. Art. 350.)

$$\Delta^n\, o^x = n^x - \frac{n}{1}(n-1)^x + \ldots \pm \frac{n}{1} \cdot 1^x,$$

this function is of the form

$$C_1 \cdot \alpha^x + C_2 \cdot \beta^x + \ldots C_n \cdot \nu^x.$$

It is, therefore, a particular integral of some equation of differences of the first degree with constant coefficients (App. Art. 389.) and is therefore (App. Art. 390.) the general term of some recurring series. Let now

$A_0 = 1,$
$A_1 = 1 + 2 + 3 + \ldots n,$

$$A_2 = 1 . 2 + 1 . 3 + 2 . 3 + \ldots . (n-1) . n$$

. . . .

$$A_n = 1 . 2 . 3 \ldots . n.$$

Then (App. Art. 389.) This equation will be

$$\Delta^n o^{x+n} - A_1 \Delta^n o^{x+n-1} + \ldots . \pm A_n \Delta^n o^x = 0$$

so that the scale of relation is

$$+ A_1, \ - A_2, \ + A_3, \ldots . \pm A_n.$$

SECTION IX.

Exercises, &c. in the Interpolation of Series.

(1). In any series of consecutive equidistant values of a function, where one is deficient, to insert that one.

Let the equidistant values be

$$v_0, \ v_1, \ v_2, \ldots . v_n,$$

and let the deficient one be v_i, so that all but v_i are given. Assume $\Delta^n v_0 = 0$, or that the $(n-1)^{\text{th}}$ differences are constant, which will almost always be nearly the case in tabulated results, except under extreme circumstances. Then we have

$$\Delta^n v_0 = v_n - \frac{n}{1} v_{n-1} + \frac{n(n-1)}{1 . 2} v_{n-2} \ldots . \pm \frac{n}{1} v_1 \mp v_0 = 0,$$

an equation of the first degree, from which any one of the values as v_i may be determined in terms of the rest.

(2). Given two values of any function, required to insert one equidistant between them.

Given v_0 and v_2, required v_1,

$$\Delta^2 v_0 = 0, \quad v_2 - 2 v_1 + v_0 = 0,$$

$$v_1 = \frac{v_0 + v_2}{2}.$$

(3). Given three values v_0, v_1, v_3 of any function to insert the deficient one v_2.

$$\Delta^3 u_0 = 0; \quad v_3 - 3 v_2 + 3 v_1 - v_0 = 0,$$

$$v_2 = \frac{v_3 + 3 v_1 - v_0}{3}.$$

In like manner, if v_1 were the deficient value, we should find

$$v_1 = \frac{v_0 + 3 v_2 - v_3}{3}.$$

(4). Given the following common logarithms,

log 510 = 2.70757018
log 511 = 2.70842090
log 513 = 2.71011737
log 514 = 2.71096312

it is required to insert the deficient value log 512.

Given $v_0 = \log 510$, $v_1 = \log 511$, $v_3 = \log 513$, and $v_4 = \log 514$. Required $v_2 = \log 512$.

$$\Delta^4 v_0 = v_4 - 4 v_3 + 6 v_2 + 4 v_1 + v_0 = 0$$

$$v_2 = \frac{4 (v_1 + v_3) - (v_0 + v_4)}{6} = 2.70926996$$

precisely as the table.

(5). In any series of consecutive equidistant values where two are deficient, to insert those two.

As before, let them be

$$v_0, v_1, \ldots \ldots v_{n+1}$$

and assuming $\Delta^n v_0 = 0$ and $\Delta^n v_1 = 0$, to obtain a continuous law of increase or diminution throughout the whole series, we have

$$\left.\begin{aligned} v_n - \frac{n}{1} v_{n-1} + \frac{n(n-1)}{1.2} v_{n-2} \ldots\ldots \pm v_0 = 0 \\ v_{n+1} - \frac{n}{1} v_n + \frac{n(n-1)}{1.2} v_{n-1} \ldots\ldots \pm v_1 = 0 \end{aligned}\right\}$$

two equations of the first degree which suffice to determine any two of the values in terms of the rest. The same principle will serve to insert any number of deficient terms.

(6). Given v_0, v_1, v_4, v_5. Required v_2 and v_3.

Assume $\Delta^4 v_0 = 0$ and $\Delta^4 v_1 = 0$, then

$$\left.\begin{aligned} v_4 - 4 v_3 + 6 v_2 - 4 v_1 + v_0 = 0 \\ v_5 - 4 v_4 + 6 v_3 - 4 v_2 + v_1 = 0 \end{aligned}\right\}$$

whence

$$v_2 = \frac{-3 v_0 + 10 v_1 + 5 v_4 - 2 v_5}{10}$$

$$v_3 = \frac{-2 v_0 + 5 v_1 + 10 v_4 - 3 v_5}{10}.$$

(7). In a table of the values of $\int dx \left(\log \frac{1}{x}\right)^{a-1}$ taken between the limits $x=0$ and $x=1$ *, we find the following values corresponding to the annexed values of a.

$a = 1.326$, $\int = 0.8938710$;
1.328,........0.8936220;
1.329,........0.8935004;
1.331,........0.8932628.

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* Legendre, *Exercises de Calcul Integral.* p. 302.

Required the values corresponding to $a = 1.327$, and $a = 1.330$.

Given v_0, v_2, v_3, v_5, required v_1 and v_4.

$$v_1 = \frac{v_5 - 10\,v_3 + 20\,v_2 + 4\,v_0}{15} = 0.8937455$$

$$v_4 = \frac{4\,v_5 + 20\,v_3 - 10\,v_2 + v_0}{15} = 0.8933807.$$

(8). In any series of consecutive equidistant values, where one or more are deficient, and the rest given, to interpolate any intermediate value whatever.

Insert the deficient equidistant values, and then interpolate the series so completed by the formula,

$$v_n = v_0 + \frac{n}{1}\,\Delta\,v_0 + \frac{n\,(n-1)}{1\,.\,2}\,\Delta^2\,v_0 + \&c.$$

For instance,

(9). In a table of the values of the function $\tan^{-}(x)$ or arc $(\tan = x)$* we have given

$\tan^{-1} 10 = 1.471127674$, $\tan^{-1} 11 = 1.480136439$

$\tan^{-1} 13 = 1.494024435$, $\tan^{-1} 15 = 1.504228163$.

Required $\tan^{-1}(11.63)$.

The values of $\tan^{-1} 12$ and $\tan^{-1} 14$, first of all inserted are respectively 1.487655094 and 1.499488856.

Let then $n = 1.63$, $v_0 = \tan^{-1} 10$, &c. and we find

$$v_n = 1.485022707$$

the number required.

(10). Given the values v_0, v_1, v_3 of a function, required to interpolate any given value as v_n,

$$v_n = v_0 - \frac{v_3 - 9\,v_1 + 8\,v_0}{6}\,.\,n + \frac{v_3 - 3\,v_1 + 2v_0}{6}\,.\,n^2.$$

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* Given by Spence in his Logarithmic Transcendents, p. 63.

(11). Two observations of a certain quantity were made at the interval of a day from each other: the first gave for its value a, the second $-b$. When was its value zero?

Given v_0 and v_1, required n so that $v_n = 0$.

1st In general $v_n = v_0 + \frac{n}{1}(v_1 - v_0)$

and making this zero,

$$n = -\frac{v_0}{v_1 - v_0} = \frac{a}{a + b},$$

which is the time in fractions of a day, from the first observation to the moment required. Having but two observations we suppose $\Delta^2 v_0$ &c. zero. This is the most ordinary instance of interpolation. To render it exact, we should, if possible, choose such opportunities for observation, as will allow of our neglecting $\Delta^2 v_0$, &c. that is, when the variation of the quantity observed is nearly uniform. Such is that of the sun's declination near the equinox, of a planet's latitude near its node, &c. The rule for proportional parts in logarithmic and other tables, depends likewise on this problem.

(12). Three observations of a certain quantity were taken at equal intervals. Its value were found to be v_0, v_1, v_2, between which its value was a. When did this happen?

Given v_0, v_1, v_2. Required n so that $v_n = a$,

make

$$v_0 + \frac{n}{1}\Delta v_0 + \frac{n(n-1)}{1 \cdot 2}\Delta^2 v_0 = a,$$

whence

$$n = \frac{1}{2\Delta^2 v_0}\left\{(\Delta^2 v_0 - 2\Delta v_0) + \sqrt{(\Delta^2 v_0 - 2\Delta v_0)^2 - 8(v_0 - a)\Delta^2 v_0}\right\}.$$

The positive sign is affixed to the radical, because the supposition $\Delta^2 v_0 = 0$, or $\Delta^2 v_0$ very small ought to reduce the value of n to that given in the last example. If this expression be developed in powers of $\Delta^2 v_0$, and its square and superior powers neglected ,we find

$$n = -\left(\frac{1}{\Delta v_0} + \frac{(v_1 - a)\,\Delta^2 v_0}{2\,(\Delta v_0)^3}\right) . (v_0 - a)$$

in which we may regard the term

$$-(v_0 - a) . (v_1 - a) . \frac{\Delta^2 v_0}{2\,(\Delta v_0)^3},$$

as a correction to be applied to the value of n calculated from the terms v_0 and v_1 alone on the supposition $\Delta^2 v_0 = 0$, and thus in certain cases dispense with a troublesome calculation. We may observe that by a proper choice of the quantities represented by v_0, v_1, &c. the quantity a may always be made zero, so that we have

$$-\frac{v_0 . v_1 . \Delta^2 v_0}{2\,(\Delta v_0)^3}.$$

for the correction to be made in this case.

(13). Given three values, not equidistant, of a function, to interpolate any intermediate value.

Given v_α, v_β, v_γ, required v_n.

Suppose the *indices* α, β, γ, to be equidistant values of some other function, thus let $\alpha = z_0$, $\beta = z_1$, $\gamma = z_2$ and let $n = z_x$, then will v_α, v_β, v_γ, v_n, be the values of the function v_{z_x} corresponding to the values 0, 1, 2, and x of the index x the independent variable. Let $v_{z_x} = u_x$, then $u_0 = v_\alpha$, $u_1 = v_\beta$, $u_2 = v_\gamma$, $u_x = v_n$, and the formula in Art. 404. gives

$$u_x \text{ or } v_n = \frac{(n-\beta)(n-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} v_\alpha + \frac{(n-\alpha)(n-\gamma)}{(\beta-\alpha)(\beta-\gamma)} v_\beta + \\ + \frac{(n-\alpha)(n-\beta)}{(\gamma-\alpha)(\gamma-\beta)} v_\gamma .$$

(14). Three observations of a quantity near its maximum or minimum, are made at given times (equidistant or not.) From the observed values, to determine when the maximum or minimum took place.

When a quantity is near its maximum or minimum, its values cannot be interpolated from two observations, because such interpolation requires the supposition of uniform variation during their interval, which cannot be made in these circumstances. In fact the function $v_0 + \frac{n}{1}(v_1 - v_0)$ does not admit a maximum or a minimum by the variation of n. In the case of three observations, however, suppose α, β, γ, to be the times (from a certain epoch) at which they were made, and v_α, v_β, v_γ, the observed values, then, since at any other time n we have $v_n =$ the expression in the last problem, if we differentiate this relative to n, and put the result $= 0$, we shall find

$$n = \frac{(\beta^2 - \gamma^2)v_\alpha - (\alpha^2 - \gamma^2)v_\beta + (\alpha^2 - \beta^2)v_\gamma}{2\,\{(\beta - \gamma)v_\alpha - (\alpha - \gamma)v_\beta + (\alpha - \beta)v_\gamma\}},$$

the value of n required, at which v_n is a maximum. By this formula may the meridian altitude of the Sun, or a Star, for example, be found when an observation precisely on the meridian cannot be had.

If the observations be equidistant, and the epoch be fixed at the first, we get

$$n = \frac{\beta}{2} \cdot \frac{3v_\alpha - 4v_\beta + v_\gamma}{2v_\alpha - 2v_\beta + v_\gamma}.$$

(15). Given any number of values of a quantity observed at given times, not equidistant, to determine its value and those of its differential coefficients at a given instant, the time being supposed to increase uniformly.

Let the given instant be fixed on for an epoch, and call t the time elapsed since that epoch, t being indeterminate, and negative for all observations preceding the epoch, also let α, β, γ, &c. be the values of t at the moments of observation, and v_α, v_β, v_γ, &c. those of the quantity observed, and we have by Append. Art. 404. for its value v at the time t,

$$v = \frac{(t-\beta)(t-\gamma)\dots}{(\alpha-\beta)(\alpha-\gamma)\dots} v_\alpha + \frac{(t-\alpha)(t-\gamma)\dots}{(\beta-\alpha)(\beta-\gamma)\dots} v_\beta + \&c.$$

The values then of the differential coefficients $\frac{dv}{dt}$, $\frac{d^2v}{dt^2}$, &c, when $t=0$, or at the epoch, will be the coefficients of t, t^2, &c. in the developement of this function, divided respectively by 1, 1 . 2, 1 . 2 . 3, &c, or calling them $\frac{dV}{dt}$, $\frac{d^2V}{dt^2}$, &c.

$$V = \pm \left\{ \frac{\beta\,.\,\gamma\,.\,\delta\dots}{(\alpha-\beta)(\alpha-\gamma)\dots} v_\alpha + \frac{\alpha\,.\,\gamma\,.\,\delta\dots}{(\beta-\alpha)(\beta-\gamma)\dots} v_\beta + \&c. \right\}$$

$$1\,.\,\frac{dV}{dt} = \mp \left\{ \frac{\beta\,.\,\gamma\,.\,\delta\dots\left(\frac{1}{\beta}+\frac{1}{\gamma}+\dots\right)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\dots} v_\alpha + \&c. \right\}$$

$$1\,.2.\,\frac{d^2V}{dt^2} = \pm \left\{ \frac{\beta.\gamma.\delta\dots\left(\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}+\dots\right)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\dots} v_\alpha + \&c. \right\}$$

The sign prefixed to the right hand members of these equations is the upper or lower, according as the number of observations, or of the letters α, β, γ, δ, ... is odd or even.

Laplace's method of computing the orbit of a comet, turns upon the application of this problem. The formulæ here deduced are somewhat different in their form from those employed in the Mecanique Celeste, and perhaps, rather more complicated to the eye. But in actual computation they will, I believe, be found more convenient, the terms of which they consist being better adapted to logarithmic computation, and in reality less intricate in their formation, and in consequence affording less room for mistakes on the part of the calculator.

SECTION X.

Application of the Calculus of Differences to the determination of Curves from properties involving consecutive points separated by a finite interval.

(1). In the circle, any line ACB (Part. III. Fig. 1.) drawn through a certain fixed point C (the center of the circle) and meeting the curve at its two extremities, is of a given length in all positions of the line. It is required to determine whether any other curves possess the same property, and if so, to include them under one general equation. In other words: Required the class of curves whose diameters are invariable.

Draw any line CM, and let the angle $MCA=\theta$, $CA=r$, $CB=r'$, and suppose $r=\phi(\theta)$ to be the polar equation of the curve sought. Then will $r'=\phi(\theta+\pi)$ and since by the condition of the question $r+r'=\text{constant}=2a$, we have the following equation for determining the form of ϕ,

$$\phi(\theta)-\phi(\theta+\pi)=2a.$$

Suppose now $z=\frac{\theta}{\pi}$, or $\theta=\pi z$, and this equation becomes

$$\phi(\pi z)+\phi\{\pi.(z+1)\}=2a.$$

This is in fact, an equation of differences; for, if we suppose $\phi(\pi z)=u_z$, we get $\phi\{\pi.(z+1)\}=u_{z+1}$, and

$$u_z+u_{z+1}=2a.$$

Now, this equation deprived of its last term $2a$, is evidently satisfied by $\cos\pi z$, because

$$\cos\pi z+\cos\pi(z+1)=0$$

Hence the complete integral is

$$u_z = a + C \,.\, \cos \pi z,$$

C being an arbitrary constant, or rather according to the remark in Appendix, Art. 368. an arbitrary function of $\cos 2 \pi z$, or in general, any quantity which does not change by the substitution of $z+1$ for z.

Hence, restoring the original denominations

$$r = a + \cos \theta \,.\, f(\cos 2 \theta),$$

where under $f(\cos 2 \theta)$ are comprehended all functions of θ, whether algebraic or transcendental, which do not change when $\theta + \pi$ is substituted for θ. Thus if $f(\cos 2 \theta) = 0$, $r = a$ the equation of the circle. If $f(\cos 2 \theta) = b$, we have

$$r = a + b \,.\, \cos \theta,$$

which represents a curve similar to that in the figure, whose algebraic equation is

$$(x^2 - b x + y^2)^2 = a^2 (x^2 + y^2).$$

(2). Instead of supposing the sum of the parts AC, CB, (Part III. Fig. 1.) constant, let their rectangle be invariable. Required the class of curves possessed of this property.

Retaining the same denominations, we have now

$$r r' = a^2, \quad \text{or} \quad \phi(\theta) \,.\, \phi(\pi + \theta) = a^2$$

which treated in the same manner, by supposing $\theta = \pi z$ and $\phi(\theta) = u_z$ gives

$$u_z \,.\, u_{z+1} = a^2$$

which is evidently satisfied by

$$u_z = a \,.\, C^{\cos \pi z},$$

because $\cos \pi (z+1) = - \cos \pi z$. This then, containing an arbitrary constant C is the complete integral, and as before replacing C by an arbitrary function of $\cos 2 \pi z$, and restoring the value of z.

$$r = \phi(\theta) = a \,.\, f(\cos 2 \theta)^{\cos \theta}.$$

Thus the oval whose equation is $r = a\, e^{\cos\theta}$ satisfies the condition, but it is also satisfied by an infinite variety of algebraic curves, as we shall now shew.

We have already remarked that $f(\cos 2\,\theta)$ may be any function which does not change by writing $\pi + \theta$ for θ. Let $F(\cos 2\,\theta)$ be any other such function, and it is evident that the expression

$$\frac{F(\cos 2\,\theta) + \cos\theta}{F(\cos 2\,\theta) - \cos\theta},$$

by that substitution has its numerator and denominator inverted, because $\cos(\pi + \theta) = -\cos\theta$, hence if this expression be raised to any power such as $\cos\theta$, or $\frac{n}{\cos\theta}$, &c. whose sign only is changed by the substitution, the function so produced will remain unaltered. We are at liberty then to suppose $(f \cos 2\,\theta)$ of the form

$$\left\{\frac{F(\cos 2\,\theta) + \cos\theta}{F(\cos 2\,\theta) - \cos\theta}\right\}^{\frac{n}{\cos\theta}}$$

which value being written for $f \cos 2\,\theta$ in the expression of r above found gives

$$r = a\left\{\frac{F(\cos 2\,\theta) + \cos\theta}{F(\cos 2\,\theta) - \cos\theta}\right\}^{n}$$

which always gives algebraic curves by assigning an algebraic form to the function F. Thus, if we suppose $F(\cos 2\,\theta) = 1$ and $n = -\frac{m}{2}$, we get for the equation of the curve

$$r = a\,.\,(\tan \tfrac{1}{2}\,\theta)^{m}.$$

If we suppose

$$F(\cos 2\,\theta) = \frac{\sqrt{(\alpha^2 - \beta^2\,.\,\sin\theta^2)}}{\beta},$$

which evidently remains unaltered by the substitution of $\pi + \theta$ for θ, we get

$$r = \frac{a}{\sqrt{(a^2-\beta^2)}} \{ \beta \, . \cos\theta + \sqrt{(a^2 - \beta^2 \, . \sin\theta^2)} \}$$

which is the equation of a circle, the pole round which the angle θ is reckoned being any point however situated. In fact this property is proved to belong to the circle in the 35th and 36th propositions of the third book of Euclid.

(3). The conchoid is produced by the revolution of a straight line round a fixed pole, one of its points being subjected to move in a straight line, while the other describes the curve. To find a curve, or class of curves, susceptible of being described like the conchoid, with this difference, however, that instead of the *directrix* being a straight line, it shall be another branch of the curve itself to be found.

The straight line CM (Part III. Fig. 2.) is to revolve about C, so that MM' cut off between the intersections M and M', with the two branches AM and BM', shall be constant and $= a$. Draw MP, $M'P'$ perpendicular to CP'. Let $CM = r$, $CM' = r'$, $CP = x$, $PM = y$, $CP' = x'$, $CM' = y$.

Therefore $r' - r = a$, or $r' = a + r$.

Assume $y = \phi \sqrt{(x^2 + y^2)} = \phi\,(r)$ for the equation of the curve. Then, since the same equation is common to both branches, we must have also $y' = \phi\,(r')$.

Now, by similar triangles CPM, $CP'M'$, we have

$$y' = y \, . \frac{r'}{r},$$

Therefore

$$\frac{\phi\,(r')}{r'} = \frac{\phi\,(r)}{r}, \text{ or } \frac{\phi\,(a+r)}{a+r} = \frac{\phi\,(r)}{r},$$

an equation from which the form of ϕ is to be determined. Let $r = az$, then $a + r = a(z+1)$ and if we suppose $\frac{\phi\,(a\,z)}{a\,z} = u_z$ we have

$$u_{z+1} = u_z,$$

whence, $u_z = f \cos 2\pi z$, and therefore

$$\frac{y}{r} = f\left(\cos 2\pi . \frac{r}{a}\right).$$

Thus if we suppose $f\left(\cos 2\pi\frac{r}{a}\right) = \sin 2\pi\frac{r}{a}$, and observe that $\frac{y}{r} = \sin MCP = \sin\theta$, we have

$$\sin\theta = \sin 2\pi\frac{r}{a}, \text{ or } r = a . \frac{\theta}{2\pi},$$

the equation to the spiral of Archimedes, in which the next inferior convolution of the curve, supplies the place of another branch. In fact, the preceding analysis does not take in the condition that M and M' should lie in *different branches*, but the following solution will apply to the strict letter of the enunciation.

Let CM be regarded as a *negative* value of r, answering to one half revolution *more* of the line CM in which case the geometrical equation $r' - r = 2a$ will be represented in analytical language by $\phi(\theta) + \phi(\pi + \theta) = 2a$, and thus the equation of the problem (1) resolves this case, provided we select only such curves as have the property described.

Thus in the result

$$r = a + \cos\theta . f(\cos 2\theta),$$

when $\theta < \pi$, r must be positive, and when greater, negative, or at least, if in any part of the variation of θ between o and π, r becomes negative, it must be negative in a higher degree in the corresponding part of its variation between π and 2π. Such a curve is

$$r = a\left(1 + \frac{1}{\sin\theta}\right), \text{ or } (x^2 + y^2)(y - a)^2 = a^2 y^2$$

and an indefinite variety of algebraic curves, among which are some which satisfy the geometrical property $r' + r = 2a$, with one pair of branches, at the same time that $r'-r=2a$ is satisfied by another pair.

(4). Required the nature of the curve AMM' (Part III. Fig. 3.) when the line AMM' revolving round A has the sum of the m^{th} powers of the segments AM, AM' constant, or yet more generally, when one segment $AM'=r'$ is any assigned function $\alpha(r)$ of the other $AM=r$.

As before, suppose $y=\phi(r)$, then we must have $y'=\phi(r')$. Now since $r'=\alpha(r)$ and by similar triangles $\frac{y'}{r'}=\frac{y}{r}$, we have

$$\frac{\phi\alpha(r)}{\alpha(r)}=\frac{\phi(r)}{r},$$

an equation for determining the form of the function ϕ. Now it is evident that if any function $f(r)$ can be found which does not change when $\alpha(r)$ is written for r, the equation

$$\frac{\phi(r)}{r}=f(r), \text{ or } \phi(r)=r\,.f(r)$$

satisfies the above, now Laplace's method explained in Appendix Art. 398.) affords a general solution of the equation

$$f\{\alpha(r)\}=f(r),$$

and thus the complete solution of the problem may be had. In the particular case proposed, however, the function $\alpha(r)$ is one of a very singular class of functions, which render the application of Laplace's method extremely delicate, and moreover unnecessary. It will be observed that since $r^m+r'^m=a^m$, therefore

$$r'=\alpha(r)=\sqrt[m]{(a^m-r^m)}.$$

Hence we have $\alpha(r')=\alpha\alpha(r)=\alpha^2(r)=\sqrt[m]{a^m-(a^m-r^m)}=$ $=r$. The function in question is therefore one of those which may properly be called *periodic functions*, under which may be comprehended all which satisfy such equations as

$\alpha^2(r) = r$, $\alpha^3(r) = r$, &c. and which are possessed of a variety of the most elegant and useful properties, which this is not the place to enumerate. However, it is here to our purpose to remark, that any symmetrical function of r and $\alpha(r)$ has the property we wish, viz. that it does not change by the substitution of $\alpha(r)$ for r, because r becoming $\alpha(r)$, and $\alpha(r)$, $\alpha^2(r)$ or r, these quantities only change places by this substitution, which, as they are similarly involved, does not alter the value of the function. Let us for instance suppose (in the proposed case)

$$f(r) = \frac{r}{b^2}\alpha(r) = r.\frac{\sqrt[m]{(a^m - r^m)}}{b^2},$$

and we find

$$b^2 y = r^2 . \sqrt[m]{(a^m - r^m)}.$$

Similarly, the equations

$$b . \left(\frac{y}{r}\right)^n = r + \sqrt[m]{(a^m - r^m)},$$

and

$$b y = a^m r^{m+1} - r^{2m+1}, \text{ \&c.}$$

may be shewn to satisfy the condition of the problem.

The cases where $r + r' = a$, $r^m + r'^m = a^m$, $r^2 . r' = a^3$, were proposed long ago in the Leipsic Acts by John Bernoulli, at the same time with the celebrated problem of the Brachystochrone, as a defiance to the mathematical world; but it does not appear that their real object, or the point where their difficulty rested was perceived, either by their proposer, or by any one of the numerous and eminent geometers who gave solutions of them. The attention of mathematicians being, however, immediately occupied by the extraordinary controversy of the Bernoullis, and the discoveries of James relative to Isoperimetrical problems, to which that of the Brachystochrone gave rise, the present questions were allowed to sink into a degree of oblivion, from which, it will not be amiss if we attempt to rescue them. They were proposed, as J. Bernoulli expressly states, with a view of calling the attention of geometers, to a case where the Cartesian methods of reducing the conditions of a geometrical problem to an equation entirely failed, while at the same time the differential calculus afforded no assistance; thus presenting a difficulty which seemed quite unexpected, and of a different kind from any which had yet been felt. This difficulty is in fact the solution of a *functional equation*, or the determination of an unknown function from an equation, such as those of (Prob 4.) where it enters under more forms than one, but Leibnitz, L'Hopital, Newton, and Jas. Bernoulli, all of whom resolved the problems *, were contented with the first particular forms of the unknown function which presented themselves, without attempting to discover any direct process by which the functional equation might be resolved, and which in cases of a little greater complexity, constitutes the only analytical difficulty to be surmounted. It is rather surprising that this was not observed by Jas. Bernoulli, who distinctly reduces the problem where $r . r' = a^2$ to the determination of the form of an equation, which shall remain unaltered by certain changes made among the variables it contains. His solution of the problem which requires that $r^2 . r'$ shall be invariable, is erroneous, and for a very obvious reason, the neglect of the constant in the integral $\int \frac{dx}{x . \log x}$, and indeed he himself calls his solution "dubiæ et suspectæ veritatis."

The subject was resumed by Clairaut in 1734, in a memoir communicated to the Academy of Paris, in which he resolves several problems by a method professedly grounded on, and equivalent to that employed by Newton in the solution of Bernoulli's problem above-mentioned, developed, however, with great ingenuity, and applied in particular to one problem of no ordinary difficulty, "To determine the nature of a curve, such that the intersection of any two of its tangents which include a given angle, shall always be found in a given curve." It was in the solution of this problem, that Clairaut first discovered the class of differential equations treated of in (Art. 270. of the text) whose general form is

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* Newton's solution, though extremely elegant, turns on a peculiarity in the case proposed. It is an application of one of his own discoveries respecting the sums of the powers of roots of an equation, and a very happy one, but the question seems not to have struck him in the light we are now considering it.

$$y - x\frac{dy}{dx} = f\left(\frac{dy}{dx}\right),$$

and which has procured him with some, the unmerited praise of having first discovered the particular solutions of differential equations; our countryman Brook Taylor in 1715 having deduced the same conclusion in the same manner, and made the same observation on it*, in integrating the equation

$$\left(y - x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2,$$

which is evidently a particular case of Clairaut's general form. Since that period many geometers have occupied themselves with the solution of problems of the kind in question, remarkable examples of which may be found in the writings of Euler, Voss, and Biot.

(5). To determine a class of curves possessed of the following property: viz. that supposing a system of lines, n in number, originating in a fixed point, and terminating in the curve, to revolve about this point, making always equal angles with each other, their sum shall be invariable.

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* He differentiated, and obtained an equation composed of two factors, one of which leads to a final result free from differentials, but containing no arbitrary constant, which is, says he "Singularis quædam solutio problematis." See a clear and impartial statement of the whole in Lagrange's Leçons sur le Calcul des fonctions. Lect. xvii.

The angle made by one of them (r) with some fixed line being θ, those made by the others will be respectively

$$\theta + \frac{2\pi}{n},\ \theta + \frac{4\pi}{n},\ \ldots\ \theta + \frac{2(n-1)\pi}{n}.$$

Consequently, if $r = \phi(\theta)$ we must have

$$\phi(\theta) + \phi\left(\theta + \frac{2\pi}{n}\right) + \ldots\ \phi\left(\theta + \frac{2(n-1)\pi}{n}\right) = n\,a,$$

$n\,a$ being some given quantity. Suppose $\theta = \frac{2\pi z}{n}$, then $\theta + \frac{2\pi}{n} = \frac{2\pi.(z+1)}{n}$, and so on. If then we make $\phi\left(\frac{2\pi z}{n}\right) = u_z$, we have

$$u_z + u_{z+1} + \ldots u_{z+n-1} = n\,a.$$

The several particular integrals of this equation deprived of its constant term are

$$\cos\frac{2\pi z}{n},\ \cos\frac{4\pi z}{n},\ldots\cos\frac{2(n-1)\pi z}{n}$$

for the sum of the series

$$\cos A + \cos(A+B) + \ldots\cos\{A+(n-1)B\}$$

being

$$\cos\left\{A + \frac{n-1}{2}B\right\}.\frac{\sin\left(\frac{n}{2}B\right)}{\sin\left(\frac{1}{2}B\right)}$$

vanishes whenever $\frac{nB}{2}$ is a multiple of π. Now, if either of the above cosines be put for u_z in the expression

$$u_z + u_{z+1} + \ldots u_{z+n-1},$$

a series of this form will arise. These functions then severally satisfy the equation

$$u_z + u_{z+1} + \ldots u_{z+n-1} = 0.$$

Of course the complete value of u_z in the proposed, or of $\phi(\theta)$ is

$$u_z = a + C_1 \cdot \cos \frac{2\pi z}{n} + C_2 \cdot \cos \frac{4\pi z}{n} +$$

$$\ldots C_{n-1} \cdot \cos \frac{(2n-2)\pi z}{n};$$

but $\theta = \frac{2\pi z}{n}$, also C_1, C_2, &c. may be arbitrary functions of $\cos 2\pi z$, that is, of $\cos n\theta$.

Let them be represented by

$$f_1(\cos n\theta), \quad f_2(\cos n\theta), \ \&c.$$

then

$$\phi(\theta) = r = a + \cos\theta \,.\, f_1(\cos n\theta) + \cos 2\theta \,.\, f_2(\cos n\theta) +$$

$$\ldots\ldots + \cos(n-1)\theta \,.\, f_{n-1}(\cos n\theta).$$

It is easy then to assign an unlimited variety of algebraic curves which answer the condition. The simplest is that whose equation is

$$r = a + b \cdot \cos\theta,$$

which we have already noticed in the case of $n = 2$, and it is a very remarkable property of this curve that it answers for every value of n. In other words: "In the curve whose "equation is

$$(x^2 - bx + y^2)^2 = a^2(x^2 + y^2)$$

"if a system of any number of radii terminating in the curve, "and making equal angles with each other, be made to revolve "round the origin of the co-ordinates, their sum will be "invariable throughout the whole extent of the curve."

(6). In the parabola, any straight line being drawn through the focus meeting the curve both ways, the tangents at its two extremities include a right angle. To what class of curves does this property, viz. that two tangents so drawn shall include a *given* angle, belong.

Let P and P' (Part III. Fig. 4.) be the two points in the curve, PTQ, $P'QT'$, the tangents; MP, $M'P'$, ordinates. $SM=x$, $SM'=x'$, $SP=r$, $MSP=\theta$, $MP=y$, $M'P'=y.'$

Then, since $\angle PQP'=QTT'+QT'T=PTS+P'T'S$, and that tan $PTS=\frac{dy}{dx}$, and tan $P'T'S=-\frac{dy'}{dx'}$, because it lies on the contrary side of the axis of the abscissæ, therefore if we suppose $PQP'=A$, we have

$$\tan A=\frac{\frac{dy}{dx}-\frac{dy'}{dx'}}{1+\frac{dy}{dx}\cdot\frac{dy'}{dx'}} \qquad (a).$$

Now, the points P, P', lying both in the curve, $\frac{dy'}{dx'}$ must be the same function of $\theta+\pi$ that $\frac{dy}{dx}$ is of θ, because the same equations belong to both, hence if we suppose

$$\frac{dy}{dx}=\phi(\theta),\text{ we have }\frac{dy'}{dx'}=\phi(\pi+\theta)$$

and supposing $z=\frac{\theta}{\pi}$ and $u_z=\phi(\theta)$ the equation (a) becomes

$$\tan A=\frac{u_z-u_{z+1}}{1+u_z u_{z+1}},$$

or

$$u_{z+1}u_z-\text{cotan}\,A\,.\,(u_{z+1}-u_z)+1=0.$$

This equation we have already integrated (Sect. 4. 20.) and by applying the formula there obtained, we get

$$u_z=\tan(Az+\tan^{-1}C)$$

that is, C being replaced as before by an arbitrary function of $\cos 2\pi z$,

$$\phi(\theta)=\tan\left\{\frac{A}{\pi}\theta+\tan^{-1}f(\cos 2\theta)\right\}; \qquad (b).$$

Now we have $y = r \,.\, \sin\theta$, and $x = r \,.\, \cos\theta$, which gives

$$\frac{dy}{dx} = \frac{dr \,.\, \cos\theta + r\,d\theta \,.\, \cos\theta}{dr \,.\, \cos\theta - r\,d\theta \,.\, \sin\theta}$$

which put equal to $\phi(\theta)$ gives a differential equation between r and θ, viz.

$$\frac{dr}{r} = \frac{\cos\theta + \phi(\theta) \,.\, \sin\theta}{\phi(\theta) \,.\, \cos\theta - \sin\theta}\, d\theta.$$

Now the equation (b) gives

$$\phi(\theta) = \frac{\tan\frac{A}{\pi}\theta + f(\cos 2\theta)}{1 - \tan\frac{A}{\pi}\theta \,.\, f(\cos 2\theta)}.$$

This value of $\phi(\theta)$ substituted in the differential equation gives after all reductions

$$\log\frac{r}{a} = \int d\theta \,.\, \cot\left\{\tan^{-1} f(\cos 2\theta) - \frac{\pi - A}{\pi}\theta\right\}$$

which is the polar equation of the curve sought. Suppose, to take a particular case, $f(\cos 2\theta) = 0$, and we have

$$\log\frac{r}{a} = \int d\theta \,.\, \cot\left(\frac{A-\pi}{\pi}\theta\right)$$

$$= \frac{\pi}{A-\pi} \,.\, \log\sin\left(\frac{A-\pi}{\pi}\theta\right)$$

and consequently

$$r = a \,.\, \left\{\sin\frac{A-\pi}{\pi}\theta\right\}^{\frac{\pi}{A-\pi}}$$

which always gives algebraic curves when the angle A included between the tangents is commensurate to the whole circumference. Thus if $A = \frac{\pi}{2}$, we have $\frac{A-\pi}{\pi} = -\frac{1}{2}$, and

$$r = \frac{a}{\left(\sin\frac{\theta}{2}\right)^2},$$

the equation of a parabola, the origin of the co-ordinates being in the focus.

If $A = \pi$, or the tangents at opposite extremities of the line are parallel, the general equation gives

$$\log \frac{r}{a} = \int d\theta \, . f(\cos 2\,\theta),$$

because the form of the function f being arbitrary we may replace cot $\tan^{-1} f(\cos 2\,\theta)$ by $f(\cos 2\,\theta)$ without infringing on the generality of the equation. The equation in this form includes, 1st the logarithmic spiral, by making $f(\cos 2\,\theta) = b$, and 2dly all curves consisting of four similar parts arranged in the manner of quadrants round a center. The reader will find other solutions of this problem by Messrs. Wallace and Ivory in Leybourne's Repository, (New Series Quest. 172.) which are well worthy his attention.

Euler, and more lately Ivory, in another solution of this problem (Thomson's Annals of Philosophy, Oct. 1816.) have shewn that it admits no solution unless in the cases when the tangents are parallel or include a right angle, but this limitation arises from the assumed condition that the straight line $P S P'$ shall not cut the curve in more than two points. If we admit, however, that the points P, P', may lie in different branches of the same curve, our solution above will apply. Mr. Ivory's final equation is (in the general case) a functional equation of the form $F(\phi\, x, \phi\, \alpha\, x) = 0$, in which $\alpha^2 x = x$, and where the function F is not symmetrical. This he properly remarks is an impossible equation. If, however, we admit that different values of the function ϕ arising from radicals, &c. involved in it, may be used in $\phi\, x$ and in $\phi\, \alpha\, x$, the impossibility vanishes and the equation may be satisfied. This remark on the nature of such equations abstractly considered, is due to Mr. Babbage. The problem just solved affords an illustration of its geometrical signification.

(7). In the parabola, the two tangents PQ, $P'Q$, (See Part III. Fig. 4.) always intersect in the directrix. To generalize this property, or to find a class of curves such that tangents drawn at opposite extremities of any line PSP' passing through a given point S shall always meet in a straight line given in position.

Let AQ be the straight line, and let SA perpendicular to it be taken as the axis of the x, then retaining the construction and denominations of the last problem we have, supposing $SA = X$ and $AQ = Y$.

$$ST = x - y \cdot \frac{dx}{dy}; \quad AT = X - x + y \cdot \frac{dx}{dy},$$

Consequently $AQ = -\dfrac{dy}{dx} \cdot AT$, or

$$Y = -\frac{dy}{dx}(X - x) - y; \qquad (a).$$

Similarly we should obtain

$$Y = -\frac{dy'}{dx'}(X - x') - y',$$

and equating these we find

$$X = \frac{\left(x'\dfrac{dy'}{dx'} - x\dfrac{dy}{dx}\right) - (y' - y)}{\dfrac{dy'}{dx'} - \dfrac{dy}{dx}},$$

Now the condition of the problem requires that this shall be constant. Denoting it then by $2a$, we get

$$2a\left(\frac{dy'}{dx'} - \frac{dy}{dx}\right) = x'\frac{dy'}{dx'} - x\frac{dy}{dx} - (y' - y),$$

or,

$$(2a - x') \cdot \frac{dy'}{dx'} + y' = (2a - x) \cdot \frac{dy}{dx} + y.$$

Suppose now

$$(2\,a-x)\,.\,\frac{d\,y}{d\,x}+y=u_z,$$

z being some quantity, which changes to $z+1$ when x, y, change to x', y', then we shall have

$$u_{z+1}=u_z, \text{ or } \Delta\,u_z=0$$

and of course

$$u_z=\text{constant}.$$

But since the points P, P' are so related that (by similar triangles)

$$\frac{y'}{x'}=\frac{y}{x},$$

Therefore the function $\frac{y}{x}$ does not change by the change of x, y, to x', y', or of z to $z+1$, hence the constant in the above equation may be a function of $\frac{y}{x}$ and denoting this by $f\left(\frac{y}{x}\right)$, we have

$$(2\,a-x)\,.\,\frac{d\,y}{d\,x}+y=f\left(\frac{y}{x}\right); \qquad (b).$$

This equation is integrable at once, by putting $\frac{y}{x}=u$ which gives

$$\frac{d\,x}{x\,(2\,a-x)}+\frac{d\,u}{2\,a\,u-f(u)}=0,$$

whence

$$\log\frac{x}{2\,a-x}+2\,a\,.\int\frac{d\,u}{2\,a\,u-f(u)})=\log c, \qquad (c).$$

Suppose, for instance, $f(u) = -\frac{2a}{u}$; then we have

$$\log \frac{x}{2a - x} + \frac{1}{2} \log (u^2 + 1) = \log c,$$

$$\frac{x^2}{(2a - x)^2}(u^2 + 1) = c^2,$$

or, replacing the value of u and reducing

$$x^2 + y^2 = c^2 (2a - x)^2,$$

which is the general equation of a conic section about the focus, and it is easily seen that the straight line in which the tangents meet, is no other than what is called the directrix in some treatises on conic sections. The conic section itself will be an ellipse, parabola or hyperbola, according as the angle between the tangents is acute, right, or obtuse.

The conic sections also satisfy the conditions of the problem in another way, which, taken in conjunction with what has just been proved, may be considered as affording a very elegant property of these curves.

If we assume $f(u) = + \frac{2a}{u}$, we find

$$\log \frac{x}{2a - x} + \frac{1}{2} \log (u^2 - 1) = \log c,$$

which reduced, as before, gives

$$y^2 - x^2 = c^2 (2a - x)^2.$$

This is likewise the general equation of the conic sections, but whereas in the former case the origin of the co-ordinates was in the focus, and the straight line in which the tangents meet, the directrix; in this it is just the reverse. The origin of the co-ordinates being now in the intersection of the axis with the directrix, and the tangents meeting always in a line drawn through the focus at right angles to the axis (that is, in the *latus rectum* indefinitely produced.) The reader may consult the Mathematical Repository, iii. p. 39. Quest. 267. for another solution of this question by Mr. Lowry.

The final equation (*c*) of this problem presents a peculiarity which ought to be remarked. It is evident that by properly assuming the form of the function $f(u)$, the integral in the first member may be made to have any form we please, and therefore the equation may express any conceivable relation between x and u, or x and y. Yet it is equally obvious, that it is not every possible curve which satisfies the conditions of the problem, but only those of a certain class. The function $f(u)$ then cannot be absolutely arbitrary, but must be subject to certain limitations. Nevertheless, if we recur to the equation (*b*) in which the function f was first introduced, we see no obvious reason for admitting any limitation of its generality; for the first member is merely the analytical expression for the distance AQ (which is easily proved) and as the second remains unaltered so long as the ratio of x to y remains the same, or the point P' lies in the same straight line with P and S, this equation appears to be nothing more than a mere translation of the condition of the question into algebraic language. The elucidation of this delicate point depends upon the theory of eliminations. Whatever may be the nature of a curve, if we put $\frac{y}{x} = u$, we may eliminate either y or x between this equation and that of the curve, and thus both x and y are expressible in functions of u. For the same reason $\frac{dy}{dx}$ is so expressible, and therefore the first member of (*b*) is *in all cases* reducible, by the theory of elimination to a function of u or $\frac{y}{x}$. But it does not thence follow that every curve possesses the property in question, for this function may have several values, and in all the excepted cases actually has so. It appears therefore that we are not at liberty, in assuming $f(u)$, to substitute any form susceptible of more than one value, but provided this limitation be attended to, it is in all other respects arbitrary. If we put therefore for f, only rational functions, we are always sure to arrive at satisfactory solutions, but in all other cases it is indispensably necessary to *try* the solution obtained before it can be relied on. Considerations of this, or a similar kind, apply to most problems of the nature now under examination, and will obviate any objections arising from the necessity of limiting functions which seem at their first introduction perfectly arbitrary.

(8). Required the class of curves which possess the following property, that any ordinate PM (Part III. fig. 5.) being erected, and normal MP_1 drawn, and at the foot of this normal, another ordinate P_1M_1 erected, and another normal M_1P_2 drawn, and so on, then the subnormals PP_1, P_1P_2, P_2P_3, &c. shall all be equal to each other in the same series, however they may differ in different series, arising from a different position of the first ordinate.

Let the abscissæ AP &c. &c. be represented by x_0, x_1, x_2, &c. the general term x_z being some certain unknown function of the rank it holds in the series or of z, and let the ordinates be y_0, y_1, y_2, &c. Then the subnormal

$$PP_1 = y_0 \cdot \frac{dy_0}{dx_0}; \quad P_1P_2 = y_1 \cdot \frac{dy_1}{dx_1}; \text{ and so on}$$

hence,

$$x_1 = x_0 + y_0 \cdot \frac{dy_0}{dx_0}; \quad x_2 = x_1 + y_1 \cdot \frac{dy_1}{dx_1},$$

and, in general, whatever be z.

$$x_{z+1} = x_z + y_z \cdot \frac{dy_z}{dx_z},$$

or

$$\Delta x_z = y_z \cdot \frac{dy_z}{dx_z}.$$

Now, by the condition of the problem, the series of subnormals

$$y_0 \frac{dy_0}{dx_0}, \quad y_1 \frac{dy_1}{dx_1}, \text{ \&c.}$$

are all equal, therefore this equation is to be integrated on the hypothesis of $y_z \cdot \frac{d y_z}{d x_z}$ being invariable by the change of z to $z+1$. It may therefore be supposed an arbitrary function of $\cos 2 \pi z$ or (which comes to the same thing) of $\tan \pi z$, which let us call Z, then,

$$\Delta x_z = Z, \text{ and } \Delta Z = 0,$$

whence integrating

$$x_z = Z z + Z'$$

Z' being another quantity of the same kind, or another arbitrary function of $\tan \pi z$. But we have

$$y_z \cdot \frac{d y_z}{d x_z} = Z,$$

and from these equations, we have only to eliminate z and we get a differential equation between x_z and y_z (or as we will now call them, x and y) expressing the nature of the curve. Now this is easy: for since both Z and Z' are arbitrary functions of $\tan \pi z$, we may suppose one an arbitrary function of the other.

Let then

$$Z' = f(Z),$$

and our equations become

$$x = Z z + f(Z),$$

$$y \frac{d y}{d x} = Z.$$

The first gives

$$z = \frac{x - f(Z)}{Z},$$

or for Z substituting its value given by the second

$$z = \frac{d x}{y d y} \left\{ x - f\left(\frac{y d y}{d x} \right) \right\}.$$

Now Z is an arbitrary function of $\tan \pi z$; let then

$$Z = F(\tan \pi z)$$

In this equation for Z write its value $y\frac{dy}{dx}$, and for z its value just determined, and we get

$$y\frac{dy}{dx} = F\left\{\tan\pi\left(\frac{x - f\left(y\frac{dy}{dx}\right)}{y\frac{dy}{dx}}\right)\right\}$$

the differential equation of the curve required, which we see involves two arbitrary functions. In the particular case where F denotes an absolute constant c, we have

$$y\,dy = c\,dx, \quad y^2 = 2cx + c'$$

the equation of a parabola.

(9). At any point of a certain curve, let a normal be drawn and an ordinate erected, let a second ordinate be taken equal to the first subnormal, and let a second normal be drawn. Required the nature of the curve, that the second subnormal so determined shall be equal to the first ordinate, in other words, that in any part of the curve constructing a triangle whose hypothenuse is the normal, and sides the ordinate and subnormal, if this be turned into a subcontrary position and adjusted to fit the curve, its hypothenuse shall still be a normal.

The subnormal being $y\frac{dy}{dx}$, we have the second ordinate y' equal to $y\frac{dy}{dx}$. Suppose now we take

$$y\frac{dy}{dx} = \phi(y).$$

Then will the second subnormal, or

$$y'\frac{dy'}{dx'} = \phi(y') = \phi\left(\frac{y\,dy}{dx}\right)$$

$$= \phi\ \{\phi(y)\} = \phi^2(y).$$

But by the condition of the question this is equal to the first ordinate y, hence

$$\phi^2(y) = y,$$

a functional equation from which the nature of the function ϕ is to be determined.

Take

$$y = u_z, \text{ and } \phi(y) = u_{z+1}$$

then we have

$$u_{z+1} = \phi u_z$$

but the proposed equation gives

$$\phi^2(y) = (\phi u_{z+1} =) y = u_z.$$

Hence we have by subtraction

$$\phi u_{z+1} - \phi u_z = -(u_{z+1} - u_z)$$

or,

$$\Delta \phi u_z = - \Delta u_z$$

whence,

$$\phi u_z + u_z = \text{constant}.$$

But the two equations

$$u_z = \phi u_{z+1}, \text{ and } \phi u_z = u_{z+1}$$

give by cross multiplication

$$u_z \cdot \phi u_z = u_{z+1} \cdot \phi u_{z+1}.$$

This function, therefore, does not change by the change of z to $z+1$, and the constant may therefore be an arbitrary function of it, so that

$$u_z + \phi u_z = f(u_z \cdot \phi u_z),$$

or

$$y + \phi(y) = f\{y \cdot \phi(y)\}$$

from which $\phi(y)$ may be found by the ordinary analysis, any form we please being assigned to f.

Now, integrating the equation,

$$\frac{y\,dy}{dx} = \phi(y)$$

we find

$$x = \int \frac{y\,dy}{\phi(y)},$$

the equation of the curve. Thus if we suppose $f(y \,.\, \phi y) = a + by \,.\, \varphi(y)$ we have

$$y + \phi(y) = a + by \,.\, \phi(y)$$

$$\phi(y) = \frac{y - a}{by - 1}$$

$$x = \int \frac{by^2 - y}{y - a}\,dy.$$

If $a = b = 1$, we have $x = \int y\,dy = \frac{y^2}{2}$, the equation of a common parabola, which therefore satisfies the problem, as does also the cubic parabola.

(10). Required the nature of the curve, such that an ordinate being drawn to any point, and also a radius of curvature, a second point may always be found so related to the first, that the ordinate at the second point shall be equal to the radius of curvature at the first, and the radius of curvature at the second equal to the ordinate at the first.

This problem, by supposing

$$\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-\,dx\,d^2y} = \phi(y)$$

leads to precisely the same equation

$$\phi^2(y) = y$$

from which determining $\phi(y)$, the differential equation above given suffices to determine the curve.

If we suppose $\phi(y) = \frac{a^2}{y}$ which evidently satisfies the condition, we have

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{d^2 y}{dx^2} \cdot \frac{a^2}{y},$$

or, putting $\frac{dy}{dx} = p$,

$$y\,dy = \frac{a^2 p\,dp}{(1+p^2)^{\frac{3}{2}}}$$

$$y^2 - b = -\frac{2a^2}{\sqrt{(1+p^2)}},$$

whence, restoring the value of p

$$dx = \frac{(y^2 - b)}{\sqrt{\{4a^4 - (y^2 - b)^2\}}} \cdot dy$$

the equation of an elastic curve. And in the very same manner might the problem be resolved, if instead of the ordinate and radius of curvature, we had taken any other pair of lines expressible in terms of y, $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, &c. the equations to be resolved being, first, the functional equation of the second order $\phi^2(y) = y$, and secondly, a differential equation in which the function so determined is involved. It would be easy to multiply examples of this kind, but what we have already given will suffice to indicate the method to be pursued in more difficult enquiries of the same nature. They all lead to functional equations of greater or less complexity the solution of which is sometimes easily accomplished by reducing them to equations of differences, though more frequently by considerations peculiar to themselves. This problem and similiar ones may be resolved also by a consideration of the following kind. It is evident that since the radius of curvature at the second point is equal to the ordinate at the first, and the radius of curvature at the first to the ordinate at the second, these two functions (the radius of curvature and the ordinate regarded as functions of x the abscissa) must be such, that when the ordinate changes to the radius of curvature, the radius of curvature shall change to the ordinate, and therefore any symmetrical function of them will remain unchanged. Let f be the characteristic of such a function, then if R be the radius of curvature, it is evident that

$$f(y, R) = \text{constant}$$

will satisfy the condition, because this change being made the equation becomes

$$f(R, y) = \text{constant},$$

which by the peculiar form of f is identical with the former. For R now write its value and we get

$$f\left(y, \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx\,d^2y}\right) = a$$

for the equation of the curve. Thus if we assume

$$y \,.\, R = \text{constant}, \quad R = \frac{\text{const.}}{y}$$

which is the case just resolved.

(11). To determine a curve, such that a moveable point setting off from a given place shall return to the same place after being twice reflected against the curve (the angle of incidence being supposed equal to that of reflection) in whatever direction it first sets off.

Let PAP' (Part III. Fig. 6.) be the curve required, S the given point then if $SPP'S$ be the course of the moving point and tangents, &c. be drawn, we have, supposing $SM = x$, $MP = y$,

$$\tan\, SPM = \frac{x}{y}$$

$$\tan\, MPT = \frac{dx}{dy}, \text{ } y \text{ decreasing as } x \text{ increases,}$$

whence we get

$$\tan SPT = \tan (MPT - SPM)$$

$$= \frac{\frac{y}{x} - \frac{dy}{dx}}{1 + \frac{y}{x} \cdot \frac{dy}{dx}} = \tan VPH,$$

or, putting $\frac{dy}{dx} = p$,

$$\tan VPH = \frac{y - px}{x + py}$$

by the condition of the first reflexion. But we also have,

$$\tan VPM = -\frac{dx}{dy},$$

and therefore

$$\tan HPM = \tan (VPH - VPM)$$

$$= \frac{x(1 - p^2) + 2py}{y(1 - p^2) - 2px}.$$

But, if we draw $P'N$ perpendicular to PM produced, we have $\tan HPM = \tan P'PN = \frac{MM'}{PM + P'M'} = \frac{x - x'}{y - y'}$, y' being made negative because it lies on the other side of the axis. Hence we must have

$$y' - y = (x' - x) \cdot \frac{y(1 - p^2) - 2px}{x(1 - p^2) + 2py}; \qquad (a).$$

If we regard y, y', and x, x', as the successive values of two functions y and x, of a certain independent variable z, which changes to $z + 1$ when the point P changes to P', and if we suppose the fraction in the second member of this equation equal to P the equation becomes

$$\Delta y = P \Delta x; \qquad (b).$$

This equation alone is not sufficient, as it contains two unknown functions x and y. To obtain another we must consider that by what was before proved

$$P = \text{cotan}\, H P M,$$

and therefore P' (the value of P corresponding to $z + 1$) will be cotan $H P' M'$. But the condition of the question requires P, H, and P' to lie in one straight line, consequently $H P M = H P' M'$ and

$$P' = P,\ P' - P = \Delta P = 0,$$

which integrated gives

$$P = \text{const} = \text{funct} (\tan \pi z) = Z; \qquad (c).$$

Substituting this in (b) it is integrable at once and gives

$$y = Z \,.\, x + Z',$$

Z' being another invariable function of z. Now our object being only to obtain a final equation between x and y, we have only to eliminate the auxiliary variable z, with which we have no farther concern, between the equations

$$y = x \,.\, Z + Z'$$

$$Z = \frac{y(1 - p^2) - 2px}{x(1 - p^2) + 2py}$$

in which we shall succeed by the same artifice as in Prob. 8. of this section: for Z and Z' being both arbitrary functions of $\tan \pi z$, we may suppose one an arbitrary function of the other, or $Z' = f(Z)$ when we have

$$y = x \,.\, Z + f(Z)$$

in which substituting for Z its value

$$y = x \,.\, \frac{y(1 - p^2) - 2px}{x(1 - p^2) + 2py} + f\left(\frac{y(1 - p^2) - 2px}{x(1 - p^2) + 2py}\right)$$

which is the differential equation of the curve.

If the arbitrary function be assumed equal to zero, the equation is that of a circle; and if constant, that of a conic section about the focus. In the former case no integration is required.

If we consider attentively the above solution, we shall see that it contains the general principle on which that of all such problems depends. Let us therefore take up the question generally.

(12). To determine the nature of a curve from any property whatever connecting two of its points separated by a finite interval. (See the figure of the last problem.)

Whatever be the nature of the property, it must enable us when one of the points P is assumed, and the figure of the curve known, to determine the other, P', or the direction of the line PP', hence the condition of the problem always enables us to express the cotangent of the angle MPH by some function more or less complicated of the coordinates at the points P, P', and their differential coefficients such as

$$F\left(x, y, x', y', \frac{dy}{dx}, \frac{dy'}{dx'}, \text{\&c.}\right)$$

which we will call P.

Hence by a reasoning precisely similar to that of the last problem we get

$$y' - y = P \,.\, (x' - x)\,; \text{ or } \Delta y = P \,.\, \Delta x.$$

But since the property is common to the two points and connects them with each other, the same co-tangent determined by setting out from the point P' will be represented by P' and may be had from P by changing x, y, to x', y', and *vice versâ*. But cotan $M'\,P'\,H =$ cotan $M\,P\,H$, so that

$$P' = P, \text{ or } \Delta\, P = 0.$$

Consequently P must be regarded as invariable, and we have as before

$$y = Px + f(P) \ldots . (a).$$

The problem now divides itself into two cases, 1st when the function P involves only the co-ordinates of the point P and their differentials; and 2dly, when those of both the points concerned are combined in it. In the former case the equation (a) containing only x, y, and their differentials, is itself the final differential equation of the curve sought. In the latter, however, another process is requisite. The equations

$$y' - y = P(x' - x),$$

$$y = Px + f(P),$$

$$P' = P,$$

must be combined to eliminate both x' and y' and the resulting equation will express the nature of the curve.

(13). For instance, suppose the relation of the two points P and P' such that a line drawn perpendicular to the curve at either of them, shall pass through the other.

Here PH is a normal to the curve, and therefore,

$$P = \tan PHM = \frac{dx}{-dy} = \frac{-1}{p}, \quad \left(\text{putting } p = \frac{dy}{dx}\right)$$

Therefore the differential equation to the curve is

$$y = \frac{-x}{p} + \text{funct}\left(\frac{-1}{p}\right),$$

or

$$x + py = f(p).$$

If we take $f(p) = 0$, we have

$$x\,dx + y\,dy = 0,$$

$$x^2 + y^2 = c^2$$

the equation of a circle.

(14). Required the nature of the curve in which if MH be always taken equal to $\dfrac{xy + a(y' + b)}{a^2} \times y$; the points P, P' shall be convertible, that is, that $M'H$ shall equal $\dfrac{x'y' + a(y + b)}{a^2} \times y'$.

Since $P = \dfrac{MH}{MP}$, we have $P = \dfrac{xy + a(y' + b)}{a^2}$;

therefore the equation $P' = P$ gives

$$xy + ay' + ab = x'y' + ay + ab,$$

$$a(y' - y) = x'y' - xy.$$

Again the equation $y' - y = (x' - x) \,.\, P$ gives

$$a(y' - y) = (x' - x) \,.\, \frac{xy + a(y' + b)}{a}; \ldots.(a).$$

whence

$$a x'y' - axy = xx'y - x^2 y + ax'y' - axy' + ab(x' - x),$$

that is,

$$a(y' - y) \,.\, x = (xy + ab)(x' - x).$$

This combined with the equation (a) gives

$$y' = \frac{(a - x)(xy + ab)}{ax},$$

whence it is easy to obtain

$$x' = \frac{a^2 b}{xy + ab}.$$

These values give

$$P = \frac{x\,y + a\,b}{a\,x} = P',$$

which substituted in the equation

$$y = P\,x + f\,(P)$$

gives

$$y\,(a - x) = a\,b + a\,.\,f\,(P)$$

or simply changing the form of the function f,

$$y\,(a - x) = f\left(\frac{x\,y + a\,b}{a\,x}\right).$$

If we suppose the arbitrary function constant and equal to c^2, we get $y\,(a - x) = c^2$, the equation of an hyperbola.

◆

SECTION XI.

On Circulating Equations.

(1). To find an analytical function of x, which when x is made to pass in succession through all integer values from 0 to infinity, shall assume in regular periodical rotation the n values $a, b, c \ldots . k, a, b, c \ldots . k,$ &c.

Let $\alpha, \beta, \gamma \ldots . \nu$ be the n^{th} roots of unity, and let

$$S_x = \frac{\alpha^x + \beta^x + \gamma^x \ldots\ldots + \nu^x}{n},$$

then if we take

$$P_x = a\,.\,S_x + b\,.\,S_{x-1} + \ldots\ldots k\,.\,S_{x-n+1},$$

P_x will be the function required. The reason is obvious, when x is a multiple of n, the function S_x becomes unity by reason of the property of the roots of unity, demonstrated in works on Algebra, but in all other cases its value is zero.

Now some one of the values x, $x-1, \ldots . x-n+1$, is necessarily such a multiple, and x being made to vary from 0 to infinity, this one will be either x, $x-1, \ldots . x-n+1$, x, $x-1, \ldots .$ &c. in rotation, so that the function P_x will reduce itself to a, $b, \ldots . k$, &c. in the same succession, and is therefore the function required.

(2). To find a function P_x which shall assume in regular periodical succession the same values as those of n other given functions a_x, b_x, $c_x \ldots . k_x$. I say that

$$P_x = a_x . S_x + b_x . S_{x-1} + \ldots . k_x . S_{x-n+1}.$$

For, the values of P_x corresponding to $x = 0, 1, 2, \ldots . n-1, n, n+1, \ldots .$ &c. are respectively a_0, b_1, $c_2, \ldots . k_{n-1}$, a_n, b_{n+1}, &c.

The functions described in the above paragraphs are "circulating functions," and may be distinguished into those with either constant or variable coefficients, of which we have here instances.

(3). *Theorem.* Any symmetrical function of S_x, S_{x-1}, $\ldots . S_{x-n+1}$ is invariable. For when x varies from 0 to ∞, some one of the values of these expressions is always unity and the rest zero, and, the function in question being symmetrical, it is no matter in what order this takes place, the order of its elements being of no consequence. The function therefore has the same value, whichever of its elements becomes unity, the rest being all zero. That is, it is invariable by the variation of x from integer to integer values. Thus

$$S_x + S_{x-1} + S_{x-2} + \ldots . S_{x-n+1} = 1,$$
$$S_x . S_{x-1} . S_{x-2} \ldots . S_{x-n+1} = 0.$$

(4). Every symmetrical function of the circulating functions P_x, $P_{x-1}, \ldots . P_{x-n+1}$ is in like manner invariable, provided the coefficients of P_x, &c. be constant.

For every such function is a symmetrical function of S_x, S_{x-1}, ... S_{x-n+1}, as will appear if we consider that by reason of the properties of the roots of unity, we have $S_{x-n} = S_x$, $S_{x-n-1} = S_{x-1}$, &c. and consequently,

$$P_x = a \,.\, S_x + b \,.\, S_{x-1} + \ldots k \,.\, S_{x-n+1}$$

$$P_{x-1} = k \,.\, S_x + a \,.\, S_{x-1} + \ldots j \,.\, S_{x-n+1},$$

$$P_{x-2} = j \,.\, S_x + k \,.\, S_{x-1} + \ldots i \,.\, S_{x-n+1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$P_{x-n+1} = b \,.\, S_x + c \,.\, S_{x-1} + \ldots a \,.\, S_{x-n+1}.$$

Now any symmetrical function of the second members of these equations will obviously involve S_x, S_{x-1}, ... S_{x-n+1}, symmetrically, and will therefore be invariable. Its value also will evidently be equal to that of a function similarly composed of the coefficients a, b, &c. Thus for instance, if $P_x = a \,.\, S_x + b \,.\, S_{x-1}$, ($n$ being $= 2$) we have,

$$P_x \,.\, P_{x-1} = (a \,.\, S_x + b\, S_{x-1})\,(a \,.\, S_{x-1} + b \,.\, S_x)$$

$$= a\, b\, (S_x)^2 + a\, b\, (S_{x-1})^2 = a\, b\, (S_x{}^2 + S_{x-1}{}^2) = a\, b.$$

(5). For instances we may take

$$P_x \,.\, P_{x-1} \ldots P_{x-n+1} = a \,.\, b \,.\, c \ldots k;$$

$$P_x + P_{x-1} + \ldots P_{x-n+1} = a + b + c + \ldots k; \text{ \&c.}$$

(6). Circulating equations are those whose coefficients are circulating functions. To resolve them they must be reduced to others, whose coefficients are of the ordinary form. The preceding propositions enable us to do this. To begin with a simple instance, let

$$u_{x+2} \pm P_x \,.\, u_{x+1} \pm u_x = 0,$$

where P_x is a circulating function *of the second degree* (or in which $n = 2$, $P_x = a \,.\, S_x + b \,.\, S_{x-1}$).

Assume $u_x = v_x . \sqrt{(P_x)}$,

then will $u_{x+2} = v_{x+2}\sqrt{(P_{x+2})} = v_{x+2} . \sqrt{(P_x)}$, and the equation becomes

$$v_{x+2}\sqrt{(P_x)} \pm P_x . v_{x+1} . \sqrt{(P_{x+1})} \pm v_x\sqrt{P_x} = 0,$$

or

$$v_{x+2} \pm v_{x+1} . \sqrt{(P_x . P_{x+1})} \pm v_x = 0.$$

Now the coefficient of the second term $\sqrt{(P_x P_{x+1})}$ being a symmetrical function, is invariable by (4) and equal to $\sqrt{(a b)}$. We have, therefore,

$$v_{x+2} \pm \sqrt{(a b)} . v_{x+1} \pm v_x = 0,$$

an ordinary equation with constant coefficients, and easily integrated.

(7). A more general process however, and applicable to all circulating equations is to assume for the independent variable, a circulating function with unknown and variable coefficients, as in the following equation,

$$u_x + (a . S_x + b . S_{x-1})\, u_{x-1} + (\alpha . S_x + \beta . S_{x-1}) = 0.$$

Assume $u_x = A_x . S_x + B_x . S_{x-1}$, and we have by substitution and by Art. 5. of this Section,

$$\left.\begin{array}{ll} A_x . S_x & + B_x . S_{x-1} \\ + a\, B_{x-1}\, S_x & + b . A_{x-1}\, S_{x-1} \\ + \alpha . S_x & + \beta . S_{x-1} \end{array}\right\} = 0;$$

whence, equating to zero the coefficients of S_x and S_{x-1} separately, we obtain

$$A_x + a . B_{x-1} + \alpha = 0; \qquad B_x + b . A_{x-1} + \beta = 0.$$

Eliminating B_x, we find

$$A_x - a\, b . A_{x-2} + (\alpha - a\,\beta) = 0,$$

whence, A_x is found, equal to

$$(C\,.\,S_x + C'\,.\,S_{x-1})\,\{\sqrt{(a\,b)}\}^x - \frac{\alpha - a\,\beta}{1 - a\,b},$$

and thence we derive by the second of the above equations,

$$B_x = -\,b\,\{\sqrt{(a\,b)}\}^{x-1}(C'\,.\,S_x + C\,.\,S_{x-1}) - \frac{\alpha - a\,\beta}{1 - a\,b}.$$

If these be substituted in the expression for u_x, we get

$$u_x = C\,\{\sqrt{(a\,b)}\}^x.\,\{\sqrt{a}\,.\,S_x - \sqrt{b}\,.\,S_{x-1}\}$$
$$-\,\frac{(\alpha - a\,\beta)\,S_x + (\beta - b\,\alpha)\,S_{x-1}}{1 - a\,b},$$

which contains (as it ought) only one arbitrary constant C.

(8). Suppose the equation were

$$u_x + (a\,.\,S_x + b\,.\,S_{x-1})\,u_{x-1} + c = 0.$$

Here $c = c\,.\,S_x + c\,.\,S_{x-1}$; therefore this is only a particular case of the preceding, and so of any other constant coefficient in a circulating equation. This gives, consequently,

$$u_x = C\,\{\sqrt{(a\,b)}\}^x.\,\{\sqrt{a}\,.\,S_x - \sqrt{b}\,.\,S_{x-1}\}$$
$$-\,c\,.\,\frac{(1 - a)\,S_x + (1 - b)\,S_{x-1}}{1 - a\,b}.$$

(9). Let the proposed equation be

$$u_{x+1} - R\,.\,u_x = P_x,$$

where R is constant, and P_x any circulating function the period of circulation being n, or

$$P_x = a\,.\,S_x + b\,.\,S_{x-1} + \ldots\ldots k\,.\,S_{x-n+1}.$$

(10). Let the equation to be integrated be

$$u_{x+1} - R\,.\,u_x + P_x = 0.$$

where R is constant, and P_x any circulating function of the form

$$aS_x + bS_{x-1} + cS_{x-2} + \dots kS_{x-n+1}.$$

Assume

$$u_x = A_x \,.\, S_x + B_x \,.\, S_{x-1} + \dots K_x \,.\, S_{x-n+1},$$

then

$$u_{x+1} = B_{x+1}\, S_x + C_{x+1}\, S_{x-1} + \dots A_{x+1}\, S_{x-n+1},$$

and the equation becomes by substitution

$$0 = (B_{x+1} - R A_x + a)\, S_x + (C_{x+1} - R \,.\, B_x + b)\, S_{x-1} + \dots\dots$$
$$\dots\dots\dots\dots + (A_{x+1} - R \,.\, K_x + k)\, S_{x-n+1}$$

whose terms severally equated to zero give

$$B_{x+1} = R A_x - a,$$
$$C_{x+1} = R B_x - b,$$
$$\dots\dots\dots\dots\dots$$
$$A_{x+1} = R K_x - k,$$

whence we get

$$C_{x+2} = R^2 A_x - (Ra + b),$$
$$\dots\dots\dots\dots\dots$$
$$A_{x+n} = R^n A_x - (aR^{n-1} + bR^{n-2} + \dots\dots k).$$

This last equation integrated gives

$$A^x = R^x \,.\, \{ C \,.\, S_x + C' \,.\, S_{x-1} + \dots\dots C \,.\, S_{x-n+1} \}$$
$$- \frac{aR^{n-1} + bR^{n-2} \dots + k}{1 - R^n},$$

C, C', ... being n arbitrary constants. Now since we have

$$u_x = A_x \,.\, S_x + B_x \,.\, S_{x-1} + \text{\&c.},$$

we may neglect in the value of A_x all the terms but that multiplied by S_x, in B_x all but that multiplied by S_{x-1} &c.

which comes ultimately to the same as making C', C'', &c. $= 0$, because u_x can only contain one arbitrary constant C[*]. This done we get (putting Q for the constant part of the value of A_x),

$$A_x = C \,.\, R^x S_x - Q,$$

$$B_x = C \,.\, R^x S_{x-1} - (R\,Q + a),$$

. . . .

$$K_x = C \,.\, R^x S_{x-n+1} - (R^{n-1} Q + a\,R^{n-2} + b\,R^{n-3} \dots + j)$$

which substituted give

$$u_x = C \,.\, R^x - \frac{a\,R^{n-1} + b\,R^{n-2} + \dots k}{1 - R^n} \,.\, S_x,$$

$$- \frac{b\,R^{n-1} + \dots k\,R + a}{1 - R^n} S_{x-1},$$

. . . .

$$- \frac{k\,R^{n-1} + a\,R^{n-2} + \dots j}{1 - R^n} S_{x-n+1}.$$

(11). Let the equation be

$$u_{x+1} - P_x u_x + P'_x = 0,$$

P_x and P'_x being respectively equal to

$$a\,S_x + b \,.\, S_{x-1} + \dots k \,.\, S_{x-n+1},$$

and

$$\alpha\,S_x + \beta\,S_{x-1} + \dots \kappa\,S_{x-n+1}.$$

Making the same substitution for u_x, the equations for determining A_x, &c. will be

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[*] The same result will be obtained, if we retain all the constants and investigate in general the values of A_x, B_x,......K_x. If these be then substituted in the expression for u_x, the supernumerary constants will all destroy each other, of which we have already seen an instance in Number 7, of this Section.

$$B_{x+1} = a \,.\, A_x - \alpha$$

$$C_{x+1} = b \,.\, B_x - \beta,$$

. . . .

$$K_{x+1} = j \,.\, J_x - i$$

$$A_{x+1} = k \,.\, K_x - \kappa.$$

These give

$$C_{x+2} = a b A_x - (\alpha b + \beta)$$

. . . .

$$K_{x+n-1} = a b c \ldots . j \,.\, A_x - (b c \ldots . j \,.\, \alpha + c \ldots . j \,.\, \beta + \ldots i)$$

$$A_{x+n} = a b c \ldots . k \left\{ A_x - \frac{\alpha}{a} - \frac{\beta}{a b} - \ldots . \frac{\kappa}{a b \ldots . k} \right\},$$

which integrated gives

$$A_x = (a b c \ldots . k)^{\frac{x}{n}} . \{ C \,.\, S_x + C' \,.\, S_{x-1} + \&c. \}$$

$$- \frac{\frac{\alpha}{a} + \frac{\beta}{a b} + \ldots . \frac{\kappa}{a b \ldots . k}}{1 - a b c \ldots . k} . (a b c \ldots . k)$$

and putting Q for the constant part, and N for $(a \,.\, b \ldots . k)^{\frac{1}{n}}$ and, effacing all the arbitrary constants but the first, as in the last number,

$$A_x = C N^x . S_x - Q,$$

$$B_x = C \,.\, N^{x-1} S_{x-1} - (a Q + \alpha),$$

$$C_x = C \,.\, a b N^{x-2} S_{x-2} - (a b \,.\, Q + b \alpha + \beta); \&c.$$

which substituted give

$$u_x = C \,.\, \{ N^x . S_x + a N^{x-1} S_{x-1} + \ldots \ldots$$

$$\ldots . (a b c \ldots . j) N^{x-n+1} S_{x-n+1} \},$$

$$- \frac{N}{1 - N} \left(\frac{\alpha}{a} + \frac{\beta}{a b} + \ldots . \frac{\kappa}{a b \ldots . k} \right) S_x,$$

$$-\frac{N}{1-N}\left(\frac{\beta}{b}+\frac{\gamma}{b\,c}+\dots\frac{\alpha}{b\,c\dots k\,a}\right)S_{x-1},$$

. . . .

$$-\frac{N}{1-N}\left(\frac{\kappa}{k}+\frac{\alpha}{k\,a}+\dots\frac{\nu}{k\,a\,b\dots j}\right)S_{x-n+1}.$$

(12). It is sometimes necessary to reduce two or more circulating functions with different periods of circulation to a common period. This is easily accomplished; suppose for instance the functions were

$$a\,.\,S_x+b\,.\,S_{x-1},\quad(\text{where } n=2),$$

and

$$\alpha\,.\,S_x+\beta\,.\,S_{x-1}+\gamma\,.\,S_{x-2},\ (\text{where } n=3).$$

To denote these and similar functions more readily, suppose we take

$$S_x{}^{(n)}=\frac{\text{sum of } x^{\text{th}} \text{ powers of } n^{\text{th}} \text{ roots of unity}}{n}.$$

Then will the two functions in question be represented by

$$a\,.\,S_x{}^{(2)}+b\,.\,S_{x-1}{}^{(2)},$$

$$\alpha\,.\,S_x{}^{(3)}+\beta\,.\,S_{x-1}{}^{(3)}+\gamma\,.\,S_{x-2}{}^{(3)}.$$

The first of these is obviously equal to

$$a\,.\,S_x{}^{(6)}+b\,.\,S_{x-1}{}^{(6)}+a\,.\,S_{x-2}{}^{(6)}+b\,.\,S_{x-3}{}^{(6)}$$
$$+\,a\,.\,S_{x-4}{}^{(6)}+b\,.\,S_{x-5}{}^{(6)}.$$

Because when either x, or $x-2$, or $x-4$, is a multiple of 6, it is so of 2, and therefore x is a multiple of 2, consequently both functions reduce themselves to a; again, if $x-1$, or $x-3$, or $x-5$, be a multiple of 6, and therefore of 2, $x-1$ must necessarily be a multiple of 2, so that both reduce themselves to b. In like manner it may be proved that the other given function is identical with

$$\alpha \,.\, S_x^{(6)} + \beta \,.\, S_{x-1}^{(6)} + \gamma \,.\, S_{x-2}^{(6)}$$

$$+ \alpha \,.\, S_{x-3}^{(6)} + \beta \,.\, S_{x-4}^{(6)} + \gamma \,.\, S_{x-5}^{(6)},$$

so that the two are thus reduced to the common period $6 = 2 \times 3$.

If the separate periods have a common measure, the compound period will be their product, divided by this common measure. The reason is obvious.

By this means should equations occur involving circulating functions with different periods, they may be integrated.

(13). The following general property of circulating functions may be mentioned in addition to those enumerated in 3, 4, 5.

Let

$$P_x = a_x \,.\, S_x + b_x \,.\, S_{x-1} + \ldots . \, k_x \,.\, S_{x-n+1},$$

Then

$$f(P_x) = f(a_x) \,.\, S_x + f(b_x) \,.\, S_{x-1} + \ldots f(k_x) \,.\, S_{x-n+1},$$

or *any function of a circulating function is itself a circulating function, whose coefficients are similar functions of those of the original one respectively.* In like manner, if P'_x, P''_x, &c. be other circulating functions,

$$f(P_x,\ P'_x, \ldots) = f(a_x,\ a'_x, \ldots) \,.\, S_x + f(b_x,\ b'_x, \ldots)\, S_{x-1} + \&c.$$

Thus for instance (if the coefficients be constant)

$$\frac{P_x + P'_x}{1 - P_x P'_x} = \frac{a + a'}{1 - a a'} S_x + \frac{b + b'}{1 - b b'} S_{x-1} + \&c.$$

we shall have occasion to recal this principle hereafter. It is too evident to require a formal demonstration.

(14). To determine the integrals $\Sigma\, S_x$ and $\Sigma\, P_x$.

Putting $\Sigma\, S_x = u_x$, we have $u_{x+1} - u_x = S_x$, and this may be treated as a circulating equation; thus assuming

$$u_x = A_x \, . \, S_x + B_x \, . \, S_{x-1} + \ldots \, K_x \, . \, S_{x-n+1},$$

we have

$$0 = (B_{x+1} - A_x - 1)\, S_x + (C_{x+1} - B_x)\, S_{x-1} + \ldots$$
$$\ldots\ldots (A_{x+1} - K_x)\, S_{x-n+1},$$

whose terms severally equated to zero give

$$B_{x+1} = 1 + A_x$$
$$C_{x+1} = B_x$$

. . . .

$$A_{x+1} = K_x,$$

whence we get

$$A_{x+n} = 1 + A_x.$$

A particular solution of this will suffice for our purpose, and it is evident that $A_x = \frac{x}{n}$ will satisfy it. This gives

$$B_x = \frac{x-1}{n} + 1\,; \quad C_x = \frac{x-2}{n} + 1, \ldots\ldots \&c.$$

and finally

$$u_x = \frac{1}{n} \left\{ x \, . \, S_x + (x+n-1)\, S_{x-1} + (x+n-2)\, S_{x-2} \ldots \right.$$
$$\left. \ldots (x+1)\, S_{x-n+1} \right\} + \text{const.}$$

$$= \text{const} + \frac{x}{n} (S_x + S_{x-1} + \ldots \, S_{x-n+1})$$
$$+ \frac{(n-1)\, S_{x-1} + \ldots 1 \, . \, S_{x-n+1}}{n}$$

$$= \text{const} + \frac{x}{n} + \frac{1}{n} \left\{ (n-1)\, S_{x-1} + (n-2)\, S_{x-2} \ldots \right.$$
$$\left. \ldots + 1 \, . \, S_{x-n+1} \right\}.$$

Hence, if $P_x = a \,.\, S_x + b \,.\, S_{x-1} + \dots k \,.\, S_{x-n+1}$, we get

$$\Sigma P_x = \text{const} + (a + b + \dots k) \,.\, \frac{x}{n}$$

$$+ \frac{1}{n} \{ 0 \,.\, a + 1 \,.\, b + \dots (n-1) \,.\, k \} \, S_x$$

$$+ \frac{1}{n} \{ (n-1) \,.\, a + 0 \,.\, b + \dots (n-2) \,.\, k \} \, S_{x-1}$$

$$+ \frac{1}{n} \{ (n-2) \,.\, a + (n-1) \,.\, b + 0 \,.\, c + \dots \} \, S_{x-2}$$

. . . .

$$+ \frac{1}{n} \{ 1 \,.\, a + 2 \,.\, b + \dots (n-1) \,.\, j + 0 \,.\, k \} \, S_{x-n+1}.$$

SECTION XII.

Of continued Fractions.

(1). To determine the value of the continued fraction,

$$\cfrac{c_1}{a_1 + \cfrac{c_2}{a_2 + \cfrac{c_3}{a_3 + \dots \cfrac{c_x}{a_x}}}},$$

or, as it may more conveniently be written *,

$$\frac{c_1}{a_1 +} \; \frac{c_2}{a_2 +} \; \frac{c_3}{a_3 +} \dots \frac{c^x}{a_x}.$$

Let the fraction be put equal to u_x. Then we have

$$u_1 = \frac{c_1}{a_1}, \quad u_2 = \cfrac{c_1}{a_1 + \cfrac{c_2}{a_2}} = \frac{c_1 \,.\, a_2}{a_1 \,.\, a_2 + c_2},$$

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* After the example of Burmann.

and so on. It appears then, that u_x is reducible to the form $\frac{N_x}{D_x}$, N_x and D_x being the numerator and denominator of a certain rational fraction, each composed of combinations of a_1, a_2, &c. c_1, c_2, &c. formed by multiplication and addition. Let us now examine them more closely. To this effect we have

$$u_3 = \cfrac{c_1}{a_1 + \cfrac{c_2\, a_3}{a_2\, a_3 + c_3}} = \frac{c_1\, a_2 \,.\, a_3 + c_1 \,.\, c_3}{(a_1\, a_2 + c_2)\, a_3 + a_1 \,.\, c_3}$$

$$u_4 = \cfrac{c_1}{a_1 + \cfrac{c_2\, a_3 \,.\, a_4 + c_2 \,.\, c_4}{(a_2\, a_3 + c_3)\, a_4 + a_2 \,.\, c_4}}$$

$$= \frac{(c_1\, a_2 \,.\, a_3 + c_1 \,.\, c_3) a_4 + c_1\, a_2 \,.\, c_4}{(a_1\, a_2\, a_3 + a_1 c_3 + c_2\, a_3)\, a_4 + (a_1\, a_2 + c_2)\, c_4}$$

and so on. Hence, we have the following series of equations:

$$N_1 = c_1$$
$$N_2 = c_1 \,.\, a_2$$
$$N_3 = c_1 \,.\, c_3 + c_1\, a_2 \,.\, a_3$$
$$N_4 = c_1\, a_2 \,.\, c_4 + (c_1 \,.\, c_3 + c_1\, a_2 \,.\, a_3)\, a_4 \text{; \&c.}$$

that is,

$$N_1 = N_1$$
$$N_2 = N_2$$
$$N_3 = c_3 \,.\, N_1 + a_3 \,.\, N_2$$
$$N_4 = c_4 \,.\, N_2 + a_4 \,.\, N_3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$N_{x+2} = c_{x+2} \,.\, N_x + a_{x+2} \,.\, N_{x+1}; \quad (1).$$

Similarly, for the denominators, we have

$$D_1 = a_1, \quad D_2 = c_2 + a_1\, a_2$$
$$D_3 = a_1 \,.\, c_3 + (c_2 + a_1\, a_2) \,.\, a_3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D_{x+2} = a_{x+2} D_{x+1} + c_{x+2} D_x; \quad (2).$$

The integration of the equations (1) and (2) will therefore lead to the values of N_x and D_x, and therefore to that of their quotient u_x. As these two equations are precisely the same, the complete integral of one is also that of the other, and the values of N_x and D_x of course can only differ by reason of the different values of the arbitrary constants which enter into their expressions.

The equations (1) and (2) were noticed at the first origin of the theory of continued fractions, by Wallis in his Arithmetica Infinitorum, Prop. 191. p. 192. (Opera Wallisii. Oxon. 1657.) as rules for the ready computation of the approximating limits of infinite fractions of this kind, for which purpose they are well adapted, as they enable us to deduce the successive numerators and denominators of the limiting vulgar fractions $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$, $\frac{N_3}{D_3}$, &c. one from the other very readily.

(2). Required the value of the continued fraction

$$\frac{c}{a+}\,\frac{c}{a+}\,\frac{c}{a+}\ \ldots\ldots\ \text{(to } x \text{ terms)}.$$

Here c_x and a_x being constant, the integral of the equation

$$N_{x+2} = a\,.\,N_{x+1} + c\,N_x,$$

is $N_x = C\,.\,\alpha^x + C'\,.\,\beta^x$, α and β being the roots of $z^2 = az + c$. This integral may be expressed more conveniently for the present purpose by changing the arbitrary constants C and C' into $C + \frac{c'}{\alpha}$, and $C + \frac{c'}{\beta}$ which does not diminish the generality of the equation*, but only reduces it to the more symmetrical form

$$N_x = C(\alpha^x + \beta^x) + c'(\alpha^{x-1} + \beta^{x-1}).$$

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* This change of the arbitrary constants might have been made with advantage in Art. 393. Appendix, where it would have dispensed with a good deal of pretty abstract reasoning, but it did not occur at the time.

Determining the constants then so that $N_1 = c$, $N_2 = a\,c$, we get

$$N_x = \frac{c}{a^2 + 4\,c}\,\{\,a\,(\alpha^x + \beta^x) + 2\,c\,(\alpha^{x-1} + \beta^{x-1})\,\}\,,$$

and since $D_1 = a$, and $D_2 = a^2 + c$, we find in like manner

$$D_x = \frac{1}{a^2 + 4\,c}\,\{\,(a^2 + 2\,c)\,(\alpha^x + \beta^x) + a\,c\,(\alpha^{x-1} + \beta^{x-1})\,\}\,.$$

So that the general expression for u_x is

$$u_x = \frac{a\,c\,(\alpha^x + \beta^x) + 2\,c^2\,(\alpha^{x-1} + \beta^{x-1})}{(a^2 + 2\,c)\,(\alpha^x + \beta^x) + a\,c\,(\alpha^{x-1} + \beta^{x-1})}\,.$$

Let U represent the fraction continued to infinity, and suppose α the greater of the roots of the equation $z^2 - a\,z - c = 0$, (without regard to its sign,) and we get by making x infinite

$$U = \frac{a\,c\,\alpha + 2\,c^2}{(a^2 + 2\,c)\,\alpha + a\,c}\,,$$

which, (as may easily be proved,) is one of the roots of the quadratic

$$U^2 + a\,U - c = 0.$$

Thus

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1 + \&c.}\ \textit{ad inf.} = \frac{\sqrt{5} - 1}{2}\,,$$

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2 + \&c.}\ \textit{ad inf.} = \sqrt{2} - 1\,,$$

$$\frac{2}{1+}\,\frac{2}{1+}\ \ldots\ldots(\text{to } x \text{ terms}) = \frac{2^{x+1} + 2(-1)^{x+1}}{2^{x+1} + (-1)^x}\,.$$

and to infinity = 1.

(3). Required the value of the continued fraction

$$u_x = \frac{1}{a+}\;\frac{1}{b+}\;\frac{1}{a+}\;\frac{1}{b+}\;\&\text{c.}$$

continued to x terms, that is, containing x fractional terms.

The period of the denominators being a, b, we may assume $c_x = 1$, $a_x = b\,S_x + a\,S_{x-1}$, where S_x is the sum of the x^{th} powers of the roots of $z^2 - 1 = 0$, and we have by (Art. 1.)

$$N_{x+2} - N_x = (b\,S_x + a\,S_{x-1})\,N_{x+1}.$$

This equation is the same with that integrated in 6, Sect. 11. and taking

$$\begin{aligned} N_x &= v_x \,.\, \sqrt{(b\,.\,S_x + a\,.\,S_{x-1})} \\ &= v_x \,.\, (\sqrt{b}\,.\,S_x + \sqrt{a}\,.\,S_{x-1}), \quad \text{by 12, Sect. 11.} \end{aligned}$$

we have seen that v_x is given by the equation

$$v_{x+2} - \sqrt{(a\,b)}\,.\,v_{x+1} - v_x = 0.$$

Let then α, β, be the two roots of

$$z^2 - \sqrt{(a\,b)}\,.\,z - 1 = 0,$$

and we have

$$N_x = \{\sqrt{a}\,.\,S_{x-1} + \sqrt{b}\,.\,S_x\}\,[c\,(\alpha^x + \beta^x) + c'\,(\alpha^{x-1} + \beta^{x-1})].$$

To determine the constants we have $N_1 = 1$, $N_2 = b$, and since the value of D_x is precisely similar, the constants only being determined by making $D_1 = a$, $D_2 = a\,b + 1$, we obtain after substitution and all reductions the following value of the continued fraction

$$\frac{N_x}{D_x} = \sqrt{\frac{b}{a}}\,.\,\frac{\sqrt{(a\,b)}\,(\alpha^x + \beta^x) + 2\,(\alpha^{x-1} + \beta^{x-1})}{(a\,b+2)\,(\alpha^x + \beta^x) + \sqrt{(a\,b)}\,(\alpha^{x-1} + \beta^{x-1})}.$$

As before, let α be the greatest root of the equation $z^2 - \sqrt{(a\,b)}\,.\,z - 1 = 0$, without regard to its sign, and let U

represent the value of the fraction ad infinitum. Then by making x infinite, we get

$$U = \sqrt{\frac{b}{a}} \cdot \frac{2 + a \cdot \sqrt{(a b)}}{\sqrt{(a b)} + a (a b + 2)},$$

which is readily shewn to be a root of the equation

$$U^2 + b U = \frac{b}{a},$$

by substituting for $\sqrt{(a b)}$ its value $a - \frac{1}{a}$.

(4). To determine the value of the fraction

$$1 + \frac{c_x}{1 +} \frac{c_{x-1}}{1 +} \cdots\cdots \frac{c_1}{1},$$

regarded as a function of x.

If we would employ the preceding investigations we must regard x as constant, and assume another independent variable z, and another function of it p_z, such that

$$c_z = p_{x-z+1}.$$

This gives

$$c_x = p_1, \quad c_{x-1} = p_2, \cdots\cdots c_{x-z+1} = p_z.$$

If then we enquire by the methods above delivered the value of the expression

$$1 + \frac{p_1}{1 +} \frac{p_2}{1 +} \cdots\cdots \frac{p_z}{1}$$

in its general form, as a function of z, and then write in the result for p_z and its derivative functions, the functions c_{x-z+1} and others similarly derived from it *as a function of* z, and finally put $z = 1$, we have the value required. But the following process is simpler, and less liable to mistake.

Assume u_x for the value required. Then

$$u_{x+1} = 1 + \frac{c_{x+1}}{u_x}$$

$$u_{x+1} u_x = u_x + c_{x+1},$$

or, taking

$$u_x = \frac{v_{x+1}}{v_x}$$

$$v_{x+2} = v_{x+1} + c_{x+1} v_x$$

an equation of differences of the second order, the form of which it will be remarked is precisely that which determines N_{x-1} and D_{x-1} in the value $\frac{N_{x-1}}{D_{x-1}}$ of the function

$$\frac{c_1}{1+} \; \frac{c_2}{1+} \; \ldots\ldots \; \frac{c_{x-1}}{1}. \qquad \text{(See Prob. 1.)}$$

But though the equation of differences is the same, the nature of the functions derived from it will be essentially modified by the different constants required to adapt its integral to the two cases. Still, however, this coincidence assigns a relation between the two functions sufficiently remarkable.

In fact, let $(A_x + C \,.\, B_x) \times C'$ be the general value of v_x. Then will

$$u_x - 1 = \frac{\Delta\, A_x + C \,.\, \Delta\, B_x}{A_x + C \,.\, B_x},$$

and

$$\frac{N_{x-1}}{D_{x-1}} = C' \,.\, \frac{A_x + C'' B_x}{A_x + C''' B_x}$$

where the constants depend on the values of c_1, c_2, A_1, A_2, B_1, B_2, and are easily determined. Now these expressions are respectively the values of

$$\frac{c_x}{1\,+}\;\frac{c_{x-1}}{1\,+}\;\cdots\cdots\;\frac{c_1}{1}$$

and

$$\frac{c_1}{1\,+}\;\frac{c_2}{1\,+}\;\cdots\cdots\;\frac{c_x}{1}.$$

(5). Having given the value of

$$1+\frac{c_x}{1\,+}\;\frac{c_{x-1}}{1\,+}\;\cdots\cdots\;\frac{c_1}{1}=V_x.$$

To determine that of the same fraction with any additional denominator at the end,

$$1+\frac{c_x}{1\,+}\;\frac{c_{x-1}}{1\,+}\;\cdots\cdots\;\frac{c_1}{a}.$$

If we proceed as above, by putting $\frac{v_{x+1}}{v_x}$ for its value we shall obtain the very same equation of differences and therefore the expressions of the two functions can only differ in the values of the arbitrary constant C in the expression $\frac{A_{x+1}+C\,.\,B_{x+1}}{A_x+C\,.\,B_x}$. Now since one value V_x of $\frac{v_{x+1}}{v_x}$ is given, one value of v_x is also known, being equal to $V_1\,.\,V_2\ldots\ldots V_{x-1}$. Hence a particular integral of the equation for v_x is given, and of course, being only of the second order, the complete integral may be ascertained by the methods delivered in the text (Append. Art. 382.): and the constant must then be adapted to the case in question.

(6). To find, for instance, the value u_x of the fraction

$$1+\frac{c}{1\,+}\;\frac{c}{1\,+}\;\cdots\cdots\;\frac{c}{a},$$

when the letter c occurs x times

$$u_x = \frac{\alpha^{x+1} + C \,.\, \beta^{x+1}}{\alpha^x + C \,.\, \beta^x}\,; \quad C = -\frac{\alpha\left(1 + \frac{c}{a}\right) - \alpha^2}{\beta\left(1 + \frac{c}{a}\right) - \beta^2}$$

α and β being the roots of $v^2 - v - c = 0$.

◆

SECTION XIII.

Application of the Calculus of Differences to various Problems.

(1). WHAT are the respective amounts of a given sum for x years, at simple, and at compound interest?

Let P_x be the amount at the end of the x^{th} year, then 1st, at simple interest, if A be the original sum, and r the interest of £.1 for 1 year, rA will be that of £A, and therefore the increase in one year being rA, $P_x + rA$ is the amount at the end of the $(x+1)^{th}$ year, but this amount is also represented by P_{x+1}. Hence

$$P_{x+1} = P_x + rA, \text{ or } \Delta P_x = rA,$$

and integrating

$$P_x = rA \,.\, x + C.$$

Now the original sum or value of P_x, when $x = 0$, is A, hence

$$P_0 = rA \,.\, 0 + C = A, \text{ or } C = A,$$

and therefore,

$$P_x = A(1 + rx).$$

2d, At compound interest, P_x being the capital at the end of the x^{th} year, $r P_r$ is the interest in the $(x + 1)^{\text{th}}$ therefore,

$$P_{x+1} = P_x + r P_x = (1 + r) . P_x,$$

and integrating,

$$P_x = C (1 + r)^x.$$

Now $P_0 = A$, hence $C = A$, and $P_r = A (1 + r)^x$.

(2). A person places money in the funds, but gradually contracting expensive habits, he spends the first year the whole interest, the second twice that of the remaining stock, the third three times that of what is left, and so on. How long will his property last, and in what year is his expenditure greatest?

As before, let P_x=his stock at the end of the x^{th} year, r=interest of £1, for 1 year.

Then $r P_x =$ that upon P_x, and consequently his expenditure in the $(x + 1)^{\text{th}}$ year is $(x + 1) r P_x$. Therefore at the end of the $(x + 1)^{\text{th}}$ year his stock will be

$$P_x + r P_x - (x + 1) r P_x = P_x (1 - x r).$$

Hence,

$$P_{x+1} = (1 - x r) P_x,$$

and integrating on the hypothesis $P_0 = A$,

$$P_x = A \cdot 1 (1 - r) (1 - 2 r) \ldots\ldots \{ 1 - (x - 1) . r \}.$$

This vanishes when $x = 1 + \frac{1}{r}$, which is the number of years his stock will last. Also, his expenditure in the x^{th} year being $x r P_{x-1}$, and in the next, $(x + 1) r P_x$ will be greatest just before $\Delta (x r P_{x-1})$ becomes negative, because then having reached its maximum it begins to decrease.

Now

$$\Delta\,(x\,r\,P_{x-1}) = 1\,.\,(1-r)\,\ldots\ldots$$
$$\ldots\ldots\,\{\,1-(x-1).\,r\,\}\;\{\,(x+1)\,(1-x\,r)-x\,\}\,.$$

Suppose then

$$(x+1)\,(1-x\,r) - x = 0,$$

this gives

$$x = \sqrt{\left(\frac{1}{r}+\frac{1}{4}\right)} - \frac{1}{2},$$

and the nearest integer less than this is the required number. If $r = \frac{1}{20}$, $x = 4$ exactly. Here then $\Delta\,(x\,P_{x-1}) = 0$, when $x = 4$, so that the sums spent in the 4th and 5th years are equal, and greater than in any other.

(3). A person puts out to interest a sum of money (A), he expends annually a portion (a) of the interest, and adds the remainder to the stock. What is the amount after x years?

Call it P_x; then the interest is $r\,P_x$, so that

$$P_{x+1} = P_x + r\,P_x - a, \text{ or}$$
$$P_{x+1} - (1+r)\,P_x + a = 0.$$

which integrated gives

$$P_x = C\,(1+r)^x + \frac{a}{r}.$$

The constant C, must be determined by the consideration that P_0 the original stock is equal to A which gives

$$C = A - \frac{a}{r},$$

and

$$P_x = \left(A - \frac{a}{r}\right)(1+r)^x + \frac{a}{r}.$$

If a exceed the interest, and on that supposition we would find how long the money will last, make $P_x = 0$, and we get

$$x = \frac{\log a - \log(a - rA)}{\log(1 + r)}.$$

If we would find what annual sum a stock-holder may expend so that his property shall just last out his life, on a fair calculation, call x the number of years he has a reasonable expectation of living (calculated from the tables of mortality) then we have

$$a = rA \cdot \frac{(1 + r)^x}{(1 + r)^x - 1}.$$

(4). A has £.1000 ($= A$) in the funds at 5 per cent. ($= r$). He spends the first year the full interest of his capital £.500, the next £.1000, and so on in regular arithmetical progression. How long will his property last?

As before, putting P_x for his property at the end of the x^{th} year, we have

$$P_{x+1} = P_x(1 + r) - (x + 1) \cdot rA,$$

which integrated gives

$$P_x = A\left\{(x + 1) - \frac{(1 + r)^x - 1}{r}\right\}.$$

Hence we have to find x from the exponential equation,

$$(1 + r)^x = 1 + r + rx,$$

and since $r = \frac{1}{20}$, $x = 6 \cdot 6$ nearly.

Were a more exact value required, we must proceed thus: Suppose $f(x) = 0$, and a being an approximate value of x, let $a + h$ be the true value, then will h be small, and the equation

$$f(a + h) = 0,$$

developed by Taylor's theorem gives, neglecting h^2, &c.

$$f(a) + h \cdot \frac{d f(a)}{d a} = 0,$$

whence,

$$h = -\frac{f(a) \cdot d a}{d f(a)}, \quad \text{and } x = a + h,$$

is a second approximation.

In the present case,

$$f(a) = (1 + r)^a - 1 - r - r a,$$

$$\frac{d f(a)}{d a} = (1 + r)^a \cdot \log (1 + r) - r,$$

and

$$h = \frac{(1 + r + r a) - (1 + r)^a}{(1 + r)^a \log (1 + r) - r},$$

$$= - 0.00453,$$

so that a second approximation is $x = 6.59547$.

(5). A man spends every year twice the sum he gained by a certain business the year before. That business, however, becomes every year more and more profitable, and he finds his property increase regularly, as the square of the time since he began business. In what progression do the profits of his trade increase?

If we call P_x the profit in the x^{th} year, the problem leads to the equation

$$P_{x+1} - 2 P_x = a (2 x + 1),$$

whence we find

$$P_x = a \{ 5 . 2^{x-1} - (2 x + 3) \} + P_1 . 2^{x-1}.$$

Now $P_1 = a . 1^2$, hence $P_x = a (3 . 2^x - 2 x - 3)$.

(6). An individual sets out in the world with a certain capital (A), one-half of which he places in the funds at r per cent. and the rest, ventures in a concern which produces

$2r$ per cent. per annum, but which returns the interest only once in two years; he lives at a stated rate of expenditure (a £ per annum) and puts all his gains and savings into the funds. Required his funded property after any number (x) of years.

Call his funded property at the end of the (x^{th}) year P_x. Then, if x be an even number, the interest on that part of his stock $\left(\frac{A}{2}\right)$ which is vested in trade does not accrue in the $(x+1)^{\text{th}}$ year, so that, (x being even),

$$P_{x+1} = (1+r)\,P_x - a,$$

but when x is odd the interest for two years at $2r$ per annum, accrues on the capital $\frac{A}{2}$, that is, $4r\frac{A}{2}$ or $2rA$, hence, (x being odd),

$$P_{x+1} = (1+r)\,P_x - a + 2rA.$$

It may not, perhaps, immediately appear how these equations are to be treated; because in either of them, if x be increased by unity, the equation ceases to be true, and therefore the function P_x cannot be found by integrating either of them separately, the law of continuity being broken. To supply this, and to include the odd and even values of x in one analysis, we must have recourse to the theory of circulating functions above delivered. In fact, since the circulating function of the second degree

$$a\,.\,S_x + (a - 2rA)\,.\,S_{x-1},$$

is equal either to a, or to $a - 2rA$, according as x is even or odd, the equation

$$P_{x+1} - R\,.\,P_x + \{a\,.\,S_x + (a - 2rA)\,S_{x-1}\} = 0,$$

(where $R = 1 + r$) includes both the others, and admits of any value being assigned to x. The law of continuity being thus restored we may find P_x by integration, and we get (9, Sect. 11.)

$$P_x = C \,.\, R^x - \frac{a\,R + a - 2\,r\,A}{1 - R^2}\,S_x$$

$$- \frac{(a - 2\,r\,A)\,R + a}{1 - R^2}\,S_{x-1},$$

and determining C so that $P_0 = \frac{1}{2}A$, his funded property at the outset, and writing for S_x and S_{x-1} their values

$$\frac{1 + (-1)^x}{2} \text{ and } \frac{1 - (-1)^x}{2}, \text{ we get finally,}$$

$$P_x = A \left\{ \frac{(r+6)\,(1+r)^x + 2\,r\,(-1)^x}{2\,(r+2)} - 1 \right\} - \frac{a}{r}\,\{\,(1+r)^x - 1\,\}\,.$$

(7). The same being supposed as in the last problem, only that the part of his capital vested in trade, yields r' per cent. per annum, but returns interest only once in n years. Required the amount of his funded stock after any number of years.

In this case the equation

$$P_{x+1} = R \,.\, P_x - a,$$

holds good for every value of x unless when $x + 1$ is a multiple of n (and therefore $x - n + 1$ such a multiple) when it changes to

$$P_{x+1} = R \,.\, P_x - (a - b),$$

where $b = n\,r' \,.\, \frac{A}{2}$. In this case then we have for our circulating equation,

$$o = P_{x+1} - R \,.\, P_x + \{\,a \,.\, S_x + a \,.\, S_{x-1} + \ldots\ldots$$

$$\ldots\ldots (a - b)\,S_{x-n+1}\,\},$$

which integrated as in (9, Sect. 11.) gives

$$P_x = C \, . \, R^x + \frac{a}{R-1} - \frac{b}{R^n - 1}$$

$$\{ S_x + R \, . \, S_{x-1} + \ldots\ldots R^{n-1} \, . \, S_{x-n+1} \} \, .$$

the constant C being determined as before, by taking $x = 0$, is found as follows :

$$C = \frac{A}{2} + \frac{b}{R^n - 1} - \frac{a}{R-1},$$

$$= \frac{A}{2}\left(1 + \frac{n r'}{R^n - 1}\right) - \frac{a}{R-1} \, .$$

(8). Suppose a merchant engaged in more than one such concern as those described in the two last problems. To determine his funded property after any time.

Let n represent the least time in which the interest of funded capital can be made readily to accrue, n', n'', &c. the intervals at which the several parts of his capital embarked in commerce return their interest, the least common measure of all their intervals n, n', n'', &c. or of such as differ from each other, being taken as the unit of time. Also let A', A'', &c. be the several parts of his capital so embarked, and r', r'', &c. the rates of interest they yield in the time 1. Lastly, let A be his original funded capital, P_x its amount after x such units of time have elapsed, r the rate of interest in the funds for the time 1, and a his uniform rate of expenditure in that time. Then we have

$$P_{x+1} = P_x - a,$$

unless when $x + 1$ is a multiple of either n, n', &c. in which several cases, the terms

$$n r \, . \, P_x, \quad n' r' A' \text{ \&c.}$$

expressing the respective sums accruing as interest at these moments are to be added to the second member. Employing then the notation of (11, Sect. 11.) the circulating equation embracing all these cases is,

$$P_{x+1} = P_x(1 + nr \,.\, S_{x-n+1}{}^{(n)})$$
$$- a + n'r'A' \,.\, S_{x-n'+1}{}^{(n')} + \&c.$$

Take $m =$ the product of n, n', &c. divided by all the greatest common measures of any two or more of them, and this equation is transformed to

$$0 = P_{x+1}$$
$$- P_x\{1 + nr(S_{x-n+1}{}^{(m)} + S_{x-2n+1}{}^{(m)} \ldots + S_{x-m+1}{}^{(m)})\} + a -$$
$$- n'r'A'\{S_{x-n'+1}{}^{(m)} + S_{x-2n'+1}{}^{(m)} \ldots + S_{x-m+1}{}^{(m)}\}$$
$$- n''r''A'', \&c.$$

Now, since

$$1 = S_x{}^{(m)} + S_{x-1}{}^{(m)} - \ldots S_{x-m+1}{}^{(m)},$$

and

$$a = a \,.\, S_x{}^{(m)} + \,.\, S_{x-1}{}^{(m)} + \&c.$$

If we take

$$a_1 = 1,\ a_2 = 1, \ldots a_n = 1 + nr,\ a_{n+1} = 1,\ a_{n+2} = 1, \ldots$$
$$\ldots\ldots a_{2n} = 1 - nr, \ldots \&c.$$

$$b_1 = 0,\ b_2 = 0, \ldots\ldots b_{n'} = n'r'A',\ b_{n'+1} = 0, \ldots\ldots$$
$$\ldots\ldots b_{2n'} = n'r'A',\ \&c.$$

$$c_1 = 0, \ldots\ldots c_{n''} = n''r''A'',\ c_{n''+1} = 0,\ \&c.\ \&c.$$

and finally

$$- a + b_1 + c_1 + \ldots = \alpha,\ - a + b_2 + c_2 + \ldots = \beta,\ \&c.\ \&c.$$

our equation will become

$$P_{x+1} - P_x\{a_1 S_x{}^{(m)} + \ldots a_m S_{x-m+1}{}^{(m)}\}$$
$$- \{\alpha \,.\, S_x{}^{(m)} \beta \,.\, S_{x-1}{}^{(m)} \ldots \kappa \,.\, S_{x-m+1}{}^{(m)}\} = 0,$$

whose integral is given at full length in (10, Sect. 11.)

In this case we have

$$N = (1 + nr)^{\frac{1}{n}},$$

and taking $P_0 = A$, we find for the arbitrary constant,

$$C = A - \frac{N}{1 - N}\left(\frac{\alpha}{\alpha_1} + \frac{\beta}{a_1\, a_x} + \ldots\ldots \frac{\kappa}{a_1 \ldots\ldots a_n}\right),$$

but till the particular values of n, n', &c. are assigned in numbers, no farther reductions in the form of this integral are practicable.

(9). *A* and *B* engage in play, on the following condition, viz. that whenever *A* wins a game, the stake shall be doubled for the next game, but whenever *B* wins, it shall be tripled. When they left off (after x games) it was found that they had won and lost alternately, *A* winning the first game. What are their respective gains and losses?

Put u_x to represent *A*'s total gain at the end of the x^{th} game, and suppose P_x to equal the stake for which that game is played. Then, provided *A* win the x^{th} game, we have $P_{x+1} = 2\, P_x$, but if *B* win it $P_{x+1} = 3\, P_x$. Now *A* wins it if x be odd, but *B* if even; hence in all cases, the numerical value of x being undetermined,

$$P_{x+1} = (3 \,.\, S_x + 2 \,.\, S_{x-1})\, P_x; \qquad (a.)$$

where $S_x = \frac{1}{2}$ the sum of the x^{th} powers of the roots of $z^2 - 1 = 0$.

Again, $\triangle\, u_x$ is *A*'s gain or loss by the event of the $(x+1)^{\text{th}}$ game. It is evidently equal in value to the stake for which that game was played (P_{x+1}) being a *gain* if *A* win, or $x + 1$ be odd, but a *loss* if he lose, or $x + 1$ be even, hence

$$u_{x+1} - u_x = (S_x - S_{x-1})\, P_{x+1}; \qquad (b.)$$

It only remains therefore to integrate the equations (*a*), (*b*). Now the equation (*a*) is that of (8, Sect. 12.) whence, determining the constant by taking P_1 the first stake for a given quantity, we find

$$P_x = P_1 . \sqrt{2} . \{ \sqrt{2} . S_x + \sqrt{3} . S_{x-1} \} (\sqrt{6})^{x-2}.$$

This given, we get, by substituting the value of P_{x+1} in (*b*),

$$u_{x+1} - u_x = P_1 . \sqrt{2} \{ \sqrt{3} . S_x - \sqrt{2} . S_{x-1} \} . (\sqrt{6})^{x-1}.$$

To integrate this, we take $u_x = A_x . S_x + B_x . S_{x-1}$, which gives by substitution and equation of like terms

$$B_{x+1} - A_x = P_1 . (\sqrt{6})^x$$

$$A_{x+1} - B_x = - 2 P_1 . (\sqrt{6})^{x-1}, \text{ or}$$

$$A_{x+2} - B_{x+1} = - 2 P_1 . (\sqrt{6})^x.$$

Adding this and the first together, we find

$$A_{x+2} - A_x = - P_1 . (\sqrt{6})^x,$$

and integrating

$$A_x = C\, S_x + C'\, S_{x-1} - \frac{P_1}{5} . (\sqrt{6})^x,$$

consequently

$$B_x = C'\, S_x + C\, S_{x-1} + \frac{4}{5} . P_1 (\sqrt{6})^{x-1},$$

and

$$u_x = C - \frac{1}{5} P_1 . \{ S_x . (\sqrt{6})^x - 4\, S_{x-1} . (\sqrt{6})^{x-1} \} .$$

But $u_0 = 0$, hence $C = \frac{1}{5} P$, and finally

$$u_x = \tfrac{1}{5} P_1 \{ 1 - (\sqrt{6})^x . S_x + 4 . (\sqrt{6})^{x-1} . S_{x-1} \} ,$$

for the total amount of A's gain or B's loss at the end of their play.

(10). The last problem may be generalized by supposing that when A wins, the stake of the succeeding game shall become any function whatever of the former stake, and of the number of games elapsed since the beginning, and when B wins, any other functions. A and B winning alternately, what is the total amount of A's gain or loss?

Here we have $P_{x+1} = f(P_x, x)$,

when x is odd, and $= f_1(P_x, x)$ when even, so that in general

$$P_{x+1} = S_x . f_1(P_x, x) + S_{x-1} . f(P_x, x).$$

To integrate this, or at least to clear it of its circulating form, we take

$$P_x = A_x . S_x + B_x . S_{x-1}.$$

Then since $f(P_x, x)$ is a function of S_x, S_{x-1} it is reducible to the general form of all circulating functions (by 12, Sect. 12.) and in fact it becomes

$$f(P_x, x) = f(A_x, x) . S_x + f(B_x, x)\, S_{x-1},$$

similarly,

$$f_1(P_x, x) = f_1(A_x, x) . S_x + f_1(B_x, x) . S_{x-1},$$

therefore

$$S_x \{ B_{x+1} - f_1(A_x, x) \} + S_{x-1} A_{x+1} - f(B_x, x) \} = 0,$$

which resolves itself into the two

$$B_{x+1} = f_1(A_x, x) \qquad A_{x+1} = f(B_x, x),$$

That is,

$$A_{x+2} = f \{ f_1(A_x, x), x \}$$

an equation of the second order whose integration suffices to give A_x. If we put $x=2z$ and $A_{2z}=A_z'$, this reduces itself to the first order, after which it depends on the particular forms of f, f_1, whether the integration be practicable or not. A very extensive case of integrability is when

$$f(P_x, x) = a_x . P_x + b_x$$
$$f_1(P_x, x) = \alpha_x . P_x + \beta_x.$$

In this case the equation for finding A_z' is linear of the first order, viz.

$$\begin{aligned} A'_{z+1} &= a_x \alpha_x . A_z' + (b_x + a_x \beta_x) \\ &= a_{2z} \alpha_{2z} . A_z' + (b_{2z} + a_{2z} \beta_{2z}. \end{aligned}$$

In like manner, we may proceed when instead of winning and losing alternately, the players win and lose in any other regular order.

(11). Let there be a series of quantities A, B, C, &c. derived from one another by the following law,

$$A = a, \ B = \frac{1-A}{2}, \ C = \frac{1-2B}{3}, \ D = \frac{1-3C}{4}, \ \text{\&c.}$$

Required the general term of the series.

Call the x^{th} term u_x, then the $(x+1)^{\text{th}}$ being u_{x+1} we have

$$u_{x+1} = \frac{1 - x u_x}{x+1},$$

$$(x+1) . u_{x+1} + x u_x = 1$$

an equation of differences which integrated give,

$$x u_x = \tfrac{1}{2} + C . (-1)^x$$

and determining C so that $u_1 = a$

$$u_x = \frac{1}{2x} + \frac{(2a-1).(-1)^{x+1}}{2x}.$$

(12). Suppose we have

$$A = a,\ B = 2A^2 - 1,\ C = 2B^2 - 1,\ \&c.$$

Required the x^{th} term of this series.

Here $$u_{x+1} = 2u_x^2 - 1,$$

an equation which integrated gives

$$u_x = \tfrac{1}{2}\{(a + \sqrt{a^2 - 1})^{2^{x-1}} + (a - \sqrt{a^2 - 1})^{2^{x-1}}\},$$

the constant being properly determined. In the same way we might proceed, if we had

$$A = a,\ B = f(A),\ C = f(B),\ D = f(C)\ \&c.;$$

(13). To integrate the differential expression

$$\int \frac{x^{2n}\,dx}{\sqrt{(1-x^2)}}.$$

Assume it equal to F_n (being a function of n to be determined). Then, integrating by parts,

$$F_n = \int x^{2n-1}.\frac{x\,dx}{\sqrt{(1-x)^2}},$$

$$= -x^{2n-1}\sqrt{(1-x^2)} + (2n-1)\int x^{2n-2}\,dx.\sqrt{(1-x^2)},$$

or, putting $P_n = x^{2n-1}\sqrt{(1-x^2)}$, and writing $\frac{1-x^2}{\sqrt{(1-x)^2}}$ for $\sqrt{(1-x^2)}$

$$F_n = -P_n + (2n-1)(F_{n-1} - F_n).$$

That is,

$$F_n - \frac{2n-1}{2n}.F_{n-1} = -\frac{P_n}{2n}; \qquad (a).$$

an equation of differences (n being the independent variable) whose integral is

$$F_n = \frac{1.3.5 \ldots\ldots (2n-1)}{2.4 \ldots\ldots (2n)} \times$$

$$\left\{ C - \Sigma \frac{2.4 \ldots\ldots (2n)}{1.3.5 \ldots\ldots (2n+1)} P_{n+1} \right\}.$$

The integration denoted by the sign Σ not being practicable we must (as in 3, Sect. 4.) write at full length the series of which the integral consists, viz.

$$\frac{1}{1} P_1 + \frac{2}{1.3} P_2 + \ldots \frac{2.4 \ldots (2n-2)}{1.3 \ldots (2n-1)} P_n,$$

and determining C by the condition

$$F_0 = \int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x,$$

we get, restoring the values of P_1, P_2, &c.

$$F_n = \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)} \left\{ \sin^{-1} x - \left(\frac{x}{1} + \frac{2x^3}{1.3} + \right.\right.$$

$$\left.\left. \ldots\ldots \frac{2.4 \ldots (2n-2)}{1.3 \ldots (2n-1)} x^{2n-1} \right) . \sqrt{(1-x^2)} \right\}.$$

If we only require the value of the integral between the limits $x = 0$, and $x = 1$, since $x^{2n-1} . \sqrt{(1-x^2)}$ vanishes at both the limits, we have $P_n = 0$, and our equation of differences (a) is simply

$$F_n - \frac{2n-1}{2n} F_{n-1} = 0,$$

which gives

$$F_n = C \times \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)},$$

but $C = \int \frac{dx}{\sqrt{(1 - x^2)}} = \frac{\pi}{2}$, between the same limits; so that

$$F_n = \frac{1 . 3 \ldots (2n - 1)}{2 . 4 \ldots (2n)} . \left(\frac{\pi}{2}\right);$$

and in the same way may all similar instances in which integrals are reduced by successive steps to a less and less degree of complication till at last they are brought to a known form, be treated without going through the process of continuation, by reducing them to equations of differences.

EXAMPLES

OF THE

Solutions

OF

FUNCTIONAL EQUATIONS.

BY

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NOTICE.

The object of the following Examples of Functional Equations, is to render a subject of considerable interest, more accessible to mathematical students, than it has hitherto been. It is, perhaps, that subject of all others, which most requires the assistance of particular instances, in order fully to comprehend the meaning of its symbols, which are of the most extreme generality; that assistance is also more particularly required in this branch of science, in consequence of its never yet having found its way into an Elementary Treatise.

Oct. 20. 1820.

OF

FUNCTIONAL EQUATIONS.

If a function α is of such a form, that, when it is twice performed on a quantity, the result is the quantity itself, or if $\alpha^2(x) = x$, then it is called a periodic function of the second order, if $\alpha^n(x) = x$, then it is termed a periodic function of the n^{th} order, thus when $\alpha(x) = a - x$ the second function, or

$$\alpha(\alpha x) = \alpha(a - x) = a - (a - x) = a - a + x = x.$$

If $\alpha(x) = \dfrac{1}{1 - x}$,

then $\alpha^2 x = \alpha(\alpha x) = \dfrac{1}{1 - \dfrac{1}{1 - x}} = \dfrac{1 - x}{1 - x - 1} = \dfrac{x - 1}{x}$,

and

$$\alpha^3 x = \alpha^2 \alpha x = \frac{\alpha x - 1}{\alpha x} = \frac{\dfrac{1}{1 - x} - 1}{\dfrac{1}{1 - x}} = 1 - \overline{1 - x} = x,$$

the first of these examples is a periodic function of the second, the last is a periodic function of the third order.

Prob. 1. To find periodic functions of the second order.

Since such functions must satisfy the equation $\psi^2 x = x$, we have

$$\psi x = \psi^{-1} x,$$

or ψ must be such a function, that it shall be the same as its inverse; if therefore $y = \psi x$, we have also $x = \psi^{-1} y = \psi y$, or if x and y are connected by some equation, it must be symmetrical relative to x and y; y or ψx must then be determined from the equation

$$* F\{\overline{x}, \overline{\psi x}\} = 0,$$

for instance, if $x + \psi x - a = 0$, $\psi x = a - x$,

or if $x \psi x = a^2$, $\psi x = \frac{a^2}{x}$.

Another method of determining such functions is as follows: since ψx is of such a form that $\psi^2 x = x$ any symmetrical function of x and ψx remain constant when x is changed into ψx thus

$$F\{\overline{x}, \overline{\psi x}\} \text{ becomes } F\{\overline{\psi x}, \overline{\psi^2 x}\} = F\{\overline{\psi x}, \overline{x}\{,$$

if therefore, we can find any particular solution of the equation $\psi^2 x = x$, containing an arbitrary constant we may substitute such a function for it, but $\psi x = a - x$ is a particular solution therefore

$$\psi x = F(\overline{x}, \overline{\psi x}) - x,$$

or

$$x + \psi x = F(\overline{x}, \overline{\psi x}),$$

and by changing the arbitrary function into another of the same form, we find

$$F\{\overline{x}, \overline{\psi x}\} = 0,$$

as before.

These two methods of determining periodic functions of the second order, are not so convenient as a third process which can be extended to all orders.

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* Bars placed above quantities under the functional sign, indicate that the function is symmetrical relative to those quantities.

Assume $\psi x = \phi^{-1} f \phi x$, then

$$\psi^2 x = \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^2 \phi x,$$

this must be equal to x or

$$\phi^{-1} f^2 \phi x = x,$$

this equation will be fulfilled if $f^2 v = v$, or if f is a particular solution, and if also ϕ^{-1} is such an inverse function that $\phi^{-1} \phi v = v$. If therefore ϕ is arbitrary, and f is a particular solution of $f^2 x = x$, then the solution of $\psi^2 x = x$ is

$$\psi x = \phi^{-1} f \phi x.$$

Ex. Let $f x = \frac{a}{x}$, then $\psi x = \phi^{-1} \left(\frac{a}{\phi x}\right)$,

if $f(x) = \frac{a - b x}{b + c x}$, $\quad \psi x = \phi^{-1} \left(\frac{a - b \phi x}{b + c \phi x}\right)$;

from these may easily be derived the following periodic functions of the second order,

$$\psi x = a - x \qquad \psi x = \frac{x}{x - 1}$$

$$\psi x = \frac{x - 2}{x - 1} \qquad \psi x = \frac{a^2}{x}$$

$$\psi x = \frac{1 - x}{1 + x} \qquad \psi x = \sqrt{1 - x^2}$$

$$\psi x = \frac{x + 1}{x - 1} \qquad \psi x = \frac{x}{\sqrt{x^2 - 1}},$$

$$\psi x = \tan^{-1} \left(\frac{\sin (a - x)}{\cos a . \cos x}\right) \qquad \psi x = \log (a - \varepsilon^x)$$

$$\psi x = (a^n - x^n)^{\frac{1}{n}} \qquad \psi x = x - \log (\varepsilon^x - 1)$$

$$\psi x = \frac{x}{(x^n - a^n)^{\frac{1}{n}}} \qquad \psi x = \tan^{-1} (a - \tan x)$$

Prob. 2. Required periodic functions of the third order, or such as fulfil the equation $\psi^3 x = x$.

Assume $\psi x = \phi^{-1} f \phi x$, then the equation becomes

$$\psi^3 x = \phi^{-1} f \phi \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^3 \phi x = x,$$

which will be verified if $f(v)$ is a particular solution of $f^3 v = v$, and if ϕ^{-1} is such an inverse value that $\phi^{-1} \phi v = v$, hence the solution of the equation is

$$\psi x = \phi^{-1} f \phi x,$$

one solution is $\dfrac{1}{1-x}$ and hence $\psi x = \phi^{-1} \left(\dfrac{1}{1 - \phi x}\right)$

more particular cases are

$$\psi x = \frac{a^2}{a - x} \qquad \psi x = \frac{1 + x}{1 - 3x}$$

$$\psi x = \frac{a^2}{ac - c^2 x} \qquad \psi x = \frac{\sqrt{a x^2 - a^2}}{x}$$

$$\psi x = \frac{ax - a^2}{x} \qquad \psi x = \frac{1}{1 - x}$$

$$\psi x = \left(\frac{a^2}{a - x^n}\right)^{\frac{1}{n}} \qquad \psi x = -\log(1 - \varepsilon^x)$$

$$\psi x = \frac{(a x^n - a^2)^{\frac{1}{n}}}{x} \qquad \psi x = \log(a \varepsilon^x - a^2) - x$$

$$\psi x = \frac{a + bx}{c - \dfrac{a^2 + bc + c^2 x}{a}} \qquad \psi x = \log(\varepsilon^x - \varepsilon^c) - x + c.$$

Prob. 3. To find periodic functions of the n^{th} order, or to solve the equation $\psi^n x = x$.

Assume as before $\psi x = \phi^{-1} f \phi x$ then it becomes

$$\phi^{-1} f \phi \phi^{-1} f \phi \ldots \phi^{-1} f \phi x = \phi^{-1} f^n \phi x = x,$$

which is verified if f is a particular solution of $f^n x = x$, and if ϕ^{-1} is such an inverse function that $\phi^{-1}\phi x = x$.

It now remains to find particular solutions of $\psi^n x = x$ which may be accomplished in the following manner: let fx represent $\frac{a+bx}{c+dx}$ then the n^{th} function will be of the same form, or

$$f^n(x) = \frac{A_n + B_n x}{C_n + D_n x},$$

where A_n, B_n, C_n, D_n, are functions of a, b, c, d, and n, these may be so determined that $D_n = 0$, $A_n = 0$ and $B_n = C_n$ all which conditions are satisfied, if

$$d = -\frac{b^2 - 2bc\cos\frac{2k\pi}{n} + c^2}{\left(2 + 2\cos\frac{2k\pi}{n}\right)a}$$

hence

$$\phi x = \phi^{-1}\left\{\frac{a + b\phi x}{c - \frac{b^2 - 2bc\cos\frac{2k\pi}{n} + c^2}{\left(2 + 2\cos\frac{2k\pi}{n}\right)a}\phi x}\right\}$$

a more detailed account of this method of solution may be found in a paper by Mr. Horner in the Annals of Philosophy, Nov. 1817.

Instances of $\psi^4 x = x$ are

$$\psi x = \frac{1}{2}\,\frac{1}{1-x} \qquad \psi x = \frac{1+x}{1-x}$$

$$\psi x = \frac{2}{2-x} \qquad \psi x = \frac{2a^2}{2ac - c^2 x}$$

$$\psi x = 2\,\frac{x-1}{x} \qquad \psi x = \frac{a + bx}{c - \frac{b^2 + c^2}{2a}x}$$

$$\psi x = \frac{\sqrt{2}}{\sqrt{2 - x^2}} \qquad \psi x = -\sqrt{\frac{1 - x^2}{1 + x^2}}$$

$$\psi x = \frac{(2 x^n - 2)^{\frac{1}{n}}}{x}$$

$$\psi x = \log 2 - x + \log (\varepsilon^x - 1).$$

All those cases which satisfy the equation $\psi^2 x = x$, also fulfil that of $\psi^4 x = x$, as well as all those which fulfil any of these equations $\psi^2 x = - x$, $\psi^2 x = \frac{1}{x}$, or more generally $\psi^2 x = \alpha x$, where αx is a particular solution of the equation $\psi^2 x = x$.

The following particular cases satisfy the equation $\psi^6 x = x$.

$$\psi x = \frac{1}{3(1 - x)} \qquad \psi x = \frac{3 x - 1}{3 x}$$

$$\psi x = \frac{3}{3 - x} \qquad \psi x = \frac{3 a^2}{3 a c - c^2 x}$$

$$\psi x = 3 \frac{x - 1}{x} \qquad \psi x = \frac{3 + 3 x}{3 - x}$$

$$\psi x = \frac{a + b x}{c - \frac{b^2 - b c + c^2}{3 a} x}$$

$$\psi x = \frac{1}{x} \left(x^n - \frac{1}{3} \right)^{\frac{1}{n}}$$

$$\psi x = \log 3 - x + \log (\varepsilon^x - 1).$$

The principle on which the solution of the functional equation $F \{ x, \psi x, \psi \alpha x \} = 0$ depends, where $\alpha^2 x = x$, is that by substituting αx for x we have another equation $F \{ \alpha x, \psi \alpha x, \psi x \} = 0$, between which and the given equation we may eliminate $\psi \alpha x$ and the result will be the value of ψx a few examples will illustrate this method.

(1). Given $\psi(x) + a\,\psi(-x) = x^n$

by putting $-x$ for x this becomes

$$\psi(-x) + a\,\psi(x) = (-x)^n,$$

and eliminating $\psi(-x)$, we have

$$\psi x - a^2\,\psi x = x^n - a(-x)^n,$$

hence

$$\psi x = \frac{1-(-1)^n a}{1-a^2}\,x^n.$$

(2). Given $\psi x - a\,\psi \frac{1}{x} = \varepsilon^x$

put $\frac{1}{x}$ for x, $\psi \frac{1}{x} - a\,\psi x = \varepsilon^{\frac{1}{x}}$

and

$$\psi x - a\,\varepsilon^{\frac{1}{x}} - a^2\,\psi x = \varepsilon^x,$$

$$\psi x = \frac{\varepsilon^x + a\,\varepsilon^{\frac{1}{x}}}{1-a^2}$$

(3). Given $(\psi x)^2 \,.\, \psi \frac{1-x}{1+x} = c^2 x$

put $\frac{1-x}{1+x}$ for x, it becomes

$$\left(\psi \frac{1-x}{1+x}\right)^2 .\, \psi x = c^2\,\frac{1-x}{1+x},$$

eliminating $\psi \frac{1-x}{1+x}$ by means of the former, we find

$$\psi x = \left(\frac{1+x}{1-x}\,c^2 x^2\right)^{\frac{1}{3}}.$$

(4). Given $\psi x + \frac{1}{1 - x^2}\psi\sqrt{(1 - x^2)} = 1 + x^2$

putting $\sqrt{(1 - x^2)}$ for x, we have

$$\psi\sqrt{(1 - x^2)} + \frac{1}{x^2}\psi x = 2 - x^2,$$

and substituting this value of $\psi\sqrt{1 - x^2}$ in the former equation

$$\psi x + \frac{2 - x^2}{1 - x^2} - \frac{1}{x^2 - x^4}\psi x = 1 + x^2,$$

hence

$$\left(\frac{x^2 - x^4 - 1}{x^2 - x^4}\right)\psi x = 1 + x^2 - \frac{2 - x^2}{1 - x^2} = \frac{-1 + x^2 - x^4}{1 - x^2}$$

and $\psi x = x^2$.

(5). Given $\frac{\psi x}{1 + \psi x} + x\frac{\psi(-x)}{1 + \psi(-x)} = 1$

put $\psi_1 x = \frac{\psi x}{1 + \psi x}$ thus the equation becomes

$\psi_1 x + x\psi_1(-x) = 1$, and changing x into $-x$ we have $\psi_1(-x) - x\psi_1(x) = 1$, by which eliminating $\psi(-x)$ from the former, we find

$$\psi_1 x = \frac{1 - x}{1 + x^2},$$

hence

$$\psi x = \frac{\psi_1 x}{1 - \psi_1 x} = \frac{1 - x}{x + x^2}.$$

(6). Given $\psi x + \frac{1 + x}{x}\psi\frac{1}{x} = c,$

putting $\frac{1}{x}$ for x this becomes

$$\psi\,\frac{1}{x} + (1 + x)\,\psi\,x = x$$

and by eliminating $\psi\,\frac{1}{x}$, we have

$$\psi\,x = \frac{1}{1 + x + x^2}\,c$$

(7). Given $\psi\,x + x\,\psi\,(1 - x) = 1$,
putting $1 - x$ for x, we have

$$\psi\,(1 - x) + (1 - x)\,\psi\,(x) = 1,$$

whence, by elimination,

$$\psi\,x = \frac{1 - x}{1 - x\,(1 - x)} = \frac{1 - x}{1 - x + x^2}.$$

(8). Given $\dfrac{\psi\,x}{\psi\,x - x} + x\,\dfrac{\psi\,(1 - x)}{\psi\,(1 - x) + x - 1} = 1$,

put $\psi_1\,x = \dfrac{\psi\,x}{\psi\,x - x}$, then will $\psi_1(1 - x) = \dfrac{\psi(1 - x)}{\psi(1 - x) + x - 1}$,

and the equation becomes

$$\psi_1\,x + x\,\psi_1\,(1 - x) = 1,$$

the same as in the last example; let $f\,x$ represent the solution there found, then

$$\psi_1\,x = f\,x = \frac{\psi\,x}{\psi\,x - x},$$

whence

$$\psi\,x = \frac{x\,f\,x}{f\,x - 1},$$

if we take for $f\,x$ its value $\dfrac{1 - x}{1 - x + x^2}$, we have

$$\psi\,x = \frac{x - 1}{x}.$$

In case the equation is symmetrical with regard to $\psi\, x$ and $\psi\, a\, x$, the process of elimination apparently becomes illusory. By a peculiar artifice this difficulty may be overcome, and it happens rather singularly that in all these cases, the solution which is so obtained contains an arbitrary function, and in general the solution is the most extensive which the question admits of.

(9). Given $\psi\, x = \psi\, \frac{1}{x}$.

If we put $\frac{1}{x}$ for x, this is changed into $\psi\, \frac{1}{x} = \psi\, x$, the same as the given equation; it is therefore impossible to eliminate.

Let us now suppose $\psi\, x = a\, \psi\, \frac{1}{x} + b$,

which becomes the given equation when $a = 1$ and $b = 0$. By putting $\frac{1}{x}$ for x this is changed into

$$\psi\, \frac{1}{x} = a\, \psi\, x + b,$$

and eliminating $\psi\, \frac{1}{x}$, we have

$$\psi\, x = \frac{a\, b + b}{1 - a^2} = \frac{b}{1 - a},$$

if $b = 0$ and $a = 1$, this becomes a vanishing fraction whose value is any constant quantity c, and we have $\psi\, x = c$, which fulfils the equation. This is a very limited solution, but the following plan will lead us to much more general ones.

Take the equation

$$\psi\, x = a\, \psi\, \frac{1}{x} + v\, \phi\, x,$$

which coincides with the given one when $v = 0$ and $a = 1$; also $\phi\, x$ is any arbitrary function of x; putting $\frac{1}{x}$ for x, we have

$$\psi\frac{1}{x} = a\,\psi\,x + v\,\phi\frac{1}{x},$$

and by elimination,

$$\psi\,x = \frac{a\,\phi\frac{1}{x} + \phi\,x}{1 - a^2}\,v.$$

Let a become $1 + 0$ and v become 0 at the same time, then

$$\frac{0}{1-(1+2.0+0^2)} = \frac{0}{-2.0+0^2} = \frac{1}{-2+0} = -\frac{1}{2}$$

and the solution becomes

$$\psi\,x = -\frac{\phi\frac{1}{x} + \phi\,x}{2},$$

or changing the arbitrary function

$$\psi\,x = \phi\,x + \phi\frac{1}{x},$$

in which ϕ is indefinite.

This solution is, in fact, nothing more than an arbitrary symmetrical function of x and $\frac{1}{x}$, and may be expressed thus

$$\psi\,x = \chi\left(\overline{x}, \overline{\frac{1}{x}}\right).$$

Precisely the same course of reasoning will produce the solutions of the following equation.

(10). $\psi(x) = \psi(a - x)$

$$\psi\,x = \chi\,(\overline{x}, \overline{a - x}).$$

(11). $\psi(x) = \psi\left(\frac{1-x}{1+x}\right)$, $\psi\,x = \chi\left\{\overline{x}, \overline{\frac{1-x}{1+x}}\right\}$.

(12). $\psi(x) = \psi\left(\frac{x}{\sqrt{(x^2-1)}}\right)$, $\psi x = \chi\left\{\bar{x}, \overline{\frac{x}{\sqrt{(x^2-1)}}}\right\}$.

(13). $\psi\left(\frac{x}{2-x}\right) = \psi(1-x)$, $\psi x = \chi\left\{\bar{x}, \overline{\frac{1-x}{1+x}}\right\}$.

(14). $\psi x = \psi(\alpha x)$, $\psi x = \chi(\bar{x}, \overline{\alpha x})$, where $\alpha^2 x = x$.

(15). The objection which has just been stated occurs in the equation $\psi x + \psi\left(\frac{x}{x-1}\right) = c$,

and a similar mode of proceeding will obviate it. The given equation is a particular case of

$$\psi x + a\psi\left(\frac{x}{x-1}\right) = c + v\phi x,$$

with which it coincides, if $a = 1$ and $v = 0$; putting $\frac{x}{x-1}$ for x in this, we have

$$\psi\left(\frac{x}{x-1}\right) + a\psi x = c + v\phi\left(\frac{x}{x-1}\right),$$

and elimination produces

$$\psi x = \frac{c-ac}{1-a^2} + \left\{\phi x - a\phi\left(\frac{x}{x-1}\right)\right\}\frac{v}{1-a^2}$$

If $v = 0$ and $a = 1$, this gives

$$\psi x = \frac{c}{2} + \phi x - \phi\left(\frac{x}{x-1}\right)$$

in which the function ϕ has been changed into another similar one.

16. Given $\psi(1+x) + \psi(1-x) = 1 - x^2$,

put $x-1$ for x, then

$$\psi x + \psi(2 - x) = 1 - (x - 1)^2 = 2x - x^2,$$

this is a particular case of the equation

$$\psi x + a\psi(2 - x) = 2x - x^2 + v\phi x,$$

with which it agrees if $v=0$ and $a=1$; changing x into $2-x$ and eliminating $\psi(2 - x)$ from the result, we find

$$\psi x = \frac{2x - x^2 - a(2x - x^2)}{1 - a^2} + \{\phi x - a\phi(2-x)\}\frac{v}{1 - a^2}.$$

or

$$\psi x = \frac{2x - x^2}{1 + a} + \{\phi x - a\phi(2 - x)\}\frac{v}{1 - a^2},$$

If $a=1$ and $v=0$, we have

$$\psi x = \frac{2x - x^2}{2} + \phi x - \phi(2 - x).$$

(17.) Given $\dfrac{1}{x + \psi x} + \dfrac{x}{1 + x\psi\frac{1}{x}} = 2,$

put $\dfrac{1}{x + \psi x} = \psi_1 x$ then $\dfrac{1}{\frac{1}{x} + \psi\frac{1}{x}} = \psi_1\dfrac{1}{x} = \dfrac{x}{1 + x\psi\frac{1}{x}},$

and the equation becomes

$$\psi_1 x + \psi_1\frac{1}{x} = 2,$$

whose solution may be found by the method just explained to be

$$\psi_1 x = 1 + \phi x - \phi\frac{1}{x},$$

hence

$$\psi x = \frac{1}{1 + \phi x - \phi\frac{1}{x}} - x.$$

(18). Required the equation of that class of curves which possess the following property, (Part IV. Fig. 1.) a given abscissa $AB = a$ being taken, then the product of any two ordinates at equal distances from B, shall always be equal to the square of the abscissa a. If $y = \psi x$ represent the equation of the curve, then the condition expressed analytically is

$$\psi(a - x) \cdot \psi(a + x) = a^2.$$

Putting $a - x$ for x, and then $\log \psi(x) = \psi_1 x$, we have

$$\psi_1 x + \psi_1(2a - x) = 2 \log a,$$

whose solution is

$$\psi_1 x = \log a + \phi x - \phi(2a - x),$$

hence

$$\log \psi x = \log a + \phi x - \phi(2a - x),$$

and

$$\psi x = \varepsilon^{\log a} \times \varepsilon^{\phi x} \times \varepsilon^{-\phi(2a - x)} = \frac{a\,\varepsilon^{\phi x}}{\varepsilon^{\phi(2a - x)}}$$

and changing the arbitrary function ϕ into $\log \phi$,

$$\psi x = a \frac{\phi x}{\phi(2a - x)},$$

and the class of curves are comprehended in the equation

$$y = \frac{a \phi x}{\phi(2a - x)}.$$

(19). Given the equation

$$\{\psi x\}^2 + \left\{\psi\left(\frac{\pi}{2} - x\right)\right\}^2 = 1.$$

This is the equation on which the composition of forces is made to depend in the Mecanique Cœleste, p. 5.

Put $\psi_1 x$ for $(\psi x)^2$ then it becomes

$$\psi x + \psi_1\left(\frac{\pi}{2} - x\right) = 1.$$

which is a particular case of

$$\psi_1 x + a\,\psi_1\left(\frac{\pi}{2} - x\right) = 1 + v\,\phi\,x.$$

Substituting $\frac{\pi}{2} - x$ for x, this gives

$$\psi_1\left(\frac{\pi}{2} - x\right) + \psi_1 x = 1 + v\,\phi\left(\frac{\pi}{2} - x\right),$$

and eliminating $\psi_1\left(\frac{\pi}{2} - x\right)$, we have

$$\psi x = \frac{1}{1 + a} - \frac{\phi x - a\,\phi\left(\frac{\pi}{2} - x\right)v}{1 - a^2}$$

making $a = 1$ and $v = 0$, and changing ϕx in $2\,\phi x$, we have

$$\psi_1 x = \frac{1}{2} - \phi x + \phi\left(\frac{\pi}{2} - x\right)$$

and therefore

$$\psi x = \sqrt{\frac{1}{2} - \phi x + \phi\left(\frac{\pi}{2} - x\right)}$$

In case the coefficient of $\psi\,a\,x$ in the equation $\psi x + f x\,\psi\,a\,x = f x$, is of such a form that $f x \,.\, f a\,x = 1$, the denominator will vanish, and we must then have recourse to an artifice similar to that which has already been explained.

(20). Ex. 1. Let $\psi x + x^{2n}\,\psi\,\frac{1}{x} = x^n.$

put $\psi x + (x^{2n} + v\,\phi\,x)\,\psi\,\frac{1}{x} = x^n$

which coincides with the given equation if $v=0$; then changing x into $\frac{1}{x}$, we have

$$\psi\frac{1}{x} + \left(x^{-2n} + v\,\phi\frac{1}{x}\right)\psi x = x^{-n}$$

and by elimination

$$\psi x = \frac{x^{n} - (x^{2n} + v\,\phi x)\,x^{-n}}{1-(x^{2n}+v\,\phi x)\left(x^{-2n} + v\,\phi\frac{1}{x}\right)}$$

$$= \frac{x^{-n}\,\phi x}{x^{2n}\,\phi\frac{1}{x} + x^{-2n}\,\phi x + v\,\phi x\,.\,\phi\frac{1}{x}}$$

and when v vanishes,

$$\psi x = \frac{x^{-n}\,\phi x}{x^{2n}\,\phi\frac{1}{x} + x^{-2n}\,\phi x}, \qquad (a).$$

The equation in this example may be solved differently, as follows. Multiply by x^{-n} and it becomes

$$x^{-n}\,\psi x + x^{n}\,\psi\frac{1}{x} = 1,$$

put $\psi_1 x = x^{-n}\,\psi x$, then

$$\psi_1 x + \psi_1\frac{1}{x} = 1,$$

whose general solution found by a process already explained is

$$\psi_1 x = \frac{1}{2} - \phi x + \phi\frac{1}{x},$$

hence

$$\psi x = \frac{x^{n}}{2} - x^{n}\,\phi x + x^{n}\,\phi\frac{1}{x}.$$

This solution differs in form from that which was previously found, but it may be proved to be the same by the following substitution; since ϕ is quite an arbitrary function

$$\phi x = x^{2n}\left(1 - 2\,\phi_1 x + 2\,\phi_1 \frac{1}{x}\right),$$

this gives $x^{2n}\,\phi\frac{1}{x} + x^{-2n}\,\phi\,x = 2$ and (a) becomes

$$\psi x = \frac{x^n}{2} - x^n\,\phi_1 x + x^n\,\phi_1\frac{1}{x},$$

exactly as the last solution.

(21). Given

$$\left(\frac{\psi x + 1}{\psi x - 1}\right)^n - \left(\frac{\psi\left(\dfrac{x}{x-1}\right) + 1}{\psi\left(\dfrac{x}{x-1}\right) - 1}\right)^n = \frac{(x-1)^2 - 1}{x - 1};$$

put $\psi_1 x = \dfrac{\psi x + 1}{\psi x - 1}$, then it becomes

$$\psi_1 x - \psi_1\left(\frac{x}{x-1}\right) = \frac{(x-1)^2 - 1}{x - 1},$$

which is a particular case of

$$\psi_1 x + (a + v\,\phi\,x)\,\psi_1\left(\frac{x}{x-1}\right) = \frac{(x-1)^2 - 1}{x - 1}$$

with which it agrees, when $a = -1$ and $v = 0$.

Put $\dfrac{x}{x-1}$ for x, then

$$\psi_1 \frac{x}{x-1} + \left(a + v\,\phi\frac{x}{x-1}\right)\psi_1 x = -\frac{(x-1)^2 - 1}{x - 1};$$

whence by elimination,

$$\psi_1 x = \frac{(x-1)^2 - 1}{x-1} \cdot \frac{1 + a + v\phi x}{1 - (a + v\phi x)\left(a + v\phi \frac{x}{x-1}\right)},$$

and when $a = -1$, and $v = 0$, this becomes

$$\psi_1 x = \frac{x^2 - 2x}{x-1} \cdot \frac{\phi x}{\phi x + \phi\left(\frac{x}{x-1}\right)},$$

and restoring the value of $\psi_1 x$, we find

$$\psi x = \frac{\left\{\frac{x^2 - 2x}{x-1} \cdot \frac{\phi x}{\phi x + \phi\left(\frac{x}{x-1}\right)}\right\}^{\frac{1}{n}} + 1}{\left\{\frac{x^2 - 2x}{x-1} \cdot \frac{\phi x}{\phi x + \phi\left(\frac{x}{x-1}\right)}\right\}^{\frac{1}{n}} - 1}$$

(22). $\psi x + \frac{x}{\sqrt{(1+x^2)}} \psi \sqrt{(1-x^2)} = x;$

$$\psi x = \frac{x(1-x^2)\phi x}{x^2 \phi \sqrt{(1-x^2)} + (1-x^2)\phi x}:$$

this solution was found by pursuing the course so frequently pointed out: another but not a more general one may be obtained as follows: multiply by $\sqrt{(1-x^2)}$; then

$$\sqrt{(1-x^2)}\,\psi x + x\,\psi\sqrt{(1-x^2)} = x\sqrt{(1-x^2)},$$

putting $\sqrt{(1-x^2)}\,\psi x = \psi_1 x$, we have

$$\psi_1 x + \psi_1 \sqrt{(1-x)} = x\sqrt{(1-x^2)},$$

whose solution is $\psi_1 x = \frac{x\sqrt{(1-x^2)}}{2} + \phi x - \phi\sqrt{(1-x^2)},$

hence

$$\psi x = \frac{x}{2} + \frac{\phi x - \phi\sqrt{(1-x^2)}}{\sqrt{(1-x^2)}}.$$

It would not be difficult to shew the identity of these two apparently different solutions.

(23). Given the equation

$$\frac{\psi x}{\sqrt{[\psi(1-x)]}} + \frac{\psi(1-x)}{\sqrt{(\psi x)}} = 1,$$

put $\psi_1 x = \dfrac{\psi x}{\sqrt{[\psi(1-x)]}}$, then $\psi_1(1-x) = \dfrac{\psi(1-x)}{\sqrt{(\psi x)}}$;

and the equation becomes

$$\psi_1 x + \psi_1(1-x) = 1,$$

whose general solution is

$$\psi_1 x = \frac{\phi x}{\phi x + \phi(1-x)};$$

hence

$$\frac{(\psi x)^2}{\psi(1-x)} = \frac{(\phi x)^2}{[\phi x + \phi(1-x)]^2}$$

putting $1-x$ for x, and eliminating $\psi(1-x)$, we find

$$\psi x = \frac{\{(\phi x)^2 \phi(1-x)\}^{\frac{2}{3}}}{[\phi(1-x) + \phi x)^2}.$$

(24). Given $(\psi x)^2 + \left(\psi \dfrac{a^2}{x}\right)^2 = \dfrac{x^4 + a^4}{x^2} \psi x \,.\, \psi \dfrac{a^2}{x}$;

divide by $\psi x \,.\, \psi \dfrac{a^2}{x}$ then

$$\frac{\psi x}{\psi \frac{a^2}{x}} + \frac{\psi \frac{a^2}{x}}{\psi x} = \frac{x^4 + a^4}{x^2};$$

putting $\psi_1 x = \dfrac{x}{\psi \frac{a^2}{x}}$, this becomes

$$\psi_1 x + \psi_1 \frac{a^2}{x} = \frac{x^4 + a^4}{a^2},$$

a particular solution of which is $\psi_1 x = x^2$; hence

$$\frac{\psi x}{\psi \frac{a^2}{x}} = a^2 \text{ or } \psi x = x^2 \psi \frac{a^2}{x}$$

and the general solution of this is

$$\psi x = x \chi \left(\bar{x}, \overline{\frac{a^2}{x}} \right).$$

(25). Given $\dfrac{\psi x + x}{\psi \frac{1}{x}} + x \dfrac{1 + x \psi \frac{1}{x}}{\psi x} = 1 + x^2,$

put $\psi_1 x = \dfrac{\psi x + x}{\psi \frac{1}{x}}$, then the equation becomes

$$\psi_1 x + x^2 \psi_1 \frac{1}{x} = 1 + x^2,$$

whose solution is $\psi_1 x = \dfrac{(x^2 + 1) \phi x}{\phi x + x^4 \phi \frac{1}{x}}$;

hence $\dfrac{x + \psi x}{\psi \frac{1}{x}} = \dfrac{(x^2 + 1) \phi x}{\phi x + x^4 \phi \frac{1}{x}}$;

putting $\frac{1}{x}$ for x and eliminating $\psi \frac{1}{x}$, we find

$$\psi x = \frac{(2x + x^{-1}) \phi x + x^5 \phi \frac{1}{x}}{(x^2 + x^{-2}) \phi x \,.\, \phi \frac{1}{x} - x^{-4} (\phi x)^2 - x^{-4} \left(\phi \frac{1}{x} \right)^2} \left(\phi \frac{1}{x} + x^{-4} \phi x \right)$$

(26). Given $(\psi\, x)^m . (\psi - x)^n - (\psi\, x)^n . (\psi - x)^m = 2\, x$; putting $\psi_1 x = (\psi\, x)^m . (\psi - x)^n$, it becomes

$$\psi_1 x - \psi_1 (-x) = 2\, x,$$

whose solution is $\psi_1 x = x + \chi(\overline{x}, \overline{-x})$,

hence

$$(\psi\, x)^m . (\psi - x)^n . = x + \chi(\overline{x}, \overline{-x}),$$

and by the process for eliminating $\psi(-x)$, we shall find

$$\psi\, x = \left\{ \frac{\chi(\overline{x}, \overline{-x}) + x}{\{\chi(\overline{x}, \overline{-x}) - x\}^{\frac{m}{n}}} \right\}^{\frac{m}{m^2 - n^2}}.$$

(27). Given $\psi\, x + f\, x . \psi\, a\, x = f_1\, x$, where $a\, x$ is such a function of x that $a^2\, x = x$; putting $a\, x$ for x, we have

$$\psi\, a\, x + f\, a\, x . \psi\, x = f_1\, a\, x,$$

and by eliminating $\psi\, a\, x$

$$\psi\, x = \frac{f_1\, x - f\, x . f_1\, a\, x}{1 - f\, x . f\, a\, x}.$$

If $f\, x . f\, a\, x = 1$ and $f_1\, x - f\, x . f_1\, a\, x = 0$, then the solution becomes a vanishing fraction; also the general value of $f_1\, x$ is in that case $f_1\, x = \surd(f\, x) . f_2(\overline{x}, \overline{a\, x})$ and the equation becomes

$$\psi\, x + f\, x . \psi\, a\, x = \surd f\, x . f_2(\overline{x}, \overline{a\, x});$$

dividing this by $\surd(f\, x)$ and putting instead of $\frac{1}{\surd f\, x}$ its value $\surd(f\, a\, x)$ derived from the equation $f\, x . f\, a\, x = 1$, we have

$$\surd(f\, a\, x) . \psi\, x + \surd(f\, x) . \psi\, a\, x = f_2(\overline{x}, \overline{a\, x}),$$

which is a symmetrical equation, whose general solution is

$$\surd(f\, a\, x) . \psi\, x = \frac{f_2(\overline{x}, \overline{a\, x}) . \phi\, x}{\phi\, x + \phi\, a\, x},$$

hence

$$\psi x = \frac{\sqrt{(fx)} \cdot f_2(\overline{x}, \overline{\alpha x}) \cdot \phi x}{\phi x + \phi \alpha x}.$$

(28). Given $a + b \psi x = \psi (a + b x)$,

$$\psi x = a \frac{1 - b^n}{1 - b} + b^n x.$$

(29). Given $\dfrac{a \psi x}{b + c \psi x} = \psi \left(\dfrac{a x}{b + c x} \right)$;

$$\psi x = \frac{a^n x}{b^n + c \dfrac{a^n - b^n}{a - b} x}.$$

(30). Given $\dfrac{\psi x}{\psi x - 1} = \psi \left(\dfrac{x}{x - 1} \right)$;

$$\psi x = \frac{x}{\dfrac{1 - (-1)^n}{2} x + (-1)^n},$$

where n is arbitrary; it may therefore be changed into any symmetrical function of x and $\dfrac{x}{x - 1}$.

(31). The three last examples are particular cases of the equation

$$\alpha \psi x = \psi \alpha x,$$

whose general solution is $\psi x = \alpha^n x$.

(32). Given $\dfrac{\psi \left(\dfrac{x}{1 + x} \right)}{1 + \psi \left(\dfrac{x}{1 + x} \right)} = \psi \left(\dfrac{x}{1 + 2x} \right)$,

$$\psi x = \frac{x}{1 + n x}.$$

(33). Given $\alpha\psi\alpha x = \psi\alpha^2 x$

$$\psi x = \alpha^n x.$$

(34). Given $\psi x + a\psi\left(\frac{1}{1-x}\right) = \frac{1}{x}$;

$\frac{1}{1-x}$ is a periodic function of the third order, or of the form $\alpha^3 x = x$; putting $\frac{1}{1-x}$ for x, we have

$$\psi\left(\frac{1}{1-x}\right) + a\psi\left(\frac{x-1}{x}\right) = 1 - x,$$

and in this again putting $\frac{1}{1-x}$ for x, we find

$$\psi\left(\frac{x-1}{x}\right) + a\psi x = \frac{x}{x-1};$$

$\psi\left(\frac{1}{x-1}\right)$ and $\psi\left(\frac{x-1}{x}\right)$ being eliminated between these three equations, we have

$$\psi x = \frac{1}{1+a^3}\left\{\frac{1}{x} - a(1-x) + a^2\frac{x}{x-1}\right\}.$$

(35). Given $\psi x - a\psi\left(\frac{\sqrt{(x^2-1)}}{x}\right) = x^{2n}$,

put $\frac{\sqrt{(x^2-1)}}{x}$ for x, it becomes

$$\psi\left(\frac{\sqrt{(x^2-1)}}{x}\right) - a\psi\left(\frac{1}{\sqrt{(1-x^2)}}\right) = \frac{(x^2-1)^n}{x^{2n}}.$$

Again, put $\frac{\sqrt{(x^2-1)}}{x}$ for x, and we have

$$\psi\left(\frac{1}{\sqrt{(1-x^2)}}\right) - a\psi x = \frac{1}{(1-x^2)^n},$$

by eliminating $\psi\left(\frac{\sqrt{(x^2-1)}}{x}\right)$ and $\psi\left(\frac{1}{\sqrt{(1-x^2)}}\right)$ from these three equations, we shall find

$$\psi x = \frac{1}{1-a^3}\left\{x^{2n} + a\,\frac{(x^2-1)^n}{x^{2n}} + a^2\,\frac{1}{(1-x^2)^n}\right\}.$$

(36). Given $\psi x + \psi\left(\frac{1+x}{1-3x}\right) + \psi\left(\frac{x-1}{3x+1}\right) = a$

the function $\frac{1+x}{1-3x}$ is periodic of the third order, and by the process of elimination

$$\psi x = \frac{a\,\phi x + \phi_1 x - \phi_1\left(\frac{1+x}{1-3x}\right)}{\phi x + \phi\left(\frac{1+x}{1-3x}\right) + \phi\left(\frac{x-1}{3x+1}\right)}.$$

(37). Given $\frac{1}{\sqrt{\psi x}} + \frac{1}{\sqrt{\psi\frac{x-1}{x}}} + \frac{1}{\sqrt{\psi\frac{1}{1-x}}} = a.$

Putting $\psi_1 x = \frac{1}{\sqrt{\psi x}}$, we have

$$\psi_1 x + \psi_1\frac{x-1}{x} + \psi_1\frac{1}{1-x} = a,$$

whose solution is

$$\psi_1 x = \frac{a\,\phi x + \phi_1 x - \phi_1\frac{1}{1-x}}{\phi x + \phi\frac{1}{1-x} + \phi\frac{x-1}{x}},$$

hence

$$\psi x = \left\{\frac{\phi x + \phi\frac{1}{1-x} + \phi\frac{x-1}{x}}{a\,\phi x + \phi_1 x - \phi_1\frac{1}{1-x}}\right\}^2.$$

(38). Given $\psi x \,.\, \psi \frac{1}{1-x} \,.\, \psi \frac{x-1}{x} = c^3$.

Putting $\psi_1 x = \log \psi x$, we find

$$\psi_1 x + \psi_1 \frac{1}{1-x} + \psi_1 \frac{x-1}{x} = \log (c^3),$$

whose solution is found in the last problem. Changing ϕ_1 into $\log \phi_1$, we have

$$\psi x = \frac{\phi_1 x}{\phi_1 \frac{1}{1-x}} \log^{-1} \left(\frac{3\, \phi x \,.\, \log c}{\phi x + \phi \frac{x-1}{x} + \phi \frac{1}{1-x}} \right).$$

Similarly if αx be any periodic equation of the third order.

(39). $\psi x + \psi \alpha x + \psi \alpha^2 x = a$,

has for its solution

$$\psi x = \frac{a \phi x + \phi_1 x - \phi_1 \alpha x}{\phi x + \phi \alpha x + \phi \alpha^2 x}.$$

(40). $\psi x \,.\, \psi \alpha x \,.\, \psi \alpha^2 x = c^3$,

has for its solution

$$\psi x = \frac{\phi_1 x}{\phi \alpha x} \log^{-1} \left\{ \frac{3\, \phi x \,.\, \log c}{\phi x + \phi \alpha x + \phi \alpha^2 x} \right\}.$$

(41). Given $\psi x + f x \,.\, \psi \alpha x = f_1 x$, where $\alpha^3 x = x$,

Putting successively αx and $\alpha^2 x$ for x, we have

$$\psi \alpha x + f \alpha x \,.\, \psi \alpha^2 x = f_1 \alpha x,$$
$$\psi \alpha^2 x + f \alpha^2 x\, \psi x = f_1 \alpha^2 x,$$

and eliminating $\psi \alpha x$ and $\psi \alpha^2 x$ from these three equations, we have

$$\psi x = \frac{f_1 x - f x \,.\, f_1 \alpha x + f x \,.\, f \alpha x \,.\, f_1 \alpha^2 x}{1 + f x \,.\, f \alpha x \,.\, f \alpha^2 x}.$$

(42). Given $\psi x + f x . \psi \alpha x = f_1 x$ where $\alpha^n x = x$, a similar process of elimination will produce

$$\psi x = \frac{f_1 x - f x . f_1 \alpha x + \ldots . f x . f \alpha x . f \alpha^{n-2} x . f_1 \alpha^{n-1} x}{1 - (-1)^n f x . f \alpha x . . f \alpha^{n-1} x}.$$

(43). Given the equation

$$1 + f x . (\psi x + \psi \alpha x) - \psi x . \psi \alpha x = 0,$$

where αx is a periodic function of the second order, and $f x$ is any function symmetrical relative to x and αx

$$f x = \frac{\psi x . \psi \alpha x - 1}{\psi x + \psi \alpha x},$$

consider ψx and $\psi \alpha x$ as two variables, and differentiate with respect to them, then

$$\frac{d \psi x}{1 + (\psi x)^2} + \frac{d \psi \alpha x}{(1 + \psi \alpha x)^2} = 0,$$

and by integration,

$$\tan^{-1} \psi x + \tan^{-1} \psi \alpha x = C = -\frac{1}{f x},$$

whose complete solution is

$$\tan^{-1} \psi x = \frac{\phi x}{\phi x + \phi \alpha x} \tan^{-1} \frac{-1}{f x},$$

hence

$$\psi x = \tan \left\{ \frac{\phi x}{\phi x + \phi \alpha x} \tan^{-1} \frac{-1}{f x} \right\},$$

this process is analogous to one employed by M. Laplace, for the integration of a similar equation of differences.

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* *Journal de l'Ecole Polytecnique,* Cah. 15.

(44). Given

$$\psi x + \psi\left(\frac{1+x}{1-x}\right) + \psi\left(-\frac{1}{x}\right) + \psi\left(\frac{x-1}{x+1}\right) = 1.$$

$\frac{1+x}{1-x}$ being a periodic function of the 4th order.

$$\psi x = \frac{\phi x}{\phi x + \phi\left(\frac{1+x}{1-x}\right) + \phi\left(\frac{-1}{x}\right) + \phi\left(\frac{x-1}{x+1}\right)}$$

$$+ \frac{x^2+1}{x-1}\chi\left(\overline{x}, \overline{\frac{1+x}{1-x}}, \overline{\frac{-1}{x}}, \overline{\frac{x-1}{x+1}}\right) +$$

$$+ \frac{x^2+1}{x.(1-x)}\chi\left\{\overline{x}, \overline{\frac{1+x}{1-x}}, \overline{\frac{-1}{x}}, \overline{\frac{x-1}{x+1}}\right\}.$$

(45). Given $\psi(x, y) + \psi\left(\frac{a^2}{x}, \frac{a^2}{y}\right) = 1,$

$$\psi(x, y) = \frac{\phi(x, y)}{\phi(x, y) + \phi\left(\frac{a^2}{x}, \frac{a^2}{y}\right)}.$$

(46). $\psi(x, y) + \psi\left(\frac{a^2}{x}, -y\right) = y^2,$

$$\psi(x\ \ y) = \frac{y^2\phi(x, y)}{\phi(x, y) + \phi\left(\frac{a^2}{x}, -y\right)}.$$

(47). Given $\psi(x, y) + x^2\psi\left(\frac{a^2}{x}, \frac{y}{y-1}\right) = \frac{x y^2}{y-1},$

$$\psi(x, y) = \frac{(y-1)^{-1} x y^2 \phi(x, y)}{\phi(x, y) + \phi\left(x, \frac{y}{y-1}\right)}.$$

(48). Given $\psi(x, y) + f(x, y)\psi(\alpha x, \beta y) =$

$$\sqrt{f(x, y)}.f_1(\overline{x}, \overline{\alpha x}, \overline{y}^{1}, \overline{\beta y}^{1}),$$

where $\alpha^2 x = x$, $\beta^2 y = y$ and $f(x, y)$ is such a function that

$$f(x, y) \cdot f(\alpha x, \beta y) = 1,$$

then $$\psi(x, y) = \frac{\sqrt{f(\alpha x, \beta y)} \cdot f_1(\overline{x}, \overline{\alpha x}, \overline{y}^{1}, \overline{\beta y}^{1})}{f(x, y)\phi(\alpha x, \beta y) + f(\alpha x, \beta y)\phi(x, y)},$$

(49). Given $\psi(x, y) = \psi\left(\frac{x-y}{y}, \frac{x-y}{x}\right)$

$$\psi(x, y) = \phi\left(\frac{x}{y}\right).$$

(50). Given $\psi(x, y) = \psi\left(\frac{f(x, y)}{y}, \frac{f(x, y)}{x}\right)$,

$$\psi(x, y) = \phi\left(\frac{x}{y}\right).$$

(51). Given $\psi(x, y) = \psi\left(\frac{y}{2}\sqrt{\frac{y}{2x}}, \sqrt{2xy}\right)$,

$$\psi(x, y) = \chi\left(2xy + y^2, \frac{1}{y\sqrt{2xy}}\right).$$

(52). Given $\psi(x, y) = \psi(y, x)$

$$\psi(x, y) = \chi(x + y, xy).$$

(53). Given $\psi(x, y) = \left(\frac{x}{y}\right)^3 \psi(y, x)$

$$\psi(x, y) = \frac{x}{y^2} \cdot \phi(\bar{x}, \bar{y}) = \frac{x}{y^2} \cdot \phi_1(x + y, xy).$$

(54). Given $\psi(\pi - x) = \frac{d\psi x}{dx}$,

differentiating $\frac{d}{dx}\psi(\pi - x) = \frac{d^2\psi x}{dx^2}$, putting $\pi - x$ for x in the given equation

$$\psi(x) = -\frac{d\,\psi(\pi - x)}{dx},$$

and eliminating $\frac{d\,\psi(\pi - x)}{dx}$, we have

$$-\psi x = \frac{d^2\,\psi x}{dx^2},$$

whence by integration

$$\psi x = b\cos x + c\sin x,$$

and it will be found that $c = -b$; hence

$$\psi x = b(\cos x - \sin x).$$

(55). Given $\psi(x, y) = \frac{d\,\psi(x, a - y)}{dx}$,

put $a - y$ for y, then

$$\psi(x, a - y) = \frac{d\,\psi(x, y)}{dx},$$

differentiate this relative to x, then

$$\frac{d\,\psi(x, a - y)}{dx} = \frac{d^2\,\psi(x, y)}{dx^2},$$

which being substituted in the given equation produces

$$\psi(x, y) = \frac{d^2\,\psi(x, y)}{dx^2},$$

whose solution is

$$\psi(x, y) = \varepsilon^x \phi y + \varepsilon^{-x}\phi_1 y,$$

ϕ and ϕ_1 being two arbitrary functions so constituted as to fulfil the given equation, in order to determine them, put $a - y$ for y and differentiate relative to x, then

$$\frac{d\,\psi(x, a - y)}{dx} = \varepsilon^x \phi(a - y) - \varepsilon^{-x}\phi_1(a - y)$$

hence

$$\phi y = \phi(a - y) \text{ and } \phi_1 y = -\phi_1(a - y),$$

whose solutions are

$$\phi x = \chi(\overline{y}, \overline{a - y}) \text{ and } \phi_1 y = (a - 2y)\,\chi_1(\overline{y}, \overline{a - y}),$$

hence the general solution of the equation is

$$\psi(x, y) = \varepsilon^x \chi(\overline{y}, \overline{a - y}) + \varepsilon^{-x}(a - 2y)\,\chi_1(\overline{y}, \overline{a - y}) .$$

A similar mode of solution is applicable to the three following equations.

(56). Given $\psi(x, y) = \frac{d}{dx}\psi\left(x, \frac{1}{y}\right)$

$$\psi(x, y) = \varepsilon^x \phi\left(\overline{y}, \overline{\frac{1}{y}}\right) + \varepsilon^{-x}\frac{1 - y^2}{y}\phi_1\left(\overline{y}, \overline{\frac{1}{y}}\right).$$

(57). Given $\psi(x, y) = \frac{d}{dx}\psi\left(x, \frac{y}{y - 1}\right)$

$$\psi(x, y) = \varepsilon^x \phi\left(\overline{y}, \overline{\frac{y}{y - 1}}\right) + \varepsilon^{-x}\frac{2y - y^2}{y - 1}\phi_1\left(\overline{y}, \overline{\frac{y}{y - 1}}\right).$$

(58). Given $\psi(x, y) = \frac{d}{dx}\psi(x, \alpha y)$, where $\alpha^2 y = y$.

$$\psi(x, y) = \varepsilon^x \phi(y, \alpha y) + \varepsilon^{-x}(\alpha y - y)\,\phi_1(\overline{y}, \overline{\alpha y}).$$

(59). Given $\psi(x, y) = \frac{d\psi(x, \alpha y)}{dx}$

where α is such a function that $\alpha^4 y = y$.

Substituting successively αy, $\alpha^2 y$, $\alpha^3 y$ for y, we have

$$\psi(x, \alpha y) = \frac{d\psi(x, \alpha^2 y)}{dx},$$

$$\psi(x,\ a^2 y) = \frac{d\,\psi(x,\ a^3 y)}{d\,x},$$

$$\psi(x,\ a^3 y) = \frac{d\,\psi(x,\ y)}{d\,x}.$$

From the given equation $\frac{d\,\psi(x,\ a\,y)}{d\,x}$ may be eliminated by means of the second, and from the result $\frac{d\,\psi(x,\ a^2 y)}{d\,x}$ may be eliminated by the third equation, and continuing this, we should find

$$\psi(x,\ y) = \frac{d^4\,\psi(x,\ y)}{d\,x^4},$$

the solution of this equation is

$$\psi(x,\ y) = \varepsilon^x\,\phi\,y + \varepsilon^{-x}\,\phi_1\,y + \sin x\,.\,\phi_2\,y + \cos x\,.\,\phi_3\,y,$$

ϕ, ϕ_1, ϕ_2, ϕ_3 must be determined so as to satisfy the given equation, taking the differential and putting $a\,y$ for y, we have,

$$\frac{d\,\psi(x,\ a\,y)}{d\,x} = \varepsilon^x\,\phi\,a\,y - \varepsilon^{-x}\,\phi_1\,a\,y + \cos x\,.\,\phi_2\,a\,y - \sin x\,.\,\phi_3\,a\,y$$

the first condition to satisfy is

$$\phi\,y = \phi\,a\,y,$$

which gives

$$\phi\,y = \chi\,(\overline{y},\ \overline{a\,y},\ \overline{a^2\,y},\ \overline{a^3\,y}),$$

the next condition is

$$\phi_1 y = -\ \phi_1\,a\,y$$

whose solution is

$$\phi_1\,y = (a^3\,y - a^2\,y + a\,y - y\,(\chi_1\,(\overline{y},\ \overline{a\,y},\ \overline{a^2\,y},\ \overline{a^3\,y}),$$

the other two conditions are

$$\phi_2 y = -\ \phi_3\,a\,y, \text{ and } \phi_3\,y = \phi_2\,a\,y$$

putting αy for y in the second of these it becomes $\phi_3 \alpha y = \phi_2 \alpha^2 y$ and this substituted in the first gives,

$$\phi_2 y = - \phi_2 \alpha^2 y,$$

whose solution is $\phi_2 y = (\alpha^2 y - y) \chi_2 (\overline{y}, \overline{\alpha y}|^{1}, \overline{\alpha^2 y}, \overline{\alpha^3 y}|^{1})$

hence

$$\phi_3 y = (\alpha^3 y - \alpha y) \chi_2 (\overline{\alpha y}, \overline{\alpha^2 y}|^{1}, \overline{\alpha^3 y}, \overline{y}|^{1}),$$

and the general solution of the equation is

$$\psi(x, y) = \varepsilon^x \chi (\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}) +$$

$$+ \varepsilon^{-x} (\alpha^3 y - \alpha^2 y + \alpha y - y) \chi_1 (\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}) +$$

$$+ (\alpha^2 y - y) \chi_2 (\overline{y}, \overline{\alpha y}|^{1}, \overline{\alpha^2 y}, \overline{\alpha^3 y}|^{1}) \sin x +$$

$$+ (\alpha^3 y - \alpha y) \chi_2 (\overline{\alpha y}, \overline{\alpha^2 y}|^{1}, \overline{\alpha^3 y}, \overline{y}|^{1}) \cos x.$$

(60). Given the equation $\psi(x, y) = \dfrac{d^n \psi(x, \alpha y)}{d x^n}$,

where α is such a function that $\alpha^n x = x$.

This equation may be reduced to the solution of the partial differential equation

$$\psi(x, y) = \frac{d^{pn} \psi(x, y)}{d x^{pn}},$$

and the arbitrary functions of y which occur in its solution, must be determined by the conditions of the equation.

(61). Given the equation

$$\frac{d \psi(a - x, y)}{d y} = \frac{d \psi(x, b - y)}{d x},$$

put $a - x$ for x, also $b - y$ for y, then we have the two equations

$$\frac{d\,\psi(x,\ y)}{d\,y} = -\frac{d\,\psi(a-x,\ b-y)}{d\,x},$$

$$-\frac{d\,\psi(a-x,\ b-y)}{d\,y} = \frac{d\,\psi(x,\ y)}{d\,x}.$$

If the first of these be differentiated relative to y, and the second relative to x; then the right side of the first resulting equation will be identical with the left side of the second, and we shall have

$$\frac{d^2\,\psi(x,\ y)}{d\,y^2} = \frac{d^2\,\psi(x,\ y)}{d\,x^2};$$

the solution of this partial differential equation is

$$\psi(x,\ y) = \phi(x + y) + \phi_1(x - y);$$

the two arbitrary functions must be determined so as to satisfy the equation; we have

$$\frac{d\,\psi(a-x,\ y)}{d\,y} = \phi'(a - x + y) - \phi'_1(a - x - y),$$

$$\frac{d\,\psi(x,\ b-y)}{d\,x} = \phi'(b + x - y) + \phi'_1(-b + x + y),$$

ϕ' and ϕ'_1 being the differential coefficients of ϕ and ϕ_1 these two expressions must be identical, hence

$$\phi'(a - \overline{x - y}) = \phi'(b + \overline{x - y}),$$

and

$$-\phi'_1(a - \overline{x + y}) = \phi'_1(-b + \overline{x + y}),$$

the solutions of which equations are

$$\phi'(x + y) = \chi\,\{\overline{x + y},\ \overline{a - b - x - y}\},$$

and

$$\phi'_1(x + y) = (a - b - 2\,x - 2\,y)\,\chi_1\,\{\overline{x + y},\ \overline{a - b - x - y}\}$$

and substituting these values, we have

$$\psi(x,\,y) = \int(d\,x + d\,y)\,\chi\left\{\overline{x+y},\ \overline{a-b-x-y}\right\} +$$

$$+\int(d\,x - d\,y)(a-b-2\,x-2\,y)\,\chi_1\left\{\overline{x-y},\ \overline{a-b-2\,x+2\,y}\right\}.$$

(63). Given the equation

$$\frac{d\,\psi\left(x,\,\frac{1}{y}\right)}{d\,x} = \frac{d\,\psi\left(\frac{1}{x},\,y\right)}{d\,y}.$$

Put $\frac{1}{y}$ for y and differentiate relative to x, then

$$\frac{d^2\,\psi(x,\,y)}{d\,x^2} = -\frac{d^2\,\psi\left(\frac{1}{x},\,\frac{1}{y}\right)}{d\,x\,d\,y}\,y^2.$$

Again, put $\frac{1}{x}$ for x, and differentiate relative to y, then

$$-\frac{d^2\,\psi\left(\frac{1}{x},\,\frac{1}{y}\right)}{d\,x\,d\,y}\,x^2 = \frac{d^2\,\psi(x,\,y)}{d\,y^2}$$

hence

$$\frac{d^2\,\psi(x,\,y)}{d\,y^2} = \frac{x^2}{y^2}\,\frac{d^2\,\psi(x,\,y)}{d\,x^2},$$

the solution of this equation of partial differentials is

$$\psi(x,\,y) = x\,\phi\left(\frac{x}{y}\right) + \phi_1(x\,y):$$

to determine the form of ϕ and ϕ_1, we have

$$\frac{d\,\psi\left(x,\,\frac{1}{y}\right)}{d\,x} = \phi(x\,y) + x\,y\,\phi'(x\,y) + \frac{1}{y}\,\phi'_1\left(\frac{x}{y}\right),$$

$$\frac{d\,\psi\left(\frac{1}{x},\,y\right)}{d\,y} = -\frac{1}{x^2\,y}\,\phi'\left(\frac{1}{x\,y}\right) + \frac{1}{x}\,\phi'_1\left(\frac{y}{x}\right).$$

In order that these two expressions may coincide, we must have

$$\phi(xy) + xy\,\phi'(xy) = -\frac{1}{x^2y^2}\,\phi'\left(\frac{1}{xy}\right).$$

$$\phi'_1\left(\frac{x}{y}\right) = \frac{y}{x}\,\phi'_1\left(\frac{y}{x}\right).$$

The first of these multiplied by $d(xy)$ may be put under the form

$$d(xy).\,\phi(xy) + xy\,\frac{d\,\phi(xy)}{d(xy)}\,d(xy) = d\,\phi\left(\frac{1}{xy}\right)$$

whose integral is

$$xy\,\phi(xy) = \phi\left(\frac{1}{xy}\right),$$

the solution of which functional equation is

$$\phi(xy) = \frac{1}{\sqrt{xy}}\,\chi\left(\overline{xy,\ \frac{1}{xy}}\right)$$

the solution of the second equation is

$$\phi'_1\left(\frac{x}{y}\right) = \sqrt{\frac{y}{x}}\,.\,\chi_1\left(\overline{\frac{x}{y},\ \frac{y}{x}}\right),$$

employing these values of ϕ and ϕ_1, we have

$$\psi(x,\ y) = \sqrt{xy}\,.\,\chi\left(\overline{\frac{x}{y},\ \frac{y}{x}}\right) +$$

$$\int d(xy)\,.\left(\frac{1}{xy}\right)^{\frac{1}{2}}\chi_1\left(\overline{xy,\ \frac{1}{xy}}\right).$$

(63). Given $\dfrac{d\,\psi(x,\ \alpha y)}{dx} = \dfrac{d\,\psi(\beta x,\ y)}{dy}$,

where $\alpha^2 y = y$ and $\beta^2 x = x$ a process nearly similar to that by which the two last equations were solved will lead to the partial differential equation

$$\frac{d\,\alpha y}{dy} \cdot \frac{d^2\,\psi(x, y)}{d x^2} = \frac{d\,\beta x}{d x} \cdot \frac{d^2\,\psi(x, y)}{d y^2}.$$

(64). Given $\psi\,\alpha x = \psi\,\psi\,x = \psi^2 x$.

It is evident, that whatever be the form of α, this equation can always be satisfied by assuming $\psi\,x = \alpha\,x$, hence the solutions of the following equations,

$$\psi(-x) = \psi^2 x \qquad\qquad \psi\,x = -x$$

$$\psi\left(\frac{a x}{b + c x}\right) = \psi^2 x \qquad\qquad \psi\,x = \frac{a x}{b + c x}$$

$$\psi\sqrt{\frac{1 + x}{x}} = \psi^2 x \qquad\qquad \psi\,x = \sqrt{\frac{1 + x}{x}},$$

(65). Given $\psi(2a - x) = \psi^3 x$.

Put $\psi\,x = \phi^{-1} f \phi\,x$,

then $\psi^2 x = \phi^{-1} f\,\phi\,\phi^{-1} f\,\phi\,x = \phi^{-1} f^2\,\phi\,x$,

and $\psi^3 x = \phi^{-1} f^2\,\phi\,\phi^{-1} f\,\phi\,x = \phi^{-1} f^3\,\phi\,x$,

and the equation becomes

$$\phi^{-1} f \phi(2a - x) = \phi^{-1} f^3\,\phi\,x.$$

This equation may be satisfied in the following manner: by making f a periodic function of the second order, we have $f^2 v = v$, and the equation becomes

$$\phi^{-1} f \phi(2a - x) = \phi^{-1} f \phi\,x,$$

or

$$\phi(2a - x) = \phi\,x,$$

This is satisfied by making ϕ any symmetrical function of x and $2a - x$. As an example take $fv = -v$, also

$$\phi x = x . \overline{2a - x} = 2ax - x^2,$$

then

$$\phi^{-1} x = a \pm \sqrt{(a^2 - x)},$$

and

$$\psi x = \phi^{-1} f \phi x = a \pm \sqrt{a^2 - \frac{1}{x^2 - 2ax}},$$

(66). Given $\psi\left(\frac{1-x}{1+x}\right) = \psi^3 x$

$\psi x = \phi^{-1} f \phi x$ where ϕ and f are determined by the equations

$$\phi x = \chi \left\{ \overline{\frac{1-x}{1+x}}, \bar{x} \right\} \text{ and } f^2 x = x.$$

(67). Given $\psi \alpha x = \psi^3 x$, where $\alpha^n v = v$

putting $\psi x = \phi^{-1} f \phi x$, we have

$$\phi^{-1} f \phi \alpha x = \phi^{-1} f^3 \phi x$$

determine ϕ from the condition $\phi x = \phi \alpha x$; hence,

$$\phi x = \chi \{ \bar{x}, \overline{\alpha x}, \ldots\ldots \overline{\alpha^{n-1} x} \},$$

and let f be such a function that $f^2 x = x$, then the equation is satisfied.

(68). Given $\psi^p \alpha x = \psi^q x$ where $q > p$ and $\alpha^n x = x$, the substitution $\phi^{-1} f \phi x$ instead of ψ will give,

$$\phi^{-1} f^p \phi \alpha x = \phi^{-1} f^q \phi x,$$

and this is satisfied if $\phi x = \chi (\bar{x}, \overline{\alpha x}, \overline{\alpha^{n-1} x})$

and also $f^{q-p} v = v$, for it then becomes

$$\phi^{-1} f^p \phi \alpha x = \phi^{-1} f^p \phi x, \text{ where } \phi x = \phi \alpha x.$$

If in a function of two variables, as $\psi(x, y)$, we substitute the function itself instead of one of those quantities, the result is denoted thus,

$$\psi\{x, \psi(x, y)\} = \psi^{1,2}(x, y),$$

$$\psi\{\psi(x, y), y\} = \psi^{2,1}(x, y),$$

if the function itself is substituted simultaneously for x and y, it is denoted thus

$$\psi\{\psi(x, y), \psi(x, y)\} = \psi^{\overline{2,2}}(x, y).$$

(69). Given $\psi^{\overline{2,2}}(x, y) = a,$

By means of the substitution $\phi^{-1}f\phi x$, for ψx, we are enabled to reduce functional equations of any order to those of the first, a substitution nearly resembling it, will be of equal value for those which contain two or more variables, by assuming

$$\psi(x, y) = \phi^{-1}f(\phi x, \phi y),$$

we have

$$\psi^{\overline{2,2}}(x, y) = \phi^{-1}f\{\phi\phi^{-1}f(\phi x, \phi y), \phi\phi^{-1}f(\phi x, \phi y)\}$$
$$= \phi^{-1}f^{\overline{2,2}}(\phi x, \phi y),$$

and substituting this value in the equation

$$\phi^{-1}f^{\overline{2,2}}(\phi x, \phi y) = a.$$

Put $\phi^{-1}x$ for x, and ϕ^{-1} for y, also taking the function ϕ on both sides

$$f^{\overline{2,2}}(x, y) = \phi a.$$

If therefore we are acquainted with a particular solution, we find the general one; let the function $A\frac{x}{y}$ be tried, then

$$f^{\overline{2,2}}(x,\ y) = A\,\frac{A^{\frac{x}{y}}}{A^{\frac{x}{y}}} = \phi\,a$$

hence $A = \phi\,a$, and the solution is

$$\psi\,(x,\ y) = \phi^{-1}\left(\frac{\phi\,x}{\phi\,y}\,\phi\,a\right),$$

a variety of solutions may be found of different forms, such as

$$\psi\,(x,\ y) = \frac{a}{\phi\,(1)}\,\phi\left(\frac{x}{y}\right),\quad \psi\,(x,\ y) = \frac{a}{\phi\,(1)}\,\phi\left(\frac{\alpha\,(x,\ y)}{\beta(x,\ y)}\right),$$

where α and β are any two homogeneous functions of the same degree.

(70). If $\psi\,(x,y) = a\,x + b\,y$,

then $\psi^{\overline{n,n}}\,(x,\ y) = (a + b)^{n-1}\,(a\,x + b\,y)$,

(71). If $\psi\,(x,\ y)$ is any homogeneous function of x, and y of the degree n,

then

$$\psi^{\overline{k,k}}\,(x,\ y) = \{\,\psi\,(x,\ y)\,\}^{n^{k-1}} \times \{\,\psi\,(1,\ 1)\,\}^{\frac{1-n^{k-1}}{1-n}}.$$

(72). Given $\psi^{\overline{2,2}}\,(x,\ y) = \sqrt{\psi\,(x,\ y)}$,

$$\psi\,(x,\ y) = \sqrt{\frac{x^2 + y^2}{2\,y\,\phi\left(\frac{x}{y}\right)}}\,\phi\,(1).$$

(73). Given

$$\psi^{\overline{2,2}}\,(x,\ y) = \psi\,(x,\ y) + \frac{1}{\psi\,(x,\ y)},$$

$$\psi(x, y) = \frac{2\,\phi\left(\frac{x}{y}\right)}{(x + y)\,\phi(1)} + \frac{(x + y)\,\phi(1)}{2\,\phi\left(\frac{x}{y}\right)}.$$

(74). Given $\psi^{\overline{2,2}}(x, y) = \dfrac{1 - \psi(x, y)}{1 + \psi(x, y)}$,

$$\psi(x, y) = \frac{y\,\phi(1) - x^2\,\phi\left(\frac{x}{y}\right)}{y\,\phi(1) + x^2\,\phi\left(\frac{x}{y}\right)}.$$

(75). $\psi^{\overline{2,2}}(x, y) = F\psi(x, y)$,

$$\psi(x, y) = F\left(\frac{\alpha(x, y)}{\beta(x, y)}\right),$$

provided α and β are homogeneous with respect to x and y; the first of the $n + 1$ degree, the second of the nth, and also at the same time $\alpha(1, 1) = \beta(1, 1)$.

(76). Given $\psi^{\overline{2,2}}(x, y) = F\psi(x, y)$.

Another solution of the same equation is

$$\psi(x, y) = F\left(\frac{\alpha(x, y)}{\beta(x, y)}\,x\right),$$

where α and β are two such functions, that when $x = y$, we have also

$$\alpha(x, y) = \beta(x, y).$$

(77). Given $\psi^{\overline{3,3}}(x, y) = \psi(x, y)$,

$$\psi(x, y) = \left\{a - \left(\frac{x^2 + y^2}{x + y}\right)^n\right\}^{\frac{1}{n}}.$$

(78). Given $\psi^{\overline{n,n}}(x, y) = \{\psi(x, y)\}^{m}$

$$\psi(x, y) = \left\{\frac{2xy\,\phi(1)}{(x+y)\,\phi\left(\frac{x}{y}\right)}\right\}^{m^{\frac{1}{n-1}}}.$$

(79.) Given $\psi^{\overline{4,4}}(x, y) = \{\psi^{\overline{2,2}}(x, y)\}^{2}$

$$\psi(x, y) = \left\{\frac{(x+y)\,\phi(1)}{2\,\phi\left(\frac{x}{y}\right)}\right\}^{\sqrt{2}}.$$

(80). Given $\psi^{\overline{3,3}}(x, y) = \frac{x^2}{y}$.

$$\psi(x, y) = \phi^{-1}\left\{\frac{1 + \phi\frac{x^2}{y}}{1 - 3\,\phi\frac{x^2}{y}}\right\}.$$

(81). Given $x\,\psi^{1,2}(x, y) = y\,\psi^{2,1}(x, y)$, put $\phi^{-1}f(\phi x, \phi y)$ for ψx, then it becomes

$$x\,\phi^{-1}f^{1,2}(\phi x, \phi y) = y\,\phi^{-1}f^{2,1}(\phi x, \phi y);$$

putting $\phi^{-1}x$ for x, and $\phi^{-1}y$ instead of y, we have

$$\phi^{-1}x\,.\,\phi^{-1}f^{1,2}(x, y) = \phi^{-1}y\,.\,\phi^{-1}f^{2,1}(x, y);$$

if $f^{1,2}(x, y) = y$, and $f^{2,1}(x, y) = x$, this equation, becomes identical; but making $f(x, y) = a - x - y$, these two equations are verified; consequently the general solution is

$$\psi(x, y) = \phi^{-1}(a - \phi x - \phi y).$$

(82). Given $\psi^{1,2}(x, y)\,.\,\psi^{2,1}(x, y) = xy$,

$$\psi(x, y) = \phi^{-1}\left(\frac{a}{\phi x\,.\,\phi y}\right).$$

(83). Given $x\,\psi^{\overline{2,2}}(x, y) = a\,\psi^{2,1}(x, y)$,

$$\psi(x, y) = \phi^{-1}\left(\frac{\phi a\,.\,\phi y}{\phi x}\right).$$

Various methods for the solution of Functional Equations may be found in the following writings :

Speculationes Analytico Geometricæ, *N. Fuss.* Mem. de l'Acad. Imp. de St. Petersburg, Vol. IV. p. 225. 1811.

Memoirs of the Analytical Society, p. 96. 1813.

Observations on various points of Analysis, Phil. Trans. *J. F. W. Herschel.*

Essay towards the Calculus of Functions, *C. Babbage.* 1815.

Ditto, Part II. p. 179. 1816.

Observations on the analogy which subsists between the Calculus of Functions, and other branches of Analysis, Phil. Trans. 1817. p. 197. *C. Babbage.*

Spence's Essays, 1819. Note by *J. F. W. Herschel,* p. 151.

Annals of Philosophy, Nov. 1817. *Mr. Horner.*

Journal of the Royal Institution. *C. Babbage.*

Part III.

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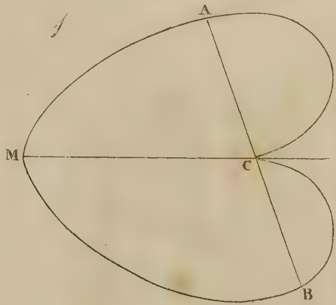

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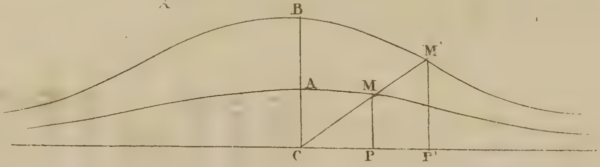

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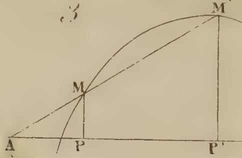

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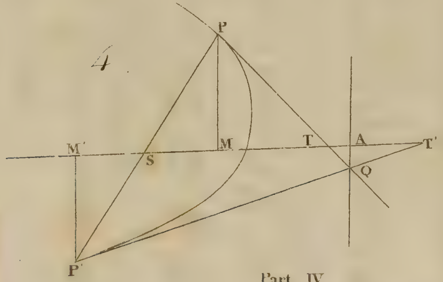

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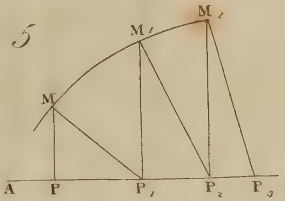
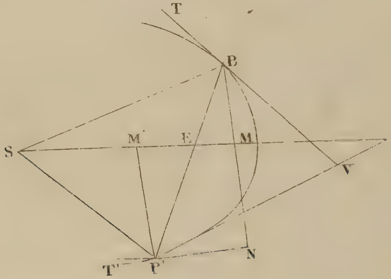
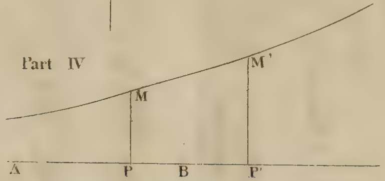




Part IV



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